This assignment is **optional** and does not need to be handed in. Attempt all questions, write out nicely written solutions (showing all your work), and the solutions will be posted on Fri, Feb 24, 2017, at which point you can mark your own work. If you have any questions regarding differences between what you wrote and what the solution key says, please contact your professor. **At least one question from this assignment will be found on Quiz 2.**

1. Simplify and express the complex numbers in Cartesian form

   (a) $2e^{i\frac{\pi}{4}} + 3e^{i\frac{-\pi}{4}}$

   Solution: $2e^{i\frac{\pi}{4}} + 3e^{i\frac{-\pi}{4}} = 2\left[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right] + 3\left[\cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4})\right] = 2\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] + 3\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \frac{5}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

   (b) $\overline{\left(\frac{(6-2i)^4}{(1+3i)^4}\right)}$

   Solution: $\overline{\left(\frac{(6-2i)^4}{(1+3i)^4}\right)} = \overline{\left(\frac{6-2i}{1+3i}\right)^4} = \overline{\left(\frac{(6-2i)(1-3i)}{(1+3i)(1-3i)}\right)^4} = \overline{\left(\frac{6-18i-2i-6}{1+9}\right)^4} = \overline{\left(\frac{-20i}{10}\right)^4} = \overline{2^4(i^2)^2} = \overline{16} = 16$

   (c) $\frac{(i-1)^{10}}{(i+1)^{13}}$

   Solution: $\frac{(i-1)^{10}}{(i+1)^{13}} = \frac{(\sqrt{2}(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))^{10}}{(\sqrt{2}(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}))^{13}} = \frac{(\sqrt{2}e^{i3\pi/4})^{10}}{(\sqrt{2}e^{i\pi/4})^{13}} = \frac{1}{2\sqrt{2}}e^{i(30\pi/4-13\pi/4)} = \frac{1}{2\sqrt{2}}e^{i(17\pi/4)} = \frac{1}{2\sqrt{2}}e^{i(5\pi/4)} = \frac{1}{2\sqrt{2}}\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = -\frac{1}{4} - \frac{i}{4}$

   (d) $\left(\frac{i}{e^{i\pi}}\right)^{25}$

   Solution: $\left(\frac{i}{e^{i\pi}}\right)^{25} = \left(\frac{i}{-1}\right)^{25} = -i\left((-i)^2\right)^{12} = -i$

2. Simplify and express the complex numbers in polar and exponential forms using the principal value of the argument θ, $\theta \in (-\pi, \pi]$

   (a) $\left(\overline{\sqrt{3} + 3i}\right)^2$

   (b) $(-12+i)^3(-12-i)^3$

   (c) $\dfrac{(1-\sqrt{3}i)^{10}}{(1+\sqrt{3}i)^{10}}$

   Solution:

(a) $\left(\overline{\sqrt{3}+3i}\right)^2 = \left(\sqrt{3}-3i\right)^2 = \left(2\sqrt{3}\left(\frac{\sqrt{3}}{2\sqrt{3}} - i\frac{3}{2\sqrt{3}}\right)\right)^2 = \left(2\sqrt{3}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)^2 = \left(2\sqrt{3}(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))\right)^2 = 12(\cos(-2\pi/3) + i\sin(-2\pi/3))$

Polar form: $12(\cos(-2\pi/3) + i\sin(-2\pi/3))$
Exponential form: $12e^{-i2\pi/3}$

