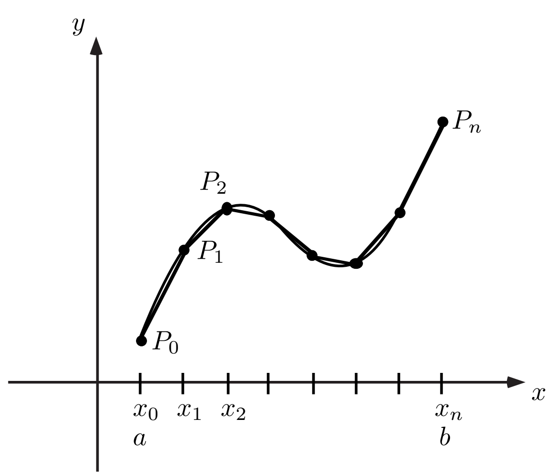
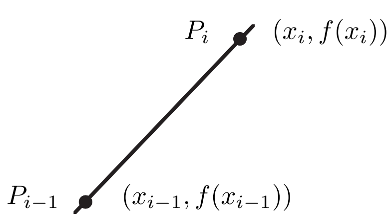
Chapter 9. Arc Length and Surface Area

In which

We apply integration to study the lengths of curves and the area of surfaces.

9.1 Arc Length (Text 547–553)





- distance from P_{i-1} to P_i

$$\boxed{\sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}}$$

- let $\Delta x = x_i - x_{i-1}$ be the same for all i

$$s \sim \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2} \cdot \Delta x$$

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2} \cdot \Delta x$$

- for some $x_{i-1} < c_i < x_i$ by the M. V. Th.

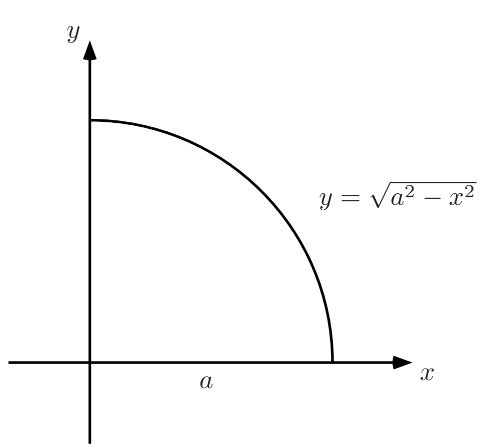
- take the limit as $\Delta x \to 0$

$$\boxed{s = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx}$$

- notice that for the limit (integral) to exist we need some hypothesis on the integrand

  – continuity of f' will do

Example 9.1.1 Find the circumference of a circle.



$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
s &= 4\int_0^a \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx \\
&= 4\int_0^a \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\, dx = 4a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \\
&= 4a \lim_{R \to a^-} \left[\text{Arcsin}\frac{x}{a}\right]_0^R \\
&= 4a[\text{Arcsin}(1) - \text{Arcsin}(0)] = 4a\frac{\pi}{2} \\
&= 2\pi a
\end{aligned}$$

Example 9.1.2 Find the length along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 9$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4x}}$$

$$L = \int_0^9 \sqrt{1 + \frac{1}{4x}}\, dx = \int_0^9 \frac{\sqrt{4x+1}}{2\sqrt{x}}\, dx$$

- this is not an easy integral
- substitute $t^2 = 4x$; $\qquad 2t\,dt = 4\,dx$

x	$t = 2\sqrt{x}$
9	6
0	0

$$\begin{aligned}
L &= \int_0^6 \frac{\sqrt{t^2+1}}{t}\,\frac{t\,dt}{2} \\
&= \frac{1}{2}\int_0^6 \sqrt{t^2+1}\, dt
\end{aligned}$$

- now the substitution is $t = \tan\theta$
- the integral that results is $\displaystyle\int \sec^3\theta\, d\theta$

- alternate method – interchange the roles of x and y
- Find the length of $x = y^2$ from $y = 0$ to $y = 3$

$$\frac{dx}{dy} = 2y$$

$$L = \int_0^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^3 \sqrt{1+4y^2}\, dy$$

- substitute $2y = \tan\theta$ to give $\displaystyle\int \sec^3\theta\, d\theta$

