# Exams

## Final: April, 1992

Values

1. Let $f(x) = \dfrac{x^5}{x^2+1}$: note that $f'(x) = \dfrac{3x^6+5x^4}{(x^2+1)^2}$.

2 (a) Find the domain and range of $f(x)$.

2 (b) Using $f'(x)$, show that $f^{-1}(x)$ exists.

2 (c) What are the domain and range of $f^{-1}(x)$?

2 (d) What is $f(1)$? What is $f^{-1}\left(\dfrac{1}{2}\right)$?

2 (e) Find $(f^{-1})'\left(\dfrac{1}{2}\right)$.

6 2. (a) Show that $(\sin^{-1})'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

4 (b) Find $\dfrac{d}{dx}\left(\dfrac{\sin^{-1}(x)}{\cos^{-1}(x)}\right)$.

6 (c) Show that $\tan^{-1}(x) + \cot^{-1}(x) = \dfrac{\pi}{2}$ for $x > 0$.

3. Find the following limits:

4 (a) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{2x} - \frac{1}{\sin 3x}\right)$.

4 (b) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

6 (c) $\displaystyle\lim_{x\to 0^+}(1-x)^{1/x}$.

4. Find each of the following integrals:

4 (a) $\displaystyle\int \frac{dx}{x \ln x}$.

6 (b) $\displaystyle\int \sin^3\theta \cos^2\theta \, d\theta$.

4 (c) $\displaystyle\int \frac{dx}{4x^2 + 16x + 24}$.

6 (d) $\displaystyle\int \frac{dx}{\sqrt{1 + x^2}}$.

8 (e) $\displaystyle\int e^x \cos x \, dx$.

8 (f) $\displaystyle\int \frac{x+3}{x^2 + 2x} \, dx$.

4 5. (a) For what values of $p$ does $\displaystyle\int_1^\infty x^p \, dx$ converge?

4 (b) Determine whether $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x + x^3}}$ converges or diverges.

6 6. Find $\displaystyle\frac{d}{dx} F(\sqrt{x})$ if $\displaystyle F(t) = \int_1^{t^4} \frac{\sin(u)}{u} \, du$.

7. Let $R$ be the region bounded by $y = x^2$ and $y = x^3$.
Express the following in terms of integrals; <u>do not evaluate the integrals</u>.

4 (a) The area of $R$.

4 (b) The length of the perimeter of $R$.

4 (c) The volume of the solid obtained by rotating $R$ around the $x$-axis.

6 (d) The total surface area of the solid obtained by rotating $R$ around the $x$-axis.

4 8. (a) Sketch the curve $r = 1 + \cos\theta$ for $0 \le \theta \le 2\pi$.

4 (b) Express the arc length of the curve in part (a) in terms of an integral; <u>do not evaluate</u>.

4 (c) Express the area enclosed by the curve in part (a) in terms of an integral; <u>do not evaluate</u>.

9. An investment in the stock market decreases at a rate proportional to its value.
After 2 months, its value is $16,000 and after 6 months is $4,000.

6 (a) What is the initial investment?

4 (b) What is the value of the investment after 1 year?

## Final: December, 1992

Values

4 1. Find $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{1}{\sqrt{7-x^2}}\right)$.

6 2. Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{3+x^2}$.

4 3. Evaluate $\displaystyle\int \frac{dx}{x\ln x}$.

5 4. Evaluate $\displaystyle\int \cos^2(3x)\,dx$.

4 5. Evaluate $\displaystyle\int \sec^3 x\tan^3 x\,dx$.

10 6. Evaluate $\displaystyle\int e^{2x}\sin 3x\,dx$.

10 7. Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

10 8. Evaluate $\displaystyle\int \frac{x^2}{(x-3)(x^2+1)}\,dx$.

6 9. Evaluate $\displaystyle\int \frac{dx}{x^2+2x+10}$.

12 10. Evaluate $\displaystyle\int \frac{dx}{x^3+5x^2}$.

8 11. Evaluate $\displaystyle\int \frac{dx}{(x+2)(x-3)}$.

4 12. Find the value of the following limit:

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}.$$

13 13. Find the value of the following limit:

$$\lim_{x \to 0^+} x^{\sin x}.$$

5 14. Set up the definite integral to find the area of the plane region bounded by $y = x^2 - 2x$ and $y = 2x - 3$. DO NOT EVALUATE THIS INTEGRAL.