(b) $(-12+i)^3(-12-i)^3 = ((-12+i)(-12-i))^3 = ((-12)^2 + 1^2)^3 = 145^3$

Polar form: $145^3\,(\cos(0) + i\sin(0))$
Exponential form: $145^3 e^{i0}$

(c)

$$\begin{aligned}
\frac{(1-\sqrt{3}i)^{10}}{(1+\sqrt{3}i)^{10}} = \left(\frac{1-\sqrt{3}i}{1+\sqrt{3}i}\right)^{10} &= \left(\frac{(1-\sqrt{3}i)(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)}\right)^{10} = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{10} \\
&= \left(\cos(4\pi/3) + i\sin(4\pi/3)\right)^{10} \\
&= \cos(40\pi/3) + i\sin(40\pi/3) \quad \text{[DeMoivre's Theorem]} \\
&= \cos(40\pi/3 - 12\pi) + i\sin(40\pi/3 - 12\pi) \\
&= \cos(4\pi/3) + i\sin(4\pi/3)
\end{aligned}$$

Polar form: $\cos(-2\pi/3) + i\sin(-2\pi/3)$
Exponential form: $e^{-i2\pi/3}$

3. Find all solutions of the equation

   $x^6 + x^3 + 1 = 0.$

Solution: The polynomial is a polynomial of degree 6. From the Fundamental Theorem of Algebra II, there are 6 solutions to this polynomial equation.

Consider $u = x^3$. The polynomial equation

$$x^6 + x^3 + 1 = (x^3)^2 + (x^3)^1 + 1 = 0$$

can be rewritten as follows

$$u^2 + u + 1 = 0.$$

Roots of the quadratic equation are $u_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$.

Now, we have to find the cubic roots of $\frac{-1 \pm i\sqrt{3}}{2}$. Solve

$$x^3 = \frac{-1 - i\sqrt{3}}{2},$$

and

$$x^3 = \frac{-1 + i\sqrt{3}}{2}.$$

First, we solve

$$x^3 = \frac{-1 - i\sqrt{3}}{2} = e^{i(-2\pi/3 + 2k\pi)} \Rightarrow (x^3)^{1/3} = \left(e^{i(-2\pi/3+2k\pi)}\right)^{1/3}$$

with $k = 0, 1, 2$. We obtain 3 roots, equally spaced on the circle of radius 1, with an angle $2\pi/3$ between successive roots; the first root x_0 has an argument $-2\pi/9$. The roots are then

$$\begin{aligned}
x_0 =& e^{-i2\pi/9}, \\
x_1 =& e^{i4\pi/9}, \\
x_2 =& e^{i10\pi/9} \quad \text{or with the principal value } x_2 = e^{-8\pi/9}.
\end{aligned}$$

Second, we solve

$$x^3 = \frac{-1 + i\sqrt{3}}{2} = e^{i(2\pi/3 + 2k\pi)} \Rightarrow (x^3)^{1/3} = \left(e^{i(2\pi/3+2k\pi)}\right)^{1/3}$$

with $k = 0, 1, 2$. We obtain 3 roots, equally spaced on the circle of radius 1, with an angle $2\pi/3$ between successive roots; the first root x_3 has an argument $2\pi/9$. The roots are then

$$\begin{aligned}
x_3 =& e^{i2\pi/9}, \\
x_4 =& e^{i8\pi/9}, \\
x_5 =& e^{i14\pi/9} \quad \text{or with the principal value } x_5 = e^{-4\pi/9}.
\end{aligned}$$

The 6 solutions are x_i with $i \in \{0, 1, 2, 3, 4, 5\}$.

4. Find all solutions of the equation $z^8 = -1$. Express your answers with the argument between $-\pi$ and π.

Solution: Find the 8^{th} roots of -1:

$$z^8 = -1 = e^{i(-\pi+2k\pi)} \Rightarrow (z^8)^{1/8} = \left(e^{i(-\pi+2k\pi)}\right)^{1/8} = e^{i(-\pi/8+k\pi/4)}$$

with $k = 0, 1, 2, 3, 4, 5, 6, 7$.