Example 9.1.3 Find the length along the curve $y = \ln x$ from $x = 1$ to $x = e$.

$$\frac{dy}{dx} = \frac{1}{x}$$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx$$

- substitute $t^2 = x^2 + 1$ to give $\displaystyle\int \frac{t^2}{t^2-1}\, dt$

9.2 Arc Length of Parametric Curves (Text 663–665)

- approximate length by straight lines

$$L \approx \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$x_i = f(t_i) \Rightarrow \Delta x_i = f'(c_i)\Delta t \quad \text{by MVTh.}$$

$$y_i = g(t_i) \Rightarrow \Delta y_i = g'(d_i)\Delta t \quad \text{by MVTh.}$$

$$L \approx \sum_i \sqrt{(f'(c_i)^2 + (g'(d_i)^2}\Delta t$$

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

- requires
  - $f'(t)$, $g'(t)$ continuous for $\alpha \le t \le \beta$
  - the curve is traversed once
- gives old formula if parameter is x

Example 9.2.1 The circumference of a circle.

- parametric equations of a circle

$$x = a\cos t,\ y = a\sin t; \quad 0 \le t \le 2\pi$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt \\ &= \int_0^{2\pi} a\, dt = at\Big|_0^{2\pi} = 2\pi a \end{aligned}$$

Example 9.2.2 Find the length of the curve
$x = t^2 - t$; $y = t^2 + t$ from $t = -1/2$ to $t = 3/2$.

$$L = \int_{-1/2}^{3/2} \sqrt{(2t-1)^2 + (2t+1)^2}\, dt$$

$$= \int_{-1/2}^{3/2} \sqrt{8t^2 + 2}\, dt = \sqrt{2}\int_{-1/2}^{3/2} \sqrt{(2t)^2 + 1}\, dt$$

- to do this integral substitute $\tan\theta = 2t$
- recall $\displaystyle\int \sec\theta = \ln|\sec\theta + \tan\theta|$

Example 9.2.3 Find the length of the curve
$x = 4\cos^3 t;\ y = 4\sin^3 t.$

$$L = \int_0^{2\pi} \sqrt{[12\cos^2 t(-\sin t)]^2 + [12\sin^2 t(\cos t)]^2}\, dt$$

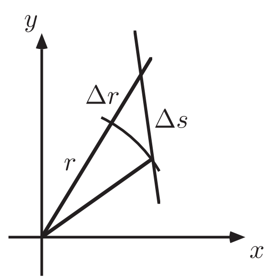
$$= 12\int_0^{2\pi} |\cos t \sin t|\, dt$$

$$= 12 \times 4 \left(\frac{\sin^2 t}{2}\right)\Bigg|_0^{\pi/2} = 24$$

! • What happens if we ignore the absolute value signs?

9.3 Arc Length of Polar Curves (Text 682–683)

• length of a sector of a circle



• compare length along the curve with length along the arc of a circle

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2$$

$$= \left(\left(\frac{\Delta r}{\Delta \theta}\right)^2 + r^2\right)(\Delta\theta)^2$$

$$\boxed{\begin{aligned} s &= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{\big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2}\, d\theta \end{aligned}}$$

- formula can also be obtained from

$$\int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

- using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example 9.3.1 Find the circumference of a circle.

- circle, centre $(0,0)$, radius a is $r = a$
- circumference

$$C = \int_0^{2\pi} \sqrt{0^2 + a^2}\, d\theta$$

$$= a\int_0^{2\pi} d\theta$$

$$= a\theta\Big|_0^{2\pi} = 2\pi a$$

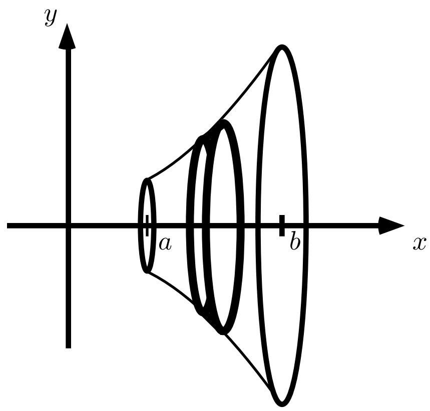
Example 9.3.2 Find the arclength of one leaf of $r = \sin 2\theta$.

