

MATH 1210 Assignment 3 Solutions Fall 2012

1. Given the vectors  $\mathbf{u} = \langle 3, 1 \rangle$ ,  $\mathbf{v} = \langle 2, -4 \rangle$ , and  $\mathbf{w} = \langle 5, -2 \rangle$ , find scalars  $a$  and  $b$  such that  $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ .

If we substitute components for the three vectors, we obtain

$$\langle 5, -2 \rangle = a\langle 3, 1 \rangle + b\langle 2, -4 \rangle.$$

We now equate  $x$ - and  $y$ -components of these vectors,

$$5 = 3a + 2b, \quad -2 = a - 4b.$$

The solution of these equations is  $a = 8/7$  and  $b = 11/14$ .

2. Show that the lines

$$\begin{array}{lll} x = 1 - t, & & x = 3 + 5s/2, \\ y = -3 + 3t, & \text{and} & y = 2 + 7s/2, \\ z = 2 + 4t, & & z = 5 + s, \end{array}$$

intersect, and find the acute angle between them.

When we equate  $x$  and  $y$ -values, we obtain

$$1 - t = 3 + \frac{5s}{2}, \quad -3 + 3t = 2 + \frac{7s}{2}.$$

The solution of these equations is  $t = 1/2$  and  $s = -1$ . These give  $x = 1/2$  and  $y = -3/2$ . Both equations for  $z$  gives  $z = 4$  and the point of intersection of the lines is therefore  $(1/2, -3/2, 4)$ . Vectors along the lines are  $\langle -1, 3, 4 \rangle$  and  $\langle 5, 7, 2 \rangle$ . If  $\theta$  is the angle between the lines, we can write that

$$\langle -1, 3, 4 \rangle \cdot \langle 5, 7, 2 \rangle = |\langle -1, 3, 4 \rangle| |\langle 5, 7, 2 \rangle| \cos \theta.$$

Thus,

$$\cos \theta = \frac{-1(5) + 3(7) + 4(2)}{\sqrt{1 + 9 + 16} \sqrt{25 + 49 + 4}} = \frac{24}{\sqrt{26} \sqrt{78}} \implies \theta = \text{Cos}^{-1} \frac{12}{13\sqrt{3}} = 1.01 \text{ radians.}$$

3. Find the components of all vectors  $\mathbf{v}$  which have length 2 and are perpendicular to both the lines  $x = 4 + 3t$ ,  $y = 2 - t$ ,  $z = 1 + 5t$  and  $x - y + z = 2$ ,  $3x + 2y - 4z = 6$ .

A vector along the second line is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \langle 2, 7, 5 \rangle.$$

Since a vector along the first line is  $\mathbf{u} = \langle 3, -1, 5 \rangle$ , a vector perpendicular to both lines is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -1 & 5 \\ 2 & 7 & 5 \end{vmatrix} = \langle -40, -5, 23 \rangle.$$

A unit vector in this direction is

$$\frac{\langle -40, -5, 23 \rangle}{\sqrt{1600 + 25 + 529}} = \frac{\langle -40, -5, 23 \rangle}{\sqrt{2154}}.$$

There are two vectors of length 2 perpendicular to both lines, namely,

$$\pm \frac{2\langle -40, -5, 23 \rangle}{\sqrt{2154}}.$$

4. Find the equation of the plane, simplified as much as possible, that contains the point where the line  $x = -1 + 2t$ ,  $y = -4 + 2t$ ,  $z = 1 - 4t$  intersects the  $xz$ -plane, and is perpendicular to the line

$$\frac{x+1}{3} = \frac{3y+1}{6} = \frac{1-2z}{4}.$$

To find where the line intersects the  $xz$ -plane, we set

$$0 = y = -4 + 2t \quad \implies \quad t = 2.$$

This gives the point  $(3, 0, -7)$ . When we write the symmetric equations of the second line in the form

$$\frac{x+1}{3} = \frac{y+1/3}{2} = \frac{z-1/2}{-2},$$

we see that a vector along the line is  $\langle 3, 2, -2 \rangle$ . Since this vector is normal to the plane, the equation of the plane is

$$3(x-3) + 2(y) - 2(z+7) = 0 \quad \implies \quad 3x + 2y - 2z = 23.$$

5. Find, if possible, the equation of a plane containing the two lines

$$\begin{array}{l} x - y + 2z = 9, \\ 2x + y - 3z = -9, \end{array} \quad \text{and} \quad \begin{array}{l} 2x + y - 4z = -12, \\ x + 3y + 5z = 10. \end{array}$$

