Total of marks, indicated in brackets, is 30.

[10] **1.** Prove, using mathematical induction (no marks for any other method), that for every positive integer n, P_n:

$$\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots+\frac{1}{5^{2n}}=\frac{1}{4}\left(1-\frac{1}{5^{2n}}\right).$$

2. [3] **(a)** State the left side of P_n above using sigma notation.

[4] **(b)** Adjust, if necessary, your formula from part (a) to allow you to obtain P_n from the following identity by matching the left sides:

$$\sum_{j=0}^{m-1} r^j=\frac{1-r^m}{1-r}.$$

[5] **3.** By translating $\sqrt{3}+i$ into polar or exponential form, find its fifth power and translate that power into Cartesian form.

4. [6] **(a)** Find the six sixth roots of $-8i$ in any form.

[2] **(b)** State one root from part (a) in Cartesian form.

Solutions and commentary

It is a good idea to read over a test/exam paper before tackling any questions. If one did that here, one might think of doing the sigma question first. Putting a sum into sigma notation—correctly—is a good way to see how it actually works. Question 1 can be done with the sums in sigma form.

1. The really important—and too common—error in this question is to indicate in any way, often by an equal sign, that P_n is a number. If it were a number, it could not be proved. P_n and all of its instances when n is given values are statements of equality between two expressions. Since P_n is true, the two expressions have the same numerical value, but neither of the expressions nor their common value is P_n. To say that it is one of these or all of them indiscriminately is to indicate confusion.

The case of P_n for $n=1$ must be verified. Left side is $\frac{1}{5}+\frac{1}{5^2}=\frac{5+1}{25}=\frac{6}{25}$. Right side is

$$\frac{1}{4}\left(1-\frac{1}{5^2}\right)=\frac{1}{4}\left(\frac{25-1}{25}\right)=\frac{6}{25}=\text{LS}.$$

A common error was taking the left side to consist of only the final term; this is not to notice how the sum works, deadly at the inductive step. The arithmetic for the right side was found difficult by an embarrassing number.

Now it must be assumed that k is a positive integer such that P_k is true.

$$\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots+\frac{1}{5^{2k}}=\frac{1}{4}\left(1-\frac{1}{5^{2k}}\right).$$

Then it must be proved that P_{k+1} is also true.

$$\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots+\frac{1}{5^{2(k+1)}}=\frac{1}{4}\left(1-\frac{1}{5^{2(k+1)}}\right).$$

This is the stage where the thinking usually has to be done in induction problems, and it is where many persons did less thinking than was necessary, assuming that the final term was added to the left side of the P_k case, which is to ignore the second-last term. The left side equals

$$\begin{aligned}\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\cdots+\frac{1}{5^{2k}}+\frac{1}{5^{2k+1}}+\frac{1}{5^{2k+2}} &= \frac{1}{4}\left(1-\frac{1}{5^{2k}}\right)+\frac{1}{5^{2k+1}}+\frac{1}{5^{2k+2}}\\
&= \frac{1}{4}-\frac{1}{4\cdot 5^{2k}}+\frac{1}{5^{2k+1}}+\frac{1}{5^{2k+2}}\\
&= \frac{1}{4}+\frac{1}{5^{2k}}\left(-\frac{1}{4}+\frac{1}{5}+\frac{1}{5^2}\right)\\
&= \frac{1}{4}+\frac{1}{5^{2k}}\left(-\frac{25}{4\cdot 25}+\frac{20}{4\cdot 25}+\frac{4}{4\cdot 5^2}\right)\\
&= \frac{1}{4}+\frac{1}{5^{2k}}\left(-\frac{1}{4\cdot 25}\right)\\
&= \frac{1}{4}\left(1-\frac{1}{5^{2(k+1)}}\right),\end{aligned}$$

which is the right side. Aside from forgetting the second-last term, the commonest errors were failing to use the laws of indices correctly and failing to add the fractions by using a lowest common denominator—or indeed any common denominator. Once one has shown that P_k implies P_{k+1}, one invokes the principle of mathematical induction to reach the conclusion that P_n is true for all $n \geq 1$. The principle is not used to prove P_{k+1} as some wrote.

2. The obvious way to use sigma is to use an index letter for the power of 5 in the denominator. Another is to use negative indices. The commonest error was to try to use n as the index letter. Since the number of terms is $2n$, n cannot be used as the index. The index has to run, say, from 1 to $2n$, something n cannot coherently do. When is $n = 2n$? When $n = 0$. It's not just wrong; it makes no sense.

$$\sum_{i=1}^{2n}\frac{1}{5^i}=\sum_{i=1}^{2n}5^{-i}.$$

If one read the whole question, one could take account of part (b) and do the sum in the form wanted in the first place, $\sum_{j=0}^{2n-1} 1/5^{j+1}$. This would more normally be done by substituting $j+1$ for i, probably thought of in the form $i-1=j$. Then when $i=1$, $j=0$, and when $i=2n$, $j=2n-1$. In any case, to match the given identity (the sum of a geometric series) one needs to get this sum. Since there are one too many 1/5s, one of them has to be taken outside the sigma. The result then follows by substitution ($r = 1/5$ not 5) and arithmetic:

$$\frac{1}{5}\sum_{j=0}^{2n-1}\left(\frac{1}{5}\right)^j = \frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^{2n}}{1-\left(\frac{1}{5}\right)} = \frac{1}{4}\left[1-\left(\frac{1}{5}\right)^{2n}\right].$$