- review sketch (example 8.7.4)
  - one leaf is swept out when $0 \le \theta \le \dfrac{\pi}{2}$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{(2\cos 2\theta)^2 + (\sin 2\theta)^2}\, d\theta$$

- this integral is not easy to do

9.4 Surface Area (Text 554–560)

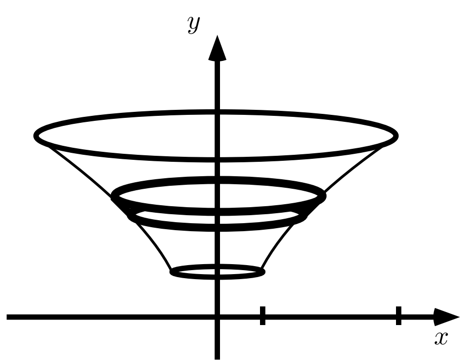


- approximate the surface using cylinders
  - radius of the cylinder is $f(c_i)$
  - height of cylinder is length along the curve
- for Δx, length is

$$\sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x \quad = \Delta s$$

$$S \approx \sum 2\pi f(c_i)\sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x \quad = \sum 2\pi f(c_i)\Delta s$$

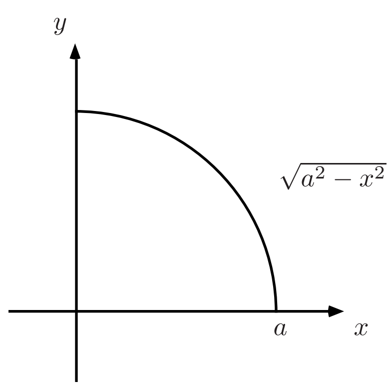
$$\boxed{S = 2\pi\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx = 2\pi\int_a^b f(x)\,ds}$$



- rotation about the y-axis
  - radius of the cylinder is x
  - length along the curve as above

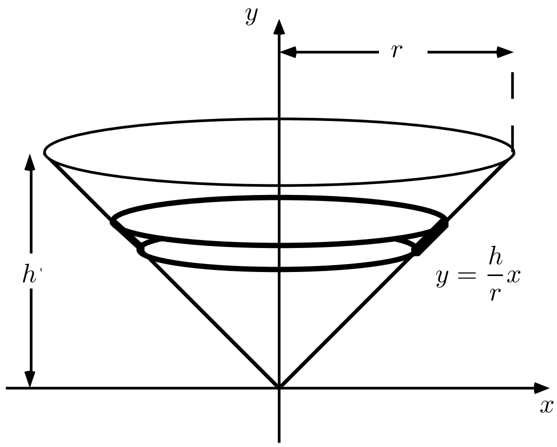
$$\boxed{S = 2\pi \int_a^b x\sqrt{1+\left(f'(x)\right)^2}\,dx = 2\pi \int_a^b x\,ds}$$

Example 9.4.1 Find the surface area of a sphere.



$$
\begin{aligned}
S &= 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx \\
&= 2\cdot 2\pi \int_0^a \sqrt{a^2-x^2}\sqrt{1+\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)^2}\,dx \\
&= 4\pi \int_0^a \sqrt{a^2-x^2}\left(\frac{a}{\sqrt{a^2-x^2}}\right)dx \\
&= 4\pi a \int_0^a dx = 4\pi a^2
\end{aligned}
$$

Example 9.4.2 Find the surface area of a cone.



$$S = 2\pi \int_0^r x\sqrt{1 + \left(\frac{h}{r}\right)^2}\, dx$$

$$= 2\pi\sqrt{1 + \left(\frac{h}{r}\right)^2}\, \left[\frac{x^2}{2}\right]_0^r$$

$$= 2\pi\sqrt{1 + \left(\frac{h}{r}\right)^2}\, \left[\frac{r^2}{2}\right]$$

$$= \pi r\sqrt{r^2 + h^2}$$

Example 9.4.3 Find the surface area of the object formed when $y = x^{3/2}$ from $x = 1$ to $x = 9$ is revolved about the y-axis.

$$S = 2\pi \int_1^9 x\sqrt{1 + \left((x^{3/2})'\right)^2}\, dx \qquad \left(2\pi \int x\, ds\right)$$

$$= 2\pi \int_1^9 x\sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= 2\pi \int_1^9 x\sqrt{1 + \frac{9}{4}x}\, dx$$