15. Set up the definite integral for finding the volume of the solid $S$ generated by rotating about the $y$-axis the region bounded by $y = \sqrt{x}$, $y = 1$, and $x = 0$.

5 (a) using the disk method (DO NOT EVALUATE THIS INTEGRAL).

5 (b) using the shell method (DO NOT EVALUATE THIS INTEGRAL).

13 16. Evaluate the given integral or show it diverges:

$$\int_0^\infty xe^{-x}\, dx.$$

6 17. Set up the definite integral for finding the length of the curve $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$. DO NOT EVALUATE THIS INTEGRAL.

## Final: April, 1993

Values

1. Integrate

8 (a) $\displaystyle\int e^{\sqrt{x}}\, dx$

7 (b) $\displaystyle\int \sin^{-1} x\, dx$

7 (c) $\displaystyle\int \cos^5 x\, dx$

8 (d) $\displaystyle\int \frac{x+1}{x^3+x}\, dx$

2. Evaluate the limits:

7 (a) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{\sin x}$

7 (b) $\lim_{x\to 0^+} (1+\tan x)^{1/x}$

3. Calculate the following integrals or show that they diverge.

8 (a) $\int_{3/4}^{1} \frac{dx}{\sqrt{1-x}}$

8 (b) $\int_{1}^{\infty} \frac{x+1}{x^2+1}\,dx$

4. Express the following in terms of integrals. Do not evaluate the integrals.

8 (a) The area of the region bounded by the line

$y = x - 1$ and the parabola $y^2 = 2x + 6$.

10 (b) The volume of the solid formed when the region bounded by the curves

$y = x$ and $y = x^2$ is rotated about

(i) the x-axis,

(ii) the y-axis.

8 (c) The surface area generated by rotating the curve

$y = \frac{1}{3}x^3$, $1 \le x \le 2$ about the line $x = -2$.

6 5. (a) Express the length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x} \text{ over } 1 \le x \le 2,$$

in the form of a definite integral.

2 (b) Evaluate the integral obtained in part (a).

8 6. (a) Sketch the graph of the curve given in polar coordinates by

$r = \sin(2\theta), \quad 0 \le \theta \le 2\pi.$

8 (b) Set up a definite integral for the length of the curve given in part (a).

[Do not attempt to evaluate this integral.]

7. Let $f(x) = (x+2)^{5/2}.$

8 (a) Find the second degree Taylor polynomial $P_2(x)$ for $f(x)$ around $-1$.

3 (b) Give the Lagrange form of the remainder $R_2(x) = f(x) - P_2(x)$.

5 (c) Use $P_2(x)$ from the part (a) to find an approximation to $(0.8)^{5/2}$.

4 (d) Show that the error involved in using the approximation of part (c) is less than $5 \times 10^{-3} = 0.005$. (Hint: $\sqrt{0.8} > \frac{1}{2}$).

## Final: December, 1993

Values

DO NOT SIMPLIFY ANY OF THE QUESTIONS ON THIS EXAM.

6 1. Find $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)\right)$.

5 2. Find $\displaystyle\int x(1+5x^2)^{7/3}\,dx$.

8 3. Find $\displaystyle\int \sec^3 x\,dx$.

12 4. Find $\displaystyle\int x^3\sqrt{9-x^2}\,dx$.

9 5. Find $\displaystyle\int \frac{7x+5}{x^2+6x+12}\,dx$.

12 6. Find the value of $\displaystyle\int_0^{\pi/2} e^x \cos x\,dx$.

12 7. Find $\displaystyle\int \frac{2x^2-2x+4}{(x-1)^3(x+1)}\,dx$.

10 8. Find $\displaystyle\int \frac{3x^2-7x+6}{x(x^2+9)}\,dx$.

5 9. Find the value of $\displaystyle\lim_{x\to\infty} xe^{-x^2}$.

8 10. Find the value of $\displaystyle\lim_{x\to\infty} x^{1/x}$.

8 11. Evaluate the integral or show it diverges

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx.$$

7 12. Evaluate the integral or show it diverges

$$\int_0^2 \frac{1}{(x-1)^3}\,dx.$$

6 13. Set up the definite integral to find the area $A$ of the plane region bounded by $y = 2x^3$ and $y = x^2$. DO NOT EVALUATE THIS INTEGRAL.

