MATH 1210 Assignment 2 Solutions Fall 2012

1. Find all exponential representations for

(a) $(-\sqrt{3}-i)^6$ (b) $\dfrac{(1+i)^{14}(2+2\sqrt{3}i)^4}{4^6 i(1-i)}$

(a) $(-\sqrt{3}-i)^6 = (2e^{-5\pi i/6})^6 = 2^6 e^{-5\pi i} = 2^6 e^{\pi i + 2k\pi i} = 64e^{(2k+1)\pi i}$

(b)
$$\begin{aligned}\frac{(1+i)^{14}(2+2\sqrt{3}i)^4}{4^6 i(1-i)} &= \frac{(1+i)^{14}(2+2\sqrt{3}i)^4}{4^6(1+i)} = \frac{1}{4^6}(1+i)^{13}(2+2\sqrt{3}i)^4\\ &= \frac{1}{4^6}(\sqrt{2}e^{\pi i/4})^{13}(4e^{\pi i/3})^4 = \frac{1}{4^6}(2^6\sqrt{2}e^{13\pi i/4})(4^4 e^{4\pi i/3}) = 4\sqrt{2}e^{55\pi i/12}\\ &= 4\sqrt{2}e^{7\pi i/12} = 4\sqrt{2}e^{7\pi i/12+2k\pi i} = 4\sqrt{2}e^{(24k+7)\pi i/12}\end{aligned}$$

2. Find exact values for the six sixth roots of $\sqrt{3}-i$. Express final answer in Cartesian form, simplified as much as possible.

We write $\sqrt{3}-i$ in exponential form,
$$\sqrt{3}-i = 2e^{-\pi i/6} = 2e^{-\pi i/6+2k\pi i} = 2e^{(12k-1)\pi i/6}.$$
We now take sixth roots,
$$(\sqrt{3}-i)^{1/6} = 2^{1/6}e^{(12k-1)\pi i/36}.$$
For $k = 0, 1, 2, 3, 4, 5$, we obtain the roots
$$\begin{aligned}
z_0 &= 2^{1/6}e^{-\pi i/36} = 2^{1/6}[\cos(-\pi/36) + \sin(-\pi/36)], i] = 2^{1/6}\cos(\pi/36) - \sin(\pi/36)\, i,\\
z_1 &= 2^{1/6}e^{11\pi i/36} = 2^{1/6}[\cos(11\pi/36) + \sin(11\pi/36)\, i] = 2^{1/6}\cos(11\pi/36) + 2^{1/6}\sin(11\pi/36)\, i,\\
z_2 &= 2^{1/6}e^{23\pi i/36} = 2^{1/6}[\cos(23\pi/36) + \sin(23\pi/36)\, i] = 2^{1/6}\cos(23\pi/36) + 2^{1/6}\sin(23\pi/36)\, i,\\
z_3 &= 2^{1/6}e^{35\pi i/36} = 2^{1/6}[\cos(35\pi/36) + \sin(35\pi/36)\, i] = 2^{1/6}\cos(35\pi/36) + 2^{1/6}\sin(35\pi/36)\, i,\\
z_4 &= 2^{1/6}e^{47\pi i/36} = 2^{1/6}[\cos(47\pi/36) + \sin(47\pi/36)\, i] = 2^{1/6}\cos(47\pi/36) + 2^{1/6}\sin(47\pi/36)\, i,\\
z_5 &= 2^{1/6}e^{59\pi i/36} = 2^{1/6}[\cos(59\pi/36) + \sin(59\pi/36)\, i] = 2^{1/6}\cos(59\pi/36) + 2^{1/6}\sin(59\pi/36)\, i.
\end{aligned}$$

3. What is the remainder when $P(x) = (1-2i)x^3 + 3ix^2 + 4x - 2i$ is divided by $2x - 1 + 3i$?

The remainder is
$$\begin{aligned}P\left(\frac{1-3i}{2}\right) &= (1-2i)\left(\frac{1-3i}{2}\right)^3 + 3i\left(\frac{1-3i}{2}\right)^2 + 4\left(\frac{1-3i}{2}\right) - 2i\\
&= \frac{(-5-5i)(-8-6i)}{8} + \frac{3i}{4}(-8-6i) + 2(1-3i) - 2i\\
&= \frac{1}{8}(10+70i) + \frac{3}{4}(6-8i) + 2 - 8i\\
&= \frac{31}{4} - \frac{21}{4}i.\end{aligned}$$

4. Find h and k so that remainders are 1291/2 and 123/16 when $x^4 + hx^2 - x + k$ is divided by $x + 5$ and $2x - 3$, respectively.