The 8 solutions are equally spaced on the circle of radius 1, with an angle $\pi/4$ between successive roots; the first root has an argument $-\pi/8$. The solutions are

$$\begin{aligned}
z_0 =& e^{-i\pi/8}, \\
z_1 =& e^{i\pi/8}, \\
z_2 =& e^{i3\pi/8}, \\
z_3 =& e^{i5\pi/8},
\end{aligned}$$

$$\begin{aligned}
z_4 &= e^{i7\pi/8}, \\
z_5 &= e^{i9\pi/8}, \quad \text{or with the principal value } z_5 = e^{-7\pi/8}, \\
z_6 &= e^{i11\pi/8}, \quad \text{or with the principal value } z_6 = e^{-5\pi/8}, \\
z_7 &= e^{i13\pi/8} \quad \text{or with the principal value } z_7 = e^{-3\pi/8}.
\end{aligned}$$

5. Find all solutions of the equation $z^4 = i$.

Solution: The modulus of i is 1. Since i is on the positive imaginary axis, $\arg(i) = \pi/2$. Then,

$$z = i^{1/4} = \left(e^{i(\frac{\pi}{2}+2k\pi)}\right)^{1/4} = e^{i\frac{\pi+4k\pi}{8}}, \; k = 0, 1, 2, 3$$

Thus, all the solutions of the given equation are:

$$\begin{aligned}
z_0 &= e^{i\frac{\pi}{8}} \\
z_1 &= e^{i\frac{\pi+4\pi}{8}} = e^{i\frac{5\pi}{8}} \\
z_2 &= e^{i\frac{\pi+8\pi}{8}} = e^{i\frac{9\pi}{8}} \\
z_3 &= e^{i\frac{\pi+12\pi}{8}} = e^{i\frac{13\pi}{8}}
\end{aligned}$$

6. Let z_1 and z_2 be 2 complex numbers. Show that

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Solution: Define $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Their sum is

$$z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2).$$

Take the conjugate on both sides:

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{a_1 + a_2 + i(b_1 + b_2)} \\
&= a_1 + a_2 - i(b_1 + b_2) \\
&= a_1 - ib_1 + a_2 - ib_2 \\
&= \overline{z_1} + \overline{z_2}.
\end{aligned}$$

7. Let z be a complex number. Using mathematical induction prove that

$\overline{z^n} = \overline{z}^n$, for all $n \geq 1$.

Solution: Preliminary result: Define two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$; multiply:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Take the conjugate on both sides:

$$\begin{aligned}\overline{z_1 z_2} &= \overline{r_1 r_2 e^{i(\theta_1+\theta_2)}} \\ &= r_1 r_2 e^{-i(\theta_1+\theta_2)} = r_1 r_2 e^{-i\theta_1} e^{-i\theta_2} = r_1 e^{-i\theta_1} r_2 e^{-i\theta_2} = \overline{z_1}\,\overline{z_2}.\end{aligned}$$

Now, we want to prove that, for all $n \geq 1$, $\overline{z^n} = \overline{z}^n$. Define the proposition P_n as

$$P_n : \quad \overline{z^n} = \overline{z}^n.$$

Base case: When $n = 1$, we have $\overline{z^1} = \overline{z} = \overline{z}^1$. Therefore P_1 is true.

Inductive step: For $k \geq 1$, assume that $P_k : \overline{z^k} = \overline{z}^k$ is true. It remains to show that P_{k+1} hold; that is, that

$$\overline{z^{k+1}} = \overline{z}^{k+1}.$$

$$\begin{aligned}\overline{z^{k+1}} &= \overline{z^k z} \\ &= \overline{z^k}\,\overline{z} && [\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad z_1, z_2 \in \mathbb{C}] \\ &= \overline{z}^k \overline{z} && [\overline{z^k} = \overline{z}^k, \text{use } P_k] \\ &= \overline{z}^{k+1}\end{aligned}$$

P_{k+1} holds. We have shown that P_{k+1} is true if P_k is true.

Conclusion: By the Principle of Mathematical Induction, we can conclude that for all $n \geq 1$, P_n is a true proposition.