• by first formula with $x \leftrightarrow y$

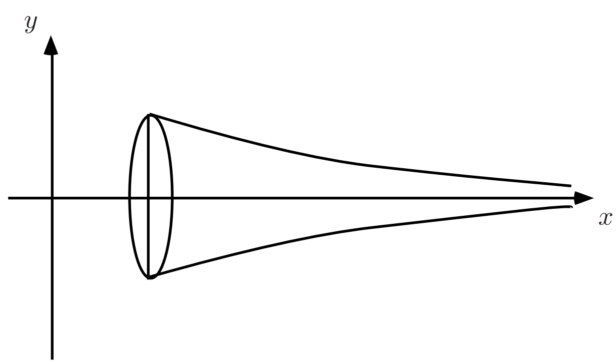
$$x = y^{2/3}$$

x	$y = x^{3/2}$
1	1
9	27

$$S = 2\pi \int_1^{27} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \left(2\pi \int x\, ds\right)$$

$$= 2\pi \int_1^{27} y^{2/3}\sqrt{1 + \left(\frac{2}{3}y^{-1/3}\right)^2}\, dy$$

Example 9.4.4 Find the volume and surface area of the infinitely long horn formed when $\frac{1}{x}$ is rotated about the x-axis from $x = 1$ to $x = \infty$.



$$V = \int_1^{\infty} \pi y^2\, dx$$

$$= \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \int_1^R \frac{dx}{x^2}$$

$$= \pi \lim_{R\to\infty} \left[\frac{-1}{x}\right]_1^R = \pi \lim_{R\to\infty} \left[\frac{-1}{R} + 1\right]$$

$$= \pi(0+1) = \pi$$

• Conclusion: The horn can be filled with π units of paint.

$$S = 2\pi \int_1^\infty y\sqrt{1+(y')^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$$

- this integral need not be evaluated

- notice

$$\frac{1}{x} \le \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} \qquad \text{for } x \ge 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{dx}{x}$$

$$= \lim_{R\to\infty} \ln x \,\Big|_1^R = \lim_{R\to\infty} \ln R - 0 = \infty$$

- by comparison $\displaystyle\int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx = \infty$
- Conclusion: The surface cannot be painted!

9.5 Surface Area and Parametric Curves (665–666)

$$S = \int 2\pi y\; ds$$

- for rotation about the x-axis
- for parametric $x = f(t)$, $y = g(t)$, just showed

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- needs f', g' continuous and $g \ge 0$

- summary

  – for rotation about the x-axis

$$\boxed{S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

  –for rotation about the y-axis

$$\boxed{S = \int_{\gamma}^{\delta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example 9.5.1 The surface area of a sphere.

- parametric equations of a semicircle

$$x = a\cos t,\ y = a\sin t; \quad 0 \le t \le \pi$$

- rotate about the x-axis to get a sphere

- $ds = a\ dt$ for the circle

$$S = \int_0^{\pi} 2\pi y\ ds = 2\pi \int_0^{\pi} (a\sin t)\ a\ dt$$

$$= 2\pi a^2(-\cos t)\Big|_0^{\pi} = 4\pi a^2$$

Example 9.5.2 The surface area of $x = e^t - t,\ y = 4e^{t/2}$, $0 \le t \le 1$, revolved about the x-axis.

$$S = 2\pi \int_0^1 4e^{t/2}\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\ dt$$

$$= 8\pi \int_0^1 e^{t/2}\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\ dt$$

$$= 8\pi \int_0^1 e^{t/2}\sqrt{(e^t+1)^2}\ dt$$

$$= 8\pi \int_0^1 e^{3t/2} + e^{t/2}\ dt = \ldots$$

Example 9.5.3 The surface area of $x = e^t - t,\ y = 4e^{t/2}$, $0 \le t \le 1$, revolved about the y-axis.

$$S = 2\pi \int_0^1 (e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt$$

$$= 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= 2\pi \int_0^1 (e^t - t)(e^t + 1)\, dt$$

$$= 2\pi \int_0^1 (e^{2t} - te^t + e^t - t)\, dt = \dots$$