12 14. Set up the definite integral that gives the volume $V$ of the solid generated by rotating about the $x$-axis the region bounded by $y = 3 - x$, $y = \dfrac{2}{x}$.

(a) using the disk method. (DO NOT EVALUATE THIS INTEGRAL.)

(b) using the shell method. (DO NOT EVALUATE THIS INTEGRAL.)

5 15. Set up the definite integral for finding the length $s$ of the curve $y = \ln(\sin x)$, from $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{3}$. DO NOT EVALUATE THIS INTEGRAL.

5 16. Set up the definite integral that gives the area $S$ of the surface obtained by rotating the curve $y = \sqrt{4-x^2}$ from $x = -1$ to $x = 1$ about the $x$-axis. DO NOT EVALUATE THIS INTEGRAL.

## Final: April, 1994

Values

20 1. (a) Simplify $\sec(\sin^{-1}\sqrt{x})$.

(b) Simplify $\cos(\cot^{-1} x^2)$.

(c) Find $\dfrac{dy}{dx}$ if $y = x(\sin^{-1} x^2)$.

2. Integrate

8 (a) $\displaystyle\int x\sqrt{2x+1}\,dx.$

7 (b) $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+10}}$.

7 (c) $\displaystyle\int_0^{\pi/2} \sin^4 x\, dx$.

8 (d) $\displaystyle\int \frac{dx}{x^2-8x+15}$.

3. Evaluate the limits:

7 (a) $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} x}{x}$.

7 (b) $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$.

4. Calculate the following integrals or show that they diverge.

8 (a) $\displaystyle\int_0^{\frac{1}{2}} \frac{1-2x}{\sqrt{x-x^2}}\, dx$.

8 (b) $\displaystyle\int_0^{\infty} xe^{-x}\, dx$.

5. Express the following in terms of integrals. Do <u>not</u> evaluate the integrals.

8 (a) The area of the region $A$ between the curves

$$y = x^2 - x - 4 \quad \text{and} \quad y = x - 1.$$

10 (b) Let $f(x) = 5x$ and $g(x) = x^2$. Let $R$ be the region between the graphs of $f(x)$ and $g(x)$ in $[0,3]$. The volume of the solid generated when $R$ is rotated around

(i) the x-axis,

(ii) the y-axis.

8 (c) The surface area generated by rotating the curve

$$y = x^3, \quad 0 \le x \le 1$$

about the $x$-axis.

6 6. (a) Express the length of the curve

$$y = x^{\frac{3}{2}} + 1 \text{ over } 0 \le x \le \frac{4}{3},$$

in the form of a definite integral.

2 (b) Evaluate the integral obtained in part (a).

8 7. (a) Sketch the graph of the curve given in polar coordinates by

$$r = 1 - \cos\theta.$$

8 (b) Set up a definite integral for the length of the curve given in part (a).
[Do not attempt to evaluate this integral.]

## Final: December, 1994

Values

4 1. Find $\dfrac{d}{dx}\left(\cos^{-1}(x^2 - 3x)\right)^3$. DO NOT SIMPLIFY YOUR ANSWER.

6 2. Find the value of $\displaystyle\int_{-\sqrt{3}/2}^{\sqrt{3}} \frac{1}{\sqrt{3 - x^2}}\,dx$.

IN QUESTIONS 3 TO 10 DO NOT SIMPLIFY YOUR ANSWERS.

4 3. Find $\displaystyle\int \frac{e^{3x}}{(e^{3x} - 1)^2}\,dx$.

4 4. Find $\displaystyle\int \sin^2(4x)\,dx$.

6 5. Find $\displaystyle\int \csc^3(2x)\cot^3(2x)\,dx$.

10 6. Find $\displaystyle\int e^x \cos(4x)\,dx$.

10 7. Find $\displaystyle\int \frac{\sqrt{x^2 - 5}}{x}\,dx$.

10 8. Find $\displaystyle\int \frac{1}{x^3 + x}\,dx$.

12 9. Find $\displaystyle\int \frac{x}{\sqrt{x^2 + 2x + 5}}\,dx$.

10 10. Find $\displaystyle\int \frac{x - 1}{x(x - 2)^2}\,dx$.