8. Consider the following polynomial $P(x) = x^5 - 2x^4 + 4x^3 + 2x^2 - 5x$.
   (a) Verify that $1 + 2i$ is a root of $P(x) = 0$.
   (b) Find all the roots of $P(x) = 0$.
   (c) Factor $P(x)$ into the product of real linear and irreducible real quadratic factors.

Solution:

(a) $P(1 + 2i) = 0$ therefore $1 + 2i$ is a root.

(b) $P(x)$ is a polynomial of degree 5. By the Fundamental Theorem of Algebra II, P has exactly 5 roots (counting multiplicities).

As P has real coefficients, if z is a complex root then $\overline{z}$ is also a root of P. Therefore, as $1 + 2i$ is a root, $1 - 2i$ is also a root of P.

Moreover, x can be factored as

$$P(x) = x(x^4 - 2x^3 + 4x^2 + 2x - 5),$$

so 0 is also a root. So far, we have:

$$P(x) = x(x - 1 - 2i)(x - 1 + 2i)Q_2(x) = x(x^2 - 2x + 5)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2. To find the last 2 roots, we first need to find $Q_2(x)$. Performing for instance the long division of $x^4 - 2x^3 + 4x^2 + 2x - 5$ by $x^2 - 2x + 5$ gives

$$x^4 - 2x^3 + 4x^2 + 2x - 5 = (x^2 - 2x + 5)(x^2 - 1),$$

where $Q_2(x) = x^2 - 1 = (x - 1)(x + 1)$.

Otherwise, to find the last 2 roots, we could have noticed that $P(1) = 0$ and $P(-1) = 0$.

Summing up, roots of P are 0, $1 \pm 2i$ and ± 1.

(c) So $P(x)$ is factored into a product of real linear and irreducible real quadratic factors as

$$P(x) = x(x - 1)(x + 1)(x^2 - 2x + 5)$$

9. (a) Show that $(x - i)$ and $(x - 1)$ are linear factors of

   $x^4 - 2(1 + i)x^3 + 4ix^2 + 2(1 - i)x - 1 = 0.$

   (b) Factor the polynomial $x^4 - 2(1 + i)x^3 + 4ix^2 + 2(1 - i)x - 1$ in linear factors.

Solution:

(a) $P(i) = i^4 - 2(1+i)i^3 + 4ii^2 + 2(1-i)i - 1 = 1 + 2(1+i)i - 4i + 2(1-i)i - 1 = 0.$
By the Factor Theorem, as $P(i) = 0$, $(x - i)$ is a linear factor of P.
$P(1) = 1 - 2(1+i) + 4i + 2(1-i) - 1 = 0$. By the Factor Theorem, as $P(1) = 0$, $(x - 1)$ is a linear factor of P.

(b) Factor the polynomial: From (a)

$$P(x) = x^4 - 2(1 + i)x^3 + 4ix^2 + 2(1 - i)x - 1 = (x - i)(x - 1)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2 that can be found by long division or by identification.

By identification: Assume that $Q_2(x) = ax^2 + bx + c$, then

$$P(x) = (x^2 - (1 + i)x + i)(ax^2 + bx + c)$$

where $x^2 - (1 + i)x + i = (x - i)(x - 1)$. Expand

$$(x^2 - (1 + i)x + i)(ax^2 + bx + c)$$

and identify the coefficients of the terms of same degree:

$$\begin{aligned}P(x) =& x^4 - 2(1+i)x^3 + 4ix^2 + 2(1-i)x - 1\\ =& ax^4 + bx^3 + cx^2 - a(1+i)x^3 - b(1+i)x^2 - c(1+i)x + aix^2 + bix + ci.\end{aligned}$$

- Terms of degree $n = 4$: $1 = a$.
- Terms of degree $n = 3$: $-2(1+i) = b - a(1+i)$.
- Terms of degree $n = 2$: $4i = c - b(1+i) + ai$.
- Terms of degree $n = 1$: $2(1-i) = -c(1+i) + bi$.
- Terms of degree $n = 0$: $-1 = ci$.

