Exercises for MATH 1210 Supplementary Notes on Geometry

T. G. Berry and M. Davidson

Department of Mathematics, University of Manitoba

1 Cartesian Coordinates

1. On a diagram locate and identify the following points:

$$P_1(2,0),\ P_2(0,-4),\ P_3(-3,4),\ P_4(2,-\pi),\ P_5(-5,-1).$$

2. On a diagram locate and identify the following points:

$$P_1(-1,0),\ P_2(0,-1),\ P_3(3,2).$$

and verify that

$$|P_1P_3|^2 = |P_1P_2|^2 + |P_2P_3|^2.$$

What does this tell you about the triangle $P_1P_2P_3$?

3. On a diagram locate and identify the following points:

$$P_1(2,0,2),\ P_2(-1,3,2),\ P_3(1,1,0).$$

and evaluate the quantities $|P_1P_2|$, $|P_3P_1|$ and $|P_3P_2|$.

What do these values tell you about the triangle $P_1P_2P_3$?

2 Vectors in and Algebraic Operations on Them

1. Consider the four points $P_1(1,1,0)$, $P_2(2,0,3)$, $P_3(0,2,5)$ and $P_4(-1,3,2)$ in $\mathbb{E}^3$.

   (a) Show that $\overrightarrow{P_1P_3} = \overrightarrow{P_1P_2} + \overrightarrow{P_1P_4}$.
   What does this tell you about the quadrilateral $P_1P_2P_3P_4$?

   (b) Verify that the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_4}$ is the same as the angle between $\overrightarrow{P_3P_2}$ and $\overrightarrow{P_3P_4}$.

   (c) Find the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_3P_4}$.

2. If $\overrightarrow{u} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\hat{\imath} + \hat{\jmath} + \frac{1}{2}\hat{k}\right)$ and $\overrightarrow{v} = \frac{1}{2\sqrt{2}}\left(\hat{\imath} - \hat{k}\right)$, find $\overrightarrow{u} + \overrightarrow{v}$, $\overrightarrow{u} - \overrightarrow{v}$ and the angle between the last two vectors.

3. Consider the vectors $\overrightarrow{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\overrightarrow{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ in $\mathbb{E}^3$, and let α denote the angle between them.

   (a) Evaluate $\|\overrightarrow{a}\|$, $\|\overrightarrow{b}\|$ and $\overrightarrow{a} \cdot \overrightarrow{b}$ and use them to find $cos(\alpha)$.

   (b) Find $\overrightarrow{a} \times \overrightarrow{b}$ and $\|\overrightarrow{a} \times \overrightarrow{b}\|$, and use these results, together with the results of part (a), in order to verify that

$$\|\overrightarrow{a} \times \overrightarrow{b}\| = \|\overrightarrow{a}\|\|\overrightarrow{b}\|sin(\alpha).$$

4. Simplify the expression $\left[\left(\left[\hat{\imath} \times \hat{\jmath}\right] \times \hat{\jmath}\right) \times \hat{\jmath}\right] \cdot \left(\hat{\imath} + 2\hat{\jmath} + 7\hat{k}\right)$.

5. You are given the vector $\overrightarrow{r} = 2\hat{\imath} + 3\hat{\jmath} + \sqrt{3}\hat{k}$.

   (a) Find a vector of **unit length** that points in the same direction as $\overrightarrow{r}$. This vector is known as **the unit vector in the direction of** $\overrightarrow{r}$, and is typically denoted by $\hat{r}$.

   (b) Use the vector of part (a) to find a vector of length 2 in the direction of $\overrightarrow{r}$.

   (c) Use the vector of part (a) to find a vector of length -3 in the direction of $\overrightarrow{r}$.

6. Throughout this exercise, suppose that P and Q are **any** two points in $\mathbb{E}^3$, other than O. Let $\overrightarrow{p} = \overrightarrow{OP}$ and $\overrightarrow{q} = \overrightarrow{OQ}$.

   (a) Use the definitions of scalar multiplication and vector addition to verify that $\overrightarrow{p} - \overrightarrow{q} = \overrightarrow{QP}$.

