Attempt all questions and show all your work. Due September 25, 2015.

1. Use Mathematical Induction to prove that for all $n \geq 1$,

$$n + (n+1) + (n+2) + \cdots + (2n) = \frac{3n(n+1)}{2}.$$

Solution: For any integer $n \geq 1$, let P_n be the statement that

$$n + (n+1) + (n+2) + \cdots + (2n) = \frac{3n(n+1)}{2}.$$

Base Case. The statement P_1 says that $1 + 2 = 3 = \frac{3(1+1)}{2} = \frac{6}{2} = 3$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$k + (k+1) + (k+2) + \cdots + (2k) = \frac{3k(k+1)}{2}$$

It remains to show that P_{k+1} holds, that is, $(k+1)+(k+2)+(k+3)+\cdots+(2k+2) = \frac{3(k+1)(k+2)}{2}$.

$$\begin{aligned}
(k+1) + (k+2) + (k+3) + \cdots + (2(k+1)) &= \frac{3(k+1)(k+2)}{2} \\
&= k + (k+1) + (k+2) + \cdots + (2k) - k + (2k+1) + (2k+2) \\
&= \frac{3k(k+1)}{2} - k + 2k + 1 + 2k + 2 \\
&= \frac{3k(k+1)}{2} + 3k + 3 \\
&= 3(k+1)\left(\frac{k}{2} + 1\right) \\
&= 3(k+1)\left(\frac{k+2}{2}\right) \\
&= \frac{3(k+1)(k+2)}{2}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

2. Use Mathematical Induction to prove that for all $n \geq 1$,

$$\sum_{i=1}^{n} (i+3)^2 = \frac{n(2n^2 + 21n + 73)}{6}$$

Solution: For any integer $n \geq 1$, let P_n be the statement that $\sum_{i=1}^{n}(i+3)^2 = \frac{n(2n^2+21n+73)}{6}$.

Base Case. The statement P_1 says that $\sum_{i=1}^{1}(i+3)^2 = (1+3)^2 = 16 = \frac{1(2(1)^2+21(1)+73)}{6} = \frac{2+21+73}{6} = \frac{96}{6} = 16$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$\sum_{i=1}^{k}(i+3)^2 = \frac{k(2k^2+21k+73)}{6}.$$

It remains to show that P_{k+1} holds, that is, $\sum_{i=1}^{k+1}(i+3)^2 = \frac{(k+1)(2(k+1)^2+21(k+1)+73)}{6} = \frac{(k+1)(2(k+1)^2+21k+94)}{6} = \frac{(k+1)(2k^2+25k+96)}{6}$.

$$\begin{aligned}
\sum_{i=1}^{k+1}(i+3)^2 &= (k+1+3)^2 + \sum_{i=1}^{k}(i+3)^2 \\
&= (k+4)^2 + \frac{k(2k^2+21k+73)}{6} \\
&= \frac{6(k+4)^2 + k(2k^2+21k+73)}{6} \\
&= \frac{6(k^2+8k+16) + k(2k^2+21k+73)}{6} \\
&= \frac{6k^2+48k+96 + k(2k^2+21k+73)}{6} \\
&= \frac{6k^2+48k+96+2k^3+21k^2+73k}{6} \\
&= \frac{2k^3+27k^2+121k+96}{6} \\
&= \frac{(2k^3+25k^2+96k)+(2k^2+25k+96)}{6} \\
&= \frac{k(2k^2+25k+96)+(2k^2+25k+96)}{6} \\
&= \frac{(k+1)(2k^2+25k+96)}{6}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

3. Use Mathematical Induction to prove that for all $n \geq 1$,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2n}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2n+1}\right).$$

Solution: For any integer $n \geq 1$, let P_n be the statement that $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2n}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2n+1}\right)$.

Base Case. The statement P_1 says that $1 + \frac{1}{3} + \frac{1}{3^2} = \frac{9}{9} + \frac{3}{9} + \frac{1}{9} = \frac{13}{9} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^3\right) = \frac{3}{2}(1 - \frac{1}{27}) = \frac{3}{2} \cdot \frac{26}{27} = \frac{13}{9}$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right).$$

It remains to show that P_{k+1} holds, that is,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2(k+1)}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2(k+1)+1}\right),$$

or in other words,

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k+2}} = \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+3}\right).$$

$$\begin{aligned}
1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{2k+2}} &= 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{2k}} + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{1}{3 \cdot 3^{2k}} + \frac{1}{3^2 \cdot 3^{2k}} \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1}\right) + \frac{3}{2}\left(\frac{2}{3^2 \cdot 3^{2k}} + \frac{2}{3^3 \cdot 3^{2k}}\right) \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1} + \frac{2}{3^2 \cdot 3^{2k}} + \frac{2}{3^3 \cdot 3^{2k}}\right) \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+1} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{2k+1} + \left(\frac{2}{3^2}\right)\left(\frac{1}{3}\right)^{2k+1}\right) \\
&= \frac{3}{2}\left(1 + \left(\frac{1}{3}\right)^{2k+1}\left(-1 + \frac{2}{3} + \frac{2}{3^2}\right)\right) \\
&= \frac{3}{2}\left(1 + \left(\frac{1}{3}\right)^{2k+1}\left(\frac{-1}{9}\right)\right) \\
&= \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{2k+3}\right).
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

4. (a) Write the sum $1 + 3 + 5 + \cdots + (4n - 1)$ using sigma notation.

Solution:

$$\sum_{i=1}^{2n} (2i - 1)$$

(b) Use Mathematical Induction to prove that for all $n \geq 1$, the above expression is equal to $(2n)^2$.