We obtain $c = i$, $a = 1$ and $b = -(1+i)$. Finally,

$$P(x) = (x-i)(x-1)(x^2 - (1+i)x + i) = (x-i)(x-1)(x-i)(x-1).$$

So $P(x)$ has 2 linear factors $(x-i)$ and $(x-1)$ of multiplicity 2.

10. Consider the following polynomial
$P(x) = x^5 - 11x^4 + 43x^3 - 73x^2 + 56x - 16$.
    (a) Show that $P(x)$ can be rewritten as $P(x) = Q(x)(x-4)$ and $P(x) = T(x)(x-1)$ where $Q(x)$ and $T(x)$ are polynomials in x. Give the degree of $Q(x)$ and $T(x)$.
    (b) Show that 4 is a root of multiplicity 2 of $P(x)$.
    (c) Factor $P(x)$.

Solution:

(a) $P(4) = 0$, so by the Factor Theorem, $(x-4)$ is a linear factor of $P(x)$. Therefore, we can write $P(x) = (x-4)Q(x)$, where $Q(x)$ is a polynomial of degree 4.

$P(1) = 0$, so by the Factor Theorem, $(x-1)$ is a linear factor of $P(x)$. Therefore, we can write $P(x) = (x-1)T(x)$, where $T(x)$ is a polynomial of degree 4.

(b) $P(x)$ can be rewritten as

$$P(x) = (x-1)(x-4)Q_3(x) = (x^2 - 5x + 4)Q_3(x)$$

where $Q_3(x)$ is a polynomial of degree 3. To find $Q_3(x)$, perform long division or identification of like parameters as in Question 8. We find

$$Q_3(x) = x^3 - 6x^2 + 9x - 4.$$

As $Q_3(4) = 0$, $(x-4)$ is a linear factor of $Q_3(x)$ and so $(x-4)$ appears for a second time in the factorization of $P(x)$:

$$P(x) = (x-1)(x-4)(x^3 - 6x^2 + 9x - 4) = (x-1)(x-4)(x-4)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2 that we can obtain by dividing $Q_3(x) = x^3 - 6x^2 + 9x - 4$ by $(x - 4)$. The result of the long division of $Q_3(x) = x^3 - 6x^2 + 9x - 4$ by $(x - 4)$ gives $Q_2(x) = x^2 - 2x + 1 = (x - 1)^2$. $x - 4$ is not a factor of $Q_2(x)$. The factor $(x - 4)$ appears only 2 times in the factorization of P, therefore the root $x = 4$ has multiplicity 2.

(c) So $P(x)$ factors as

$$P(x) = (x - 1)^3(x - 4)^2$$

11. For each of the following polynomials:

$$P_5(x) = 6x^5 + 7x^4 - 13x^3 - 85x^2 - 50x$$

$$P_9(x) = x^9 + 3x^8 + 3x^7 + 3x^6 + 6x^5 + 6x^4 + 4x^3 + 6x^2 + 6x + 2$$

(a) Use Descartes' rules of signs to state the number of possible positive and negative zeros of the polynomial;

(b) use the bounds theorem to find bounds for zeros of the polynomial;

(c) use the rational root theorem to list all possible rational zeros of the polynomial;

(d) use this information to find all the zeros of the polynomial.

Solution: $P_5(x) = 6x^5 + 7x^4 - 13x^3 - 85x^2 - 50x$

(a) There is one sign change in the sequence of coefficients, so $P_5(x)$ has exactly 1 positive real root. There are 3 sign changes in the sequence of coefficients of $P_5(-x) = -6x^5 + 7x^4 + 13x^3 - 85x^2 + 50x$, so $P_5(x)$ has 3 or 1 negative real root.