If $P(x) = x^4 + hx^2 - x + k$, then we can write that

$$\frac{1291}{2} = P(-5) = (-5)^2 + h(-5)^2 - (-5) + k, \qquad \frac{123}{16} = P(3/2) = \left(\frac{3}{2}\right)^4 + h\left(\frac{3}{2}\right)^2 - \frac{3}{2} + k.$$

These simplify to

$$25h + k = \frac{31}{2}, \qquad \frac{9h}{4} + k = \frac{33}{8}.$$

Solutions are $h = 1/2$ and $k = 3$.

5. In each part of this question: (i) use Descartes' rules of signs to state the number of possible positive and negative zeros of the polynomial; (ii) use the bounds theorem to find bounds for zeros of the polynomial; (iii) use the rational root theorem to list all possible rational zeros of the polynomial. Take the results of (i) and (ii) into account in (iii).

(a) $15x^8 - 2x^4 + 3x - 12$ (b) $24x^4 - 13x^3 + 2x^2 - 5x + 21$

(a)(i) Since $P(x) = 15x^8 - 2x^4 + 3x - 12$ has three sign changes, there is 3 or 1 positive zero. Since $P(-x) = 15x^8 - 2x^4 - 3x - 12$ has one sign change, there is one negative zero.
(ii) Since $M = 12$, the bounds theorem states that $|x| < 12/15 + 1 = 9/5$.
(iii) Possible rational zeros are $\pm 1, \pm 1/3, \pm 2/3, \pm 4/3, \pm 1/5, \pm 2/5, \pm 3/5, \pm 4/5, \pm 6/5, \pm 1/15, \pm 2/15, \pm 4/15$.

(b)(i) Since $P(x) = 24x^4 - 13x^3 + 2x^2 - 5x + 21$ has four sign changes, there is 4 or 2 or 0 positive zeros. Since $P(-x) = 24x^4 + 13x^3 + 2x^2 + 5x + 21$ has no sign change, there are no negative zeros.
(ii) Since $M = 21$, the bounds theorem states that $|x| < 21/24 + 1 = 15/8$.
(iii) Possible rational zeros are $1, 1/2, 3/2, 1/3, 1/4, 3/4, 7/4, 1/6, 7/6, 1/8, 3/8, 7/8, 1/12, 7/12, 1/24, 7/24$.

6. In each part of this question, use the procedure of Problem 5 to find all roots of the equation:

(a) $12x^4 + 7x^3 + 2x^2 + 7x - 10 = 0$ (b) $x^4 + 2x^3 - 41x^2 - 42x + 360 = 0$ (c) $2x^6 - x^5 + 4x - 2$

(a) Since $P(x) = 12x^4 + 7x^3 + 2x^2 + 7x - 10$ has one sign change, there is one positive root. Since $P(-x) = 12x^4 - 7x^3 + 2x^2 - 7x - 10$ has 3 sign changes, there is 3 or 1 negative root. Since $M = 10$, the bounds theorem states that $|x| < 10/12 + 1 = 11/6$. Possible rational roots are $\pm 1, \pm 1/2, \pm 1/3, \pm 2/3, \pm 5/3, \pm 1/4, \pm 5/4, \pm 1/6, \pm 5/6, \pm 1/12, \pm 5/12$, Trial and error shows that $x = \pm 1, \pm 1/2, \pm 1/3$ are not roots, but $x = 2/3$ is. We factor it from the quartic,

$$P(x) = (3x - 2)(4x^3 + 5x^2 + 4x + 5).$$

Possible rational zeros of the cubic are $-1/2, -3/2, -1/4, -5/4$. Trial and error shows that $x = -5/4$ is a solution. We factor it from the cubic,

$$P(x) = (3x - 2)(4x + 5)(x^2 + 1).$$

The remaining two solutions are $x = \pm i$.