4 11. Find the value of the following limit

$$\lim_{x\to\infty} \frac{\ln(1+e^x)}{1+x}.$$

11 12. Find the value of the following limit

$$\lim_{x\to 0^+} (\sin x)^{\tan x}.$$

7 13. Find the value of the given integral or show it diverges.

$$\int_0^\infty e^{-(x+1)}\,dx.$$

9 14. Find the value of the given integral or show it diverges.

$$\int_{-2}^3 \frac{1}{x^3}\,dx.$$

5 15. Set up the definite integral to find the area of the plane region bounded by $y^2 = 9x$ and the line $y = x$. DO NOT EVALUATE THIS INTEGRAL.

6 16. Set up the definite integral for finding the length of the curve

$y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ from $x = 1$ to $x = 2$. DO NOT EVALUATE THIS INTEGRAL.

17. Set up the definite integral for finding the volume of the solid $S$ generated by rotating about the $x$-axis the finite region bounded by $y^2 = 9x$ and $y = x$.

5 (a) using the disk method (DO NOT EVALUATE THIS INTEGRAL)

5 (b) using the shell method (DO NOT EVALUATE THIS INTEGRAL)

## Final: April, 1995

Values

4 1. Simplify $\sin\left(\sin^{-1}(\frac{1}{\sqrt{2}}) + \cos^{-1}(\frac{1}{\sqrt{2}})\right).$

8 2. Let $f(x) = \tan^{-1}(x) + \cot^{-1}(x)$.

(i) Show that $f'(x) = 0$ (this will imply that $f(x)$ is a constant function).

(ii) What is the (constant) value of $f(x)$?
Hint: pick a value of $x$ that makes finding the answer easy.

8 3. Evaluate $\displaystyle\lim_{x\to 0} \frac{\tan^{-1}(x)}{\sin^{-1}(x)}$.

8 4. Evaluate $\displaystyle\lim_{x\to\infty} \left( \frac{x^2}{x-2} - \frac{x^2}{x+2} \right)$.

8 5. Let $L = \displaystyle\lim_{x\to 0} \frac{\sin(ax)}{\tan(bx)}$. Answer (i) or (ii).

(i) What is the value of $L$?

(ii) Write a Mathematica expression that evaluates $L$.

8 6. A block of cheese has the following properties:

(i) Its height is 6 units.

(ii) The cross sectional area at height $t$ is

$$A(t) = \begin{cases} t+3, & 0 \le t \le 3; \\ 12-2t, & 3 \le t \le 6. \end{cases}$$

What is the volume of the cheese?

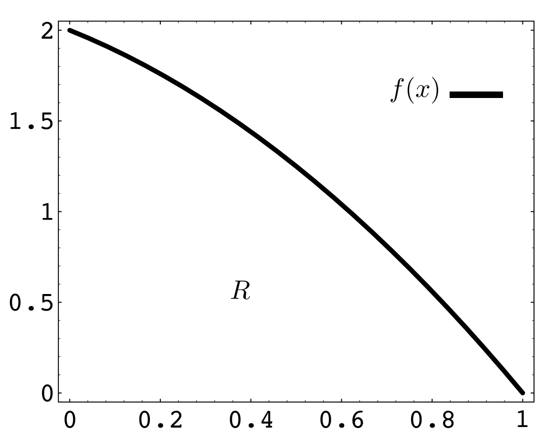



Figure 1

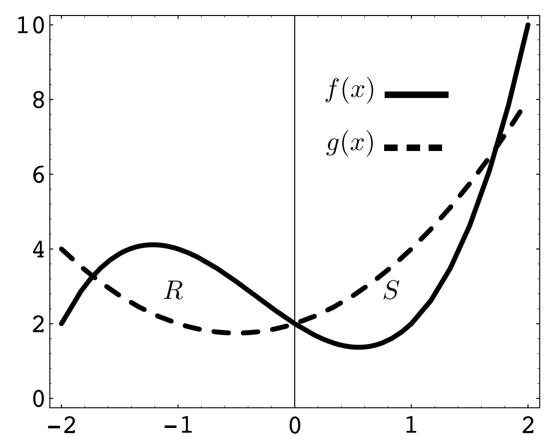



Figure 2

10 7. Let $f(x) = -(x-1)(x+2)$ and let $R$ be the region bounded by the $x$-axis, the $y$-axis and $f(x)$ as in Figure 1.
Find the volume of the solid formed when $R$ is rotated around the $x$-axis.