(b) If x is a root of $P_5(x)$, then $|x| < \frac{85}{6} + 1 = \frac{91}{6}$

(c) We cannot use the rational root theorem right away, because the last coefficient is 0. Notice that 0 is a root of $P_5(x)$, and $P_5(x) = x(6x^4 + 7x^3 - 13x^2 - 85x - 50) = xQ(x)$. Then we can use the rational root theorem for $Q(x) = 6x^4 + 7x^3 - 13x^2 - 85x - 50$.

If $\frac{p}{q}$ is a rational root of $Q(x)$, then p divides 50 and q divides 6, so $\frac{p}{q} \in \pm\{1, 1/2, 1/3, 1/6, 2, 2/3, 5, 5/2, 5/3, 5/6, 10, 10/3, 25, 25/2, 25/3, 25/6, 50, 50/3\}$

(d) Using the bounds theorem, we can limit the possible candidates for rational roots to $\pm\{1, 1/2, 1/3, 1/6, 2, 2/3, 5, 5/2, 5/3, 5/6, 10, 10/3, 25, 25/2, 25/3, 25/6\}$. By plugging first the different positive values in $Q(x)$, we eventually get that $Q(5/2) = 0$, so the only positive root is $5/2$. Also, $Q(x)$ is divisible by $(2x - 5)$. We obtain from the long division that $Q(x) = (2x - 5)(3x^3 + 11x^2 + 21x + 10) = (2x - 5)R(x)$. $R(x)$ can have rational roots from the set $\pm\{1, 2, 5, 10, 1/3, 2/3, 5/3, 10/3\}$. Since we know already the only positive root, which is $5/2$, we only try negative values from this set. By plugging different values in $R(x)$, we eventually get that $R(-2/3) = 0$, so $-2/3$ is root of $R(x)$; and $(3x+2)$ is a linear factor of $R(x)$. From the long division, $R(x) = 3x^3 + 11x^2 + 21x + 10 = (3x+2)(x^2+3x+5)$. The quadratic factor (x^2+3x+5) has roots $-3/2 \pm i\sqrt{11}/2$. Summing up, the 5 roots/zeros of $P_5(x)$ are : 0, $5/2$, $-3/2$, $-3/2 + i\sqrt{11}/2$ and $-3/2 - i\sqrt{11}/2$

Solution: $P_9(x) = x^9 + 3x^8 + 3x^7 + 3x^6 + 6x^5 + 6x^4 + 4x^3 + 6x^2 + 6x + 2$

(a) There are no sign changes in the sequence of coefficients, so $P_9(x)$ has no positive real roots. There are 9 sign changes in the sequence of coefficients of $P_9(-x) = -x^9 + 3x^8 - 3x^7 + 3x^6 - 6x^5 + 6x^4 - 4x^3 + 6x^2 - 6x + 2$, so $P_9(x)$ has 9, 7, 5, 3 or 1 negative real roots.

(b) If x is a root of $P_9(x)$, then $|x| < \frac{6}{1} + 1 = 7$.

(c) If $\frac{p}{q}$ is a rational root of $P_9(x)$, then p divides 2 and q divides 1, so $\frac{p}{q} \in \pm\{1, 2\}$.

(d) Since $P_9(x)$ has no positive real roots, the only possible rational roots are 1 and 2. We have $P_9(-2) \neq 0$ and $P_9(1) = 0$; from the long division of $P_9(x)$ by $(x+1)$, we obtain $P_9(x) = (x+1)(x^8+2x^7+x^6+2x^5+4x^4+2x^3+2x^2+4x+2) = (x+1)Q_8(x)$. We get $Q_8(-1) = 0$ and then from the long division of $Q_8(x)$ by $(x+1)$, we obtain $Q_8(x) = (x+1)(x^7 + x^6 + 2x^4 + 2x^3 + 2x + 2) = (x+1)Q_7(x)$. Again, we get $Q_7(-1) = 0$, and from the long division of $Q_7(x)$ by $(x+1)$, we obtain $Q_7(x) = (x+1)(x^6 + 2x^3 + 2)$. We verify $(-1)^6 + 2(-1)^3 + 2 \neq 0$. Therefore, -1 is a root of multiplicity 3 of $P_9(x)$. So, $P_9(x) = (x+1)^3(x^6 + 2x^3 + 2)$ and $x^6 + 2x^3 + 2$ has no rational roots.