   (b) Find a vector from O to the mid-point M of the line segment PQ, and express $\overrightarrow{m} = \overrightarrow{OM}$ as a combination of $\overrightarrow{p}$ and $\overrightarrow{q}$.

   (c) Let T be the point on the line segment PQ determined by the condition that $|PT| = \frac{1}{3}|PQ|$. Find the vector $\overrightarrow{OT}$.

   (d) In $\mathbb{E}^3$, describe the figure consisting of all vectors $\overrightarrow{x}$ at O satisfying the condition that $\|\overrightarrow{x} - \overrightarrow{p}\| = 2$.

   (e) Let P_1 and Q_1 be the midpoints of the the line segments OP and OQ respectively. Show that $\overrightarrow{Q_1P_1}$ is parallel to and half the length of $\overrightarrow{QP}$

7. Consider a cube in $\mathbb{E}^3$.

   Find the angle between a diagonal and any of the adjacent edges of the cube.

8. Consider a rectangular parallelopiped (i.e., a rectangular box) in $\mathbb{E}^3$, whose sides are of length a, b and c metres respectively.

   Find the angle between a diagonal and each of the three adjacent edges.

3 Lines and Planes

1. At $P_0(1, 2, -3)$ in $\mathbb{E}^3$, let $\overrightarrow{u} = -2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\overrightarrow{v} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

    (a) Find the scalar parametric equations of the line L through P_0 in the direction of the vector $\overrightarrow{u} \times \overrightarrow{v}$.

    (b) Find the coordinates of the point P^* on L which lies a distance of $2\|\overrightarrow{u} \times \overrightarrow{v}\|$ from P_0, to the side of P_0 determined by $\overrightarrow{u} \times \overrightarrow{v}$.

    (c) Find the symmetric equations of the line L.

2. Consider the plane $\mathcal{P}$ passing through $P_0(1, 2, -3)$ with normal vector $\overrightarrow{u} \times \overrightarrow{v}$, where $\overrightarrow{u} = -2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\overrightarrow{v} = \hat{\imath} - 3\hat{\jmath} + 4\hat{k}$, as in the previous exercise.

    (a) Find the equation of the plane $\mathcal{P}$ in point-normal form.

    (b) Find the coordinates of the point P^* on $\mathcal{P}$ which is obtained by adding $-2\overrightarrow{u}$ and $+3\overrightarrow{v}$ to $\overrightarrow{X_0} = \overrightarrow{OP_0}$.

    (c) Verify that P^* lies on $\mathcal{P}$.

3. Find all points of intersection of the following lines and/or planes in $\mathbb{E}^3$ and, in each case, explain the geometrical significance of your answer(s):

    (a) Two planes, namely, $\mathcal{P}_1$: $x + y - 2z = 3$ and $\mathcal{P}_2$: $3x + 4y + z = 5$,

    (b) Two planes, namely, $\mathcal{P}_1$: $x + y - 2z = 3$ and $\mathcal{P}_3$: $3x + 3y - 6z = 5$,

    (c) A plane $\mathcal{P}_1$: $x + y - 2z = 3$ and a line L_1: $x = 4 - 2t$, $y = -3 - t$, $z = 4 + t$ (with parameter "t"),

    (d) Two lines, namely, L_1: $x = 4 - 2t$, $y = -3 - t$, $z = 4 + t$ (with parameter "t") and L_2: $x = 2 + s$, $y = -4 + 2s$, $z = 5 - s$ (with parameter " s")

    (e) Two lines, namely, L_1: $x = 4 - 2t$, $y = -3 - t$, $z = 4 + t$ (with parameter "t") and L_3: $x = 6 - 2s$, $y = -1 + s$, $z = 5 - s$ (with parameter "s")

    (f) Two lines, namely, L_1: $x = 4 - 2t$, $y = -3 - t$, $z = 4 + t$ (with parameter "t") and L_4: $x = 6 - 2s$, $y = -1 - s$, $z = 3 + s$ (with parameter "s")