10 8. Let $f(x) = x^3 + x^2 - 2x + 2$ and $g(x) = x^2 + x + 2$. Find the area between the curves. (See regions $R$ and $S$ in Figure 2.)

8 9. Let $f(x) = 2x^{3/2} + 3$. Answer (a) or (b).

(a) Find the arc length of $f(x)$ as $0 \le x \le 1$.

(b) Write a Mathematica expression to find the arc length of $f(x)$ as $0 \le x \le 1$.

6 10. Set up the definite integral (but do not evaluate it) that finds the area of the surface of revolution of the curve $f(x) = (x-1)(x-2)$ about the $x$-axis as $1 \le x \le 2$.

4x10 11. Calculate the following integrals:

$$\int \frac{e^x}{e^{2x}+1}\,dx$$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

$$\int e^{3x}\cos(x)\,dx$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin^3(x)}{\cos(x)}\,dx$$

12 12. (i) Show that the following improper integral converges and evaluate it:

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}\,dx.$$

(ii) Determine whether the following improper integral converges or diverges:

$$\int_1^{\infty} \frac{x^2}{x^{10}+1}\,dx.$$

## Final: December, 1995

Values

4 1. (a) Simplify $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$.

4 (b) Find $\frac{d}{dx}\left(\sin^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right)$.

6 (c) Show that $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$.

(Hint. It might be useful to recall that if a function has derivative 0 then it is constant.)

2. Evaluate the following integrals. Do not simplify your answers.

7 (a) $\displaystyle\int x \sin x \, dx.$

7 (b) $\displaystyle\int \frac{x^2+1}{x-1} \, dx.$

7 (c) $\displaystyle\int \frac{dx}{x^3+x}.$

7 (d) $\displaystyle\int \cot^2 x \, dx.$

7 (e) $\displaystyle\int \frac{dx}{\sqrt{9x^2+12x-5}}.$

3. Evaluate the following limits:

7 (a) $\displaystyle\lim_{x\to 0^+} x^x.$

7 (b) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x}\right).$

10 4. Evaluate $\displaystyle\int_{-\infty}^{0} xe^x \, dx.$

5. Consider the region in the plane bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
Write integrals which will give the following. Do not evaluate these integrals.

5 (a) The area of this region.

5 (b) The volume formed when the region is rotated about the $x$-axis.

5 (c) The volume formed when the region is rotated about the $y$-axis.

5 (d) The length of the perimeter of the region.

5 (e) The total surface area of the solid formed when the region is rotated about the $x$-axis.

5 6. (a) Sketch $r = \cos 2\theta$.

5 (b) Find the area enclosed by one "leaf" in terms of an integral. Do not evaluate.

5 (c) Find the perimeter (arc length) of one "leaf" in terms of an integral. Do <u>not</u> evaluate.

8 7. (a) Find the Taylor polynomial of degree 3 for the function $e^x$ with centre 0.

3 (b) Write an approximation for $e^{0.1}$ using this polynomial.
(It is not necessary to simplify the expression.)

6 (c) Provide a reasonable estimate of the error which results from this approximation.
(Again you do not have to simplify the expression for the number.)

## Final: April, 1996

Values

4 1. (a) Let $F(x) = \displaystyle\int_{x^3}^{2} \sin(t^2)\,dt;$ find $F'(x)$.

6 (b) Find $\displaystyle\int_{-\pi/2}^{\pi/4} \sin^3 x \cos x\,dx.$

2 (c) What is the average value of $\sin^3 x \cos x$ on the interval $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{4}\right]$?

8 2. (a) Find $\displaystyle\int \sin^{-1} x\,dx$ by first using integration by parts and then a substitution.

10 (b) Find $\displaystyle\int \frac{dx}{(1-x)(1+x)(1+x^2)}$ by the method of partial fractions.

8 (c) Find $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ by using a trigonometric substitution.

4 3. (a) Evaluate $\displaystyle\lim_{x\to 0^+} \frac{\cos x}{1-\sin x}.$

6 (b) Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x-(1+x)}{x^2}.$

8 (c) Suppose $\displaystyle\lim_{x\to\infty} f(x) = +\infty$ and $f'(x) > 0$ for $x > 0$.