To find roots of $x^6 + 2x^3 + 2$, we substitute $y = x^3$ to obtain $y^2 + 2y + 2 = 0$. The 2 roots of the quadratic equation are $y_{1,2} = -1 \pm i$.

If $x^3 = y_1 = -1 + i = \sqrt{2}e^{i\left(\frac{3\pi}{4}+2k\pi\right)}$, then $x = \sqrt[6]{2}e^{i\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)}$ with $k = 0, 1, 2$. We have 3 roots: $x_1 = \sqrt[6]{2}e^{i\left(\frac{\pi}{4}\right)}$, $x_2 = \sqrt[6]{2}e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)} = \sqrt[6]{2}e^{i\left(\frac{11\pi}{12}\right)}$, $x_3 = \sqrt[6]{2}e^{i\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)} = \sqrt[6]{2}e^{i\left(-\frac{5\pi}{12}\right)}$.

If $x^3 = y_2 = -1 - i = \sqrt{2}e^{i\left(\frac{5\pi}{4}+2k\pi\right)}$, then $x = \sqrt[6]{2}e^{i\left(\frac{5\pi}{12}+\frac{2k\pi}{3}\right)}$ with $k = 0, 1, 2$. We have 3 roots: $x_4 = \sqrt[6]{2}e^{i\left(\frac{5\pi}{12}\right)}$, $x_5 = \sqrt[6]{2}e^{i\left(\frac{5\pi}{12}+\frac{2\pi}{3}\right)} = \sqrt[6]{2}e^{i\left(-\frac{11\pi}{12}\right)}$, $x_6 = \sqrt[6]{2}e^{i\left(\frac{5\pi}{12}+\frac{4\pi}{3}\right)} = \sqrt[6]{2}e^{i\left(-\frac{\pi}{4}\right)}$.

Summing up, the 9 roots/zeros of $P_9(x)$ are: -1 (with multiplicity 3), $\sqrt[6]{2}e^{i\left(\frac{\pi}{4}\right)}$, $\sqrt[6]{2}e^{i\left(\frac{11\pi}{12}\right)}$, $\sqrt[6]{2}e^{i\left(-\frac{5\pi}{12}\right)}$, $\sqrt[6]{2}e^{i\left(\frac{5\pi}{12}\right)}$, $\sqrt[6]{2}e^{i\left(-\frac{11\pi}{12}\right)}$, $\sqrt[6]{2}e^{i\left(-\frac{\pi}{4}\right)}$.

12. Let

$$A = \begin{bmatrix} 4 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} -1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}.$$

Evaluate each of the following expressions or explain why it is undefined.

(a) $(2D^T - E)A$

(b) $(4B)C + 3B$

(c) $D^T(4E^T) - 4(ED)^T$

Solution:

(a)

$$\begin{aligned}
(2D^T - E)A &= \left(2\begin{bmatrix} -1 & 1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 4 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 1 & 3 \\ 11 & -1 & 2 \\ -1 & 1 & 5 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -27 & 5 \\ 49 & 0 \\ 5 & 7 \end{bmatrix}
\end{aligned}$$

(b) Undefined, since $(4B)C$ is a 2×3 matrix, which cannot be added to the 2×2 matrix $3B$.

(c) Since $(ED)^T = D^T E^T$, we have

$$D^T(4E^T) - 4(ED)^T = 4D^T E^T - 4D^T E^T = 0.$$