For the lines to determine a plane, they must either be parallel or intersect. Since vectors along the lines are

$$\mathbf{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \langle 1, 7, 3 \rangle, \quad \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -4 \\ 1 & 3 & 5 \end{vmatrix} = \langle 17, -14, 5 \rangle,$$

and these vectors are not multiples of each other, the lines are not parallel. To determine whether the lines intersect, we solve all four equations simultaneously. The augmented matrix is

$$\begin{aligned} & \left( \begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & -3 & -9 \\ 2 & 1 & -4 & -12 \\ 1 & 3 & 5 & 10 \end{array} \right) \begin{array}{l} R_2 \rightarrow -2R_1 + R_2 \\ R_3 \rightarrow -2R_1 + R_3 \\ R_4 \rightarrow -R_1 + R_4 \end{array} \longrightarrow \left( \begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -7 & -27 \\ 0 & 3 & -8 & -30 \\ 0 & 4 & 3 & 1 \end{array} \right) \begin{array}{l} R_4 \rightarrow -R_2 + R_4 \\ \\ \\ \end{array} \\ & \longrightarrow \left( \begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -7 & -27 \\ 0 & 3 & -8 & -30 \\ 0 & 1 & 10 & 28 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_4 \\ R_4 \rightarrow R_2 \\ \\ \end{array} \longrightarrow \left( \begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & 10 & 28 \\ 0 & 3 & -8 & -30 \\ 0 & 3 & -7 & -27 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_2 + R_1 \\ R_3 \rightarrow -3R_2 + R_3 \\ R_4 \rightarrow -3R_2 + R_4 \\ \\ \end{array} \\ & \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & 12 & 37 \\ 0 & 1 & 10 & 28 \\ 0 & 0 & -38 & -114 \\ 0 & 0 & -37 & -111 \end{array} \right) \begin{array}{l} R_3 \rightarrow -R_3/38 \\ R_4 \rightarrow -R_4/37 \\ \\ \end{array} \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & 12 & 37 \\ 0 & 1 & 10 & 28 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array} \right) \begin{array}{l} R_1 \rightarrow -12R_3 + R_1 \\ R_2 \rightarrow -10R_3 + R_2 \\ R_4 \rightarrow -R_3 + R_4 \\ \\ \end{array} \\ & \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right). \end{aligned}$$

The lines therefore intersect at the point  $(1, -2, 3)$ . A vector normal to the plane is

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 7 & 3 \\ 17 & -14 & 5 \end{vmatrix} = \langle 77, 46, -133 \rangle.$$

The equation of the plane is therefore

$$77(x - 1) + 46(y + 2) - 133(z - 3) = 0 \quad \implies \quad 77x + 46y - 133z = -414.$$

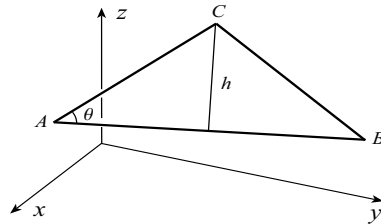
6. (a) Prove that if  $A$ ,  $B$ , and  $C$  are three points in space, then the area of triangle  $ABC$  can be calculated with the formula

$$\text{Area of } \triangle ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|.$$

- (b) Use the formula in part (a) to find the area of the triangle with vertices  $(2, 0, -3)$ ,  $(1, 5, 6)$ , and  $(-1, 3, 4)$ .

- (a) From the figure below, we can say that

$$\text{Area of } \triangle ABC = \frac{1}{2}|\mathbf{AB}|h = \frac{1}{2}|\mathbf{AB}||\mathbf{AC}|\sin\theta = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|.$$



- (b) If we denote the vertices by  $A(2, 0, -3)$ ,  $B(1, 5, 6)$  and  $C(-1, 3, 4)$ , then the area of the triangle is

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2} \left\| \begin{array}{ccc} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 5 & 9 \\ -3 & 3 & 7 \end{array} \right\| = \frac{1}{2}|\langle 8, -20, 12 \rangle| = \frac{1}{2}\sqrt{64 + 400 + 144} = \sqrt{151}.$$