Evaluate $\displaystyle\lim_{x\to\infty} (f(x))^{1/f(x)}.$

6 4. (a) Evaluate the following improper integral:

$$\int_1^\infty \frac{1}{x^2}\,dx.$$

4 (b) Using part (a) and the fact that $\dfrac{\ln x}{x^3} \le \dfrac{1}{x^2}$ for $x \ge 1$, show that the following improper integral converges:

$$\int_1^\infty \frac{\ln x}{x^3}\,dx.$$

8 (c) Evaluate the integral $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$ using integration by parts.

5. A. **(Only students in regular sections L04 and L07 should do this question.)**

8 (a) Express as a definite integral the length of the perimeter of the ellipse $4x^2 + y^2 = 1$ ; do not evaluate this integral.

8 (b) Consider the ellipsoid formed by rotating the ellipse $4x^2 + y^2 = 1$ about the $x$-axis. Express as a definite integral the surface area of this ellipsoid; do not evaluate this integral.

8 (c) Find the volume of this ellipsoid.

5. B. **(Only students in the computer sections L03 and L06 should do this question.)**

8 (a) Consider the ellipsoid formed by rotating the ellipse $4x^2 + y^2 = 1$ about the $x$-axis. Express as a definite integral the surface area of this ellipsoid; do not evaluate this integral.

6 (b) Write a Mathematica command to compute the definite integral of part a).

5 (c) Explain briefly the difference between the Integrate and NIntegrate commands in Mathematica.

5 (d) Write a Mathematica command to compute the point where the graphs of $y = \cos(x)$ and $y = e^x - 1$ cross in the interval $[0, \pi]$.

6. The curve $C$ is given by the parametric equations $x = t^2$ and $y = t^3 - t$.

4 (a) Show that the curve crosses itself only at $(1, 0)$.

4 (b) Find the slopes of the two tangent lines to $C$ at $(1, 0)$.

6 (c) Find the points of $C$ where the tangent line to the curve is horizontal or vertical.

4 (d) Sketch the curve.

4 (e) Express as a definite integral the arc length of the curve from $t = 1$ to $t = 1$; do not evaluate this integral.

## Final: December, 1996

Values

1. Calculate the following integrals:

10 (a) $$\int \frac{x^2+1}{(x-1)^2(x+2)}\,dx$$

8 (b) $$\int \frac{2x+1}{x^2+4x+5}\,dx$$

8 (c) $$\int_0^{\pi/4} \sec^4 x \tan^2 x\,dx$$

8 (d) $$\int (\sin x)\, e^{2x}\,dx$$

2. Evaluate the following limits:

8 (a) $$\lim_{x\to 0} \frac{e^x-1}{\sin 2x}$$

8 (b) $$\lim_{x\to \pi^-/2} (\sec x - \tan x)$$

8 3. (a) Show that the following improper integral diverges

$$\int_0^1 \frac{x}{x-1}\,dx$$

8 (b) Show that the following improper integral converges

$$\int_1^\infty \frac{|\sin x|}{x^2}\,dx$$

4. Consider the ellipse $x^2 + 4y^2 = 4$. Find the integrals which give the following. Do not evaluate these integrals.

6 (a) The area enclosed by the ellipse.

6 (b) The circumference of the ellipse.

6 (c) The volume of the solid obtained by revolving the ellipse about the $x$–axis.

6 (d) The volume of the solid obtained by revolving the ellipse about the $y$–axis.

6 (e) The surface generated by revolving the ellipse about the $x$–axis.

5. Consider the curve with parametric equations:

$$x(t) = 3 - 4\sin t$$
$$y(t) = 4 + 3\cos t$$

for $0 \leq t \leq 2\pi$.

6 (a) Write the equation of the tangent line to the curve at the point corresponding to $t = \pi/4$.

6 (b) Find the points on the curve where the tangent line is horizontal.

6 (c) Find the points on the curve where the tangent line is vertical.

6 (d) Write a definite integral for the length of this curve. Do **not** evaluate the integral.

## Final: April, 1997

Values

8 1. (a) **For students in L04, L05, L06 and L91 ONLY**
Find the area of a circle of radius $r$ by setting up and evaluating an appropriate definite integral.

(b) **For students in the computer section L03 ONLY**
Consider the following MAPLE statements:

```
[> A:=Int(x*log(x), x=1..exp(1));
[> B:=int(x*log(x), x=1..exp(1));
[> value(A);
[> evalf(A);
```

4 (i) Write in ordinary mathematical notation, the integral being considered.

