Instructions

1. You must join the WebEx/zoom meeting for your class with your webcam on and write the test in front of your webcam. Failure to join the WebEx/zoom meeting for invigilation or failure to follow the test guidelines may result in a grade of zero on the test. If you lose the connection to the meeting, rejoin as soon as possible and contact your instructor immediately after the test. If you have any questions during the test, you may ask your instructor using the private chat feature.

2. During quizzes, tests and exams in this class, students may use their lecture notes, the course textbook, lab worksheets, and any of the material that has been made available on the UM Learn site. No other materials are permitted. While students may consult the allowed resources, they **may not copy** from them. All work is to be completed independently.

3. By submitting this assessment, you are agreeing to follow the Honesty Declaration (available on the course website) and that you either have already or you will submit a signed copy of it (following the instructions) before the Honesty Declaration due date.

4. Student are expected to answer all questions. Any question without a submitted answer will receive a grade of zero.

5. If you can print the question papers and work on those, please do. Otherwise you may work on (preferably plain white) paper. (Hand written work done on electronic paper is permissible.)

6. You can submit more than one page for a question if your solution does not fit on one page. The total value of all questions is 26 points.

7. You have 50 minutes to complete and submit the paper. The expectation is that you are given **35 minutes to complete the questions** and 15 minutes to submit your solutions.

8. If you run into problems uploading your solutions, but you still have internet access, you should email your solutions to the instructors. You should also continue to attempt to upload.

You may find these formulae useful:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}.$$

[5] 1. Calculate the following summmation. Be sure to show all your work. Leave your final answer as a single fraction (you do not need to simplify it):

$$\sum_{j=5}^{121} \left((j-4) + 2(j-4)^2\right)$$

Solution:

$$\begin{aligned}
\sum_{j=5}^{121} \left((j-4) + 2(j-4)^2\right) &= \sum_{j=1}^{117} \left(j + 2(j)^2\right) \\
&= \left(\sum_{j=1}^{117} j\right) + 2\left(\sum_{j=1}^{117} j^2\right) \\
&= \frac{117 \cdot 118}{2} + 2\frac{117 \cdot 118(2 \cdot 117 + 1)}{6} \\
&= \frac{3 \cdot 117 \cdot 118 + 2 \cdot 117 \cdot 118(2 \cdot 117 + 1)}{6} \\
&= 1088373
\end{aligned}$$

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[12] 2. Using the principle of mathematical induction (PMI), prove that, for all $n \geq 0$, 10 divides $9^{2n+1} + 1$.

Solution: This is really just problem 25 from section 1.1 with $x = 9$ and $y = 1$. We shall prove the proposition by induction on n. Thus, for $n \geqslant 0$, let P_n be the statement: 10 divides $9^{2n+1} + 1$.

Base case The statement P_0 reads: "10 divides $9 + 1$", which is true.

Inductive step Fix $k \geqslant 0$, and assume that P_k holds, i.e. 10 divides $9^{2k+1} + 1$. Our goal is to show P_{k+1}, i.e. 10 divides $9^{2(k+1)+1} + 1$. So, we calculate

$$\begin{aligned} 9^{2(k+1)+1} + 1 &= 9^2 9^{2k+1} + 1 = 9^2(9^{2k+1} + 1 - 1) + 1 \\ &= 9^2(9^{2k+1} + 1) - 9^2 + 1. \end{aligned}$$

By the inductive hypothesis, we know that 10 divides the first term, what remains is to show that 10 divides $-9^2 + 1 = -(9^2 - 1)$, this follows from the equation

$$(9 + 1)(9 - 1) = 9^2 - 1.$$

Thus, P_{k+1} holds, and by the principle of induction, it follows that P_n holds for all non-negative integers n.

[4] 3. Let $z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, $z_2 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. Write the following in Cartesian frm:

$$\left(\frac{3+2i}{(z_1)^2}\right)\left(\frac{1-2i}{(z_2)^2}\right)$$

Solution: Method #1: We first notice that $z_2 = \bar{z}_1$, and thus $z_1 z_2 = |z_1|^2 = 1$. The solution then becomes

$$\begin{aligned}\left(\frac{3+2i}{(z_1)^2}\right)\left(\frac{1-2i}{(z_2)^2}\right) &= \frac{(3+2i)(1-2i)}{(z_1 z_2)^2} \\ &= 7-4i.\end{aligned}$$

Method #2: If you multiply everything out directly:

- $z_1^2 = i$
- $z_2^2 = -i$
- $\frac{3+2i}{z_1^2} = \frac{3+2i}{i} = 2-3i$
- $\frac{1-2i}{z_1^2} = \frac{1-2i}{-i} = 2+i$
- $(2-3i)(2+i) = 7-4i.$

[5] 4. Let $t \in \mathbb{R}$. Calculate and simplify the real part of

$$\frac{1}{2}\left(e^{it} + e^{-it}\right).$$

Solution: Method #1: We know that $e^{it} = \cos(t) + i\sin(t)$, and get

$$\begin{aligned}
\frac{1}{2}(e^{it} + e^{-it}) &= \frac{1}{2}\left(\cos(t) + i\sin(t) + \cos(-t) + i\sin(-t)\right) \\
&= \frac{1}{2}\left(\cos(t) + i\sin(t) + \cos(t) - i\sin(t)\right) \\
&= \frac{1}{2}\left(2\cos(t)\right) \\
&= \cos(t),
\end{aligned}$$

Therefore the real part of $\frac{1}{2}\left(e^{it} + e^{-it}\right)$ is $\cos t$.

Method #2: Note that if $z = e^{it} = \cos t + i\sin t$, then $\bar{z} = e^{-it}$, and therefore $e^{it} + e^{-it} = z + \bar{z} = 2\,\mathrm{re}(z)$. Therefore

$$\frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{2}2\,\mathrm{re}(z) = \mathrm{re}(z) = \cos t.$$