Most of the questions we ask you don't require any particular originality to answer. But that does not prevent the invention of interesting ways to do them. One way to get around the too-many-1/5s problem was this:

$$\sum_{i=1}^{2n}\frac{1}{5^i} = \frac{5^{2n-1}+5^{2n-2}+\cdots+5^1+5^0}{5^{2n}} = \frac{1}{5^{2n}}\sum_{j=0}^{2n-1}5^j$$
$$= \frac{1}{5^{2n}}\frac{1-5^{2n}}{1-5} = \frac{1}{5^{2n}}\frac{5^{2n}-1}{5-1} = \frac{5^{2n}-1}{4\cdot 5^{2n}} = \frac{1}{4}\left(1-\frac{1}{5^{2n}}\right).$$

The problem of wanting $m-1$ at the top was also solved this way:

$$\sum_{i=1}^{2n}\left(\frac{1}{5}\right)^i = \sum_{i=0}^{(2n+1)-1}\left(\frac{1}{5}\right)^i - \sum_{i=0}^{1-1}\left(\frac{1}{5}\right)^i$$
$$= \frac{1-\left(\frac{1}{5}\right)^{2n+1}}{1-\frac{1}{5}} - \frac{1-\left(\frac{1}{5}\right)^1}{1-\frac{1}{5}} = \frac{-\left(\frac{1}{5}\right)^{2n+1}+\frac{1}{5}}{\frac{4}{5}}$$
$$= \frac{\frac{1}{5}-\left(\frac{1}{5}\right)^{2n+1}}{4\left(\frac{1}{5}\right)} = \frac{1}{4}\left[1-\left(\frac{1}{5}\right)^{2n}\right] = \frac{1}{4}\left(1-\frac{1}{5^{2n}}\right).$$

3. Putting a complex number into polar or exponential form amounts to finding the modulus and argument of the number. Since $|\sqrt{3}+i| = \sqrt{\sqrt{3}^2+1^2} = \sqrt{4} = 2$ and $\arg(\sqrt{3}+i) = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$, we get

$$\sqrt{3}+i = 2(\cos\frac{\pi}{6}+\sin\frac{\pi}{6}i) = 2e^{\frac{\pi}{6}i}$$

The first of these is polar form, the second is exponential. Finding the fifth power using DeMoivre's Theorem:

$$(\sqrt{3}+i)^5 = 2^5(\cos\frac{5\pi}{6}+\sin\frac{5\pi}{6}i) = 2^5e^{\frac{5\pi}{6}i}.$$

Putting this into Cartesian form we get:

$$(\sqrt{3}+i)^5 = 2^5(\cos\frac{5\pi}{6}+\sin\frac{5\pi}{6}i) = 32(-\frac{\sqrt{3}}{2}+\frac{1}{2}i) = -16\sqrt{3}+16i.$$

The most common mistakes in this question were finding the modulus to be $\sqrt{\sqrt{3}^2+i^2}$ or finding the argument to be $\frac{\pi}{3}$ instead of $\frac{\pi}{6}$. Some students also had trouble converting back into Cartesian form, or simply had not bothered.

4. To find the roots of $-8i$, we first want to find $-8i$ in polar or exponential form. Since $|-8i| = 8$ and $\arg(-8i) = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$ we get $-8i = 8e^{-\frac{\pi}{2}i}$ or $= 8e^{\frac{3\pi}{2}i}$.
The modulus of any sixth root of $-8i$ will be $\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt{2}$.
The argument of a sixth root of $-8i$ will be $\frac{1}{6}(\frac{3\pi}{2} + 2k\pi)$ for some value of k. Using the values $k = 0 \ldots 5$ we get

$$\frac{3\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}.$$

Hence the six sixth roots of $-8i$ are$\sqrt{2}e^{\frac{3\pi}{12}i}, \sqrt{2}e^{\frac{7\pi}{12}i}, \sqrt{2}e^{\frac{11\pi}{12}i}, \sqrt{2}e^{\frac{15\pi}{12}i}, \sqrt{2}e^{\frac{19\pi}{12}i}, \sqrt{2}e^{\frac{23\pi}{12}i}$.
One of the more common mistakes in this question was to find the argument of one root, and then add to it 2π rather than $\frac{2\pi}{6}$.
For part (b), this was most easily done for $\sqrt{2}e^{\frac{3\pi}{12}i}$ or $\sqrt{2}e^{\frac{15\pi}{12}i}$. So

$$\sqrt{2}e^{\frac{3\pi}{12}i} = \sqrt{2}(\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})i) = \sqrt{2}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = 1 + i$$

or

$$\sqrt{2}e^{\frac{15\pi}{12}i} = \sqrt{2}(\cos(\frac{5\pi}{4}) + \sin(\frac{5\pi}{4})i) = \sqrt{2}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = -1 - i.$$