Solution: For any integer $n \geq 1$, let P_n be the statement that $1+3+5+\cdots+(4n-1) = (2n)^2$.

Base Case. The statement P_1 says that $1 + 3 = 2^2 = 4$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + 3 + 5 + \cdots + (4k - 1) = (2k)^2.$$

It remains to show that P_{k+1} holds, that is, $1 + 3 + 5 + \cdots + (4(k+1) - 1) = (2(k+1))^2$, or in other words, $1 + 3 + 5 + \cdots + (4k + 3) = (2k + 2)^2$.

$$\begin{aligned} 1 + 3 + 5 + \cdots + (4k-1) + (4k+1) + (4k+3) &= (2k)^2 + (4k+1) + (4k+3) \\ &= 4k^2 + 4k + 1 + 4k + 3 \\ &= 4k^2 + 8k + 4 \\ &= (2k+2)^2. \end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

5. Use Mathematical Induction to prove that for all $n \geq 1$, $3^n > n^2$.

Solution: For any integer $n \geq 1$, let P_n be the statement that $3^n > n^2$.

Base Case. The statement P_1 says that $3^1 = 3 > 1^2 = 1$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$3^k > k^2.$$

It remains to show that P_{k+1} holds, that is, $3^{k+1} > (k+1)^2 = k^2 + 2k + 1$.

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &= 3^k + 3^k + 3^k \\ &> k^2 + 2^k + 1 \\ &\geq k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

6. Consider the sequence of real numbers defined by the relations $x_0 = 0.5$ and for all $n \geq 1$, $x_n = 0.5x_{n-1}(1 - x_{n-1})$. Prove by induction that for all $n \geq 0$, $x_n \in (0, 1)$.

Solution: For any $n \geq 0$, let P_n be the statement that $x_n \in (0, 1)$.

Base Case. The statement P_0 says that $x_0 = 0.5 \in (0, 1)$, which is true.

Inductive Step. Fix $k \geq 0$, and suppose that P_k holds, that is, $x_k \in (0, 1)$. Then $x_k < 1$, $x_k > 0$, $-x_k < 0$, and $-x_k > -1$.

It remains to show that P_{k+1} holds, that is, that $x_{k+1} \in (0, 1)$.

$$\begin{aligned} x_{k+1} &= 0.5x_k(1 - x_k) \\ &< 0.5(1)(1) = 0.5, \text{ and} \\ x_{k+1} &= 0.5x_k(1 - x_k) \\ &> 0.5(0)(1 + -1) = 0. \end{aligned}$$

Therefore $x_{k+1} \in (0, 0.5)$, and so P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

7. Let a and d be fixed real numbers. Prove using Mathematical Induction that for each $n \geq 1$,

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d).$$

Solution: For any integer $n \geq 1$, let P_n be the statement that $a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d)$.

Base Case. The statement P_1 says that $a = \frac{1}{2}(2a) = a$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$a + (a + d) + (a + 2d) + \cdots + (a + (k - 1)d) = \frac{k}{2}(2a + (k - 1)d)$$

It remains to show that P_{k+1} holds, that is, $a + (a + d) + (a + 2d) + \cdots + (a + kd) = \frac{k+1}{2}(2a + kd)$.

$$\begin{aligned} &a + (a + d) + (a + 2d) + \cdots + (a + (k - 1)d) + (a + kd) \\ &= \frac{k}{2}(2a + (k - 1)d) + (a + kd) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2ak + k(k-1)d + 2a + 2kd}{2} \\
&= \frac{2ak + k^2d - kd + 2a + 2kd}{2} \\
&= \frac{2ak + k^2d + 2a + kd}{2} \\
&= \frac{(k+1)(2a+kd)}{2}.
\end{aligned}
$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

8. (a) Express the sum $\displaystyle\sum_{k=1}^{m}(2+3k)^2$ in terms of three simpler sums in sigma notation by expanding. Do not calculate the value.

Solution:

$$
\begin{aligned}
\sum_{k=1}^{m}(2+3k)^2 &= \sum_{k=1}^{m} 4 + 12k + 9k^2 \\
&= \sum_{k=1}^{m} 4 + \sum_{k=1}^{m} 12k + \sum_{k=1}^{m} 9k^2 \\
&= 4\sum_{k=1}^{m} 1 + 12\sum_{k=1}^{m} k + 9\sum_{k=1}^{m} k^2.
\end{aligned}
$$

(b) Find the value of the sum

$$\sum_{p=1}^{100}(2 - 10p + 3p^2).$$

HINT: Make use of the formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:

$$
\begin{aligned}
\sum_{p=1}^{100}(2 - 10p + 3p^2) &= \sum_{p=1}^{100} 2 - \sum_{p=1}^{100} 10p + \sum_{p=1}^{100} 3p^2 \\
&= 2\sum_{p=1}^{100} 1 - 10\sum_{p=1}^{100} p + 3\sum_{p=1}^{100} p^2 \\
&= 2(100) - 10\frac{100(100+1)}{2} + 3\frac{100(100+1)(200+1)}{6}
\end{aligned}
$$

$$
\begin{aligned}
&= 200 - 500(100 + 1) + 50(100 + 1)(200 + 1)\\
&= 200 - 50500 + 1015050\\
&= 964750.
\end{aligned}
$$

(c) Rewrite the sum

$$\sum_{r=12}^{122} \frac{r-6}{r+9}$$

using an index whose initial and terminal values are 1 and 111 (HINT: use a change of variables).

Solution: Let $i = r - 11$. Then, $r - 6 = i + 5$ and $r + 9 = i + 20$, and so,

$$\sum_{r=12}^{122} \frac{r-6}{r+9} = \sum_{i=1}^{111} \frac{i+5}{i+20}.$$