(b) Since $P(x) = x^4 + 2x^3 - 41x^2 - 42x + 360$ has two signs changes, there is 2 or 0 positive roots. Since $P(-x) = x^4 - 2x^3 - 41x^2 + 42x + 360$ has 2 sign change, there is also 2 or 0 negative roots. Since $M = 360$, the bounds theorem states that $|x| < 360/1 + 1 = 361$. Possible rational solutions are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 9, \pm 10, \pm 12, \pm 15, \pm 18, \pm 20, \pm 24,$$

$$\pm 30, \pm 36, \pm 40, \pm 45, \pm 60, \pm 72, \pm 90, \pm 120, \pm 180, \pm 360.$$

Trial and error shows that $\pm 1, \pm 2$ are not solutions, but $x = 3$. We factor it from the polynomial,

$$P(x) = (x - 3)(x^3 + 5x^2 - 26x - 120).$$

Possible rational solutions of the cubic are

$$\pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \pm 40, \pm 60, \pm 120.$$

Trial and error shows that $\pm 3, 4$ are not zeros of the cubic, but $x = -4$ is. We factor it from the cubic,

$$P(x) = (x - 3)(x + 4)(x^2 + x - 30) = (x - 3)(x + 4)(x - 5)(x + 6).$$

The remaining two solutions are $x = 5$ and $x = -6$.

(c) Since $P(x) = 2x^6 - x^5 + 4x - 2$ has three signs changes, there is 3 or 1 positive roots. Since $P(-x) = 2x^6 + x^5 - 4x - 2$ has one sign change, the equation has one negative solution. Since $M = 4$, the bounds theorem states that $|x| < 4/2 + 1 = 3$. Possible rational solutions are $\pm 1, \pm 2, \pm 1/2$. Since $x = 1/2$ is a solution, we factor it from the polynomial,

$$P(x) = 2x^6 - x^5 + 4x - 2 = (2x - 1)(x^5 + 2).$$

The remaining solutions satisfy

$$x^5 + 2 = 0 \qquad \text{or} \qquad x^5 = -2.$$

We write -2 in exponential form

$$x^5 = 2e^{\pi i} = 2e^{\pi i + 2k\pi i} = 2e^{(2k+1)\pi i}.$$

We now take fifth roots,

$$x = 2^{1/5} e^{(2k+1)\pi i/5}.$$

For $k = 0, 1, 2, 3, 4$, we obtain the solutions

$$\begin{aligned}
x_0 &= 2^{1/5} e^{\pi i/5} = 2^{1/5}[\cos(\pi/5) + \sin(\pi/5)\, i], \\
x_1 &= 2^{1/5} e^{3\pi i/5} = 2^{1/5}[\cos(3\pi/5) + \sin(3\pi/5)\, i], \\
x_2 &= 2^{1/5} e^{\pi i} = 2^{1/5}[\cos(\pi) + \sin(\pi)\, i] = -2^{1/5}, \\
x_3 &= 2^{1/5} e^{7\pi i/5} = 2^{1/5}[\cos(7\pi/5) + \sin(7\pi/5)\, i], \\
x_4 &= 2^{1/5} e^{9\pi i/5} = 2^{1/5}[\cos(9\pi/5) + \sin(9\pi/5)\, i].
\end{aligned}$$

7. Prove that if a_n is greater than $2|a_{n-1}|$, $2|a_{n-2}|$, ..., $2|a_0|$, then every zero of the polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ must satisfy

$$|x| < \frac{3}{2}.$$

According to the bounds theorem, $|x| < \dfrac{M}{|a_n|} + 1$. Since $|a_n| > 2M$, we can write that

$$|x| < \frac{M}{2M} + 1 = \frac{3}{2}.$$

8. Prove that if $P(x)$ is a polynomial having only even powers of x, and $P(a) = 0$, then $P(x)$ is divisible by $x^2 - a^2$.

If $P(x)$ has only even powers of x, then when $x = a$ is a zero of the polynomial, so also is $x = -a$. It follows that $x-a$ and $x+a$ are both factors of $P(x)$, and therefore so also is $(x-a)(x+a) = x^2 - a^2$; that is, $P(x)$ is divisible by $x^2 - a^2$.