4 (ii) Explain – briefly! – what each MAPLE statement does. Be careful to make clear what the differences between the four statements are. (You should need to use no more than one complete sentence for each statement.)

8 2. If $F(x) = \displaystyle\int_{x^2}^{x^3} e^{-t^2}\,dt$ , find $F'(1)$ .

8 3. (a) Find $\displaystyle\int \frac{\sqrt{1+\ln(x)}}{x}\,dx.$

8 (b) Find $\displaystyle\int \frac{dx}{(1+x)(1+x^2)}.$

8 (c) Find $\int x \ln(x)\, dx$.

4 4. (a) Evaluate $\lim_{x \to 1^+} \dfrac{\ln(x)}{1+x}$.

6 (b) Evaluate $\lim_{x \to 0} \dfrac{\sin(x) - x}{x^2}$.

6 (c) Evaluate $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$.

8 5. (a) Evaluate the improper integral $\int_1^\infty e^{-x}\, dx$.

4 (b) Determine whether the improper integral $\int_0^\infty \cos(x)\, dx$ converges.

4 (c) Use the Comparison test and the results of 5(a) to decide whether

$\int_1^\infty \dfrac{e^{-x}}{1+x^4}\, dx$ converges. Briefly explain your steps.

6 (d) Determine whether the improper integral $\int_0^{\pi/4} \dfrac{\sec^2(x)}{\tan(x) - 1}\, dx$ converges.

If it does, find the value to which it converges.

5 6. (a) Consider the curve $C$ given by $y = (x^2+4)^{1/2}$ for $0 \le x \le 2$.
Express as a definite integral the length of the curve $C$. Do not evaluate this integral.

5 (b) Consider the surface $S$ obtained by rotating the curve $C$ around the $x$-axis. Express as a definite integral the surface area of $S$. Do not evaluate this integral.

8 (c) Find the volume of the solid obtained by rotating about the $x$-axis the region bounded by $x = 0$, $x = 2$, $y = 0$ and the curve $C$.

6 7. (a) Let the curve $C$ be given parametrically by $y = t^3$, $x = t^2$ for $0 \le t \le 5$. Express the length of the curve $C$ as a definite integral; do not evaluate this integral.

6 (b) Let $S$ be the surface obtained by rotating the curve $C$ about the $x$-axis. Express the surface area of $S$ as a definite integral; do not evaluate this integral.

4 8. (a) Sketch the graph of the curve given by the polar equation $r = 2\sin(\theta)$ for $0 \le \theta \le \pi$.

4 (b) Find $\dfrac{dy}{dx}$ for the curve in part (a).

4 (c) Find a point on the curve in part (a) where the tangent line is horizontal, and a point where it is vertical.

## Final: December, 1997

Values

6 1. Sketch $\sin^{-1}(\sin x)$. (Notice that the function is periodic because $\sin x$ is periodic. It is enough, therefore, to sketch the curve on the interval $[-\pi/2, 3\pi/2]$, say, and then extend it periodically.)

2. Evaluate the following integrals.

5 x 8 (a) $$\int \frac{\sin^3 x}{\cos^7 x}\, dx.$$

(b) $$\int \frac{\cos x}{1+\sin^2 x}\, dx.$$

(c) $$\int \frac{x^2}{\sqrt{2-x^2}}\, dx.$$

(d) $$\int \frac{x^3-3}{x^3-9x}\, dx.$$

(e) $$\int_0^\infty xe^{-x}\, dx.$$

3. Evaluate these limits or show they do not exist.

2 x 7 (a) $$\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}.$$

(b) $$\lim_{x\to\infty} x \sin\frac{1}{x}.$$

7 4. Determine whether the following integral converges or diverges.

$$\int_0^\infty \frac{x^2}{x^5+1}\, dx.$$

5. Answers for this question are to be given in terms of integrals. Do **not** evaluate the integrals.

4 x 8 (a) Find the area bounded by $x = y^2 - 2y$ and $x = 4 - y^2$.

(b) A hemispherical bowl of radius 30 cm. contains water of maximum depth 20 cm. How much water is in the bowl?

(c) Find the circumference of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(d) A parabolic reflector is formed by rotating the curve $y = x^2$, $0 \leq x \leq 1$, about the y-axis. Find the surface area of the reflector.

6. A curve is given parametrically by $x = t^3 - 3t$, $y = t^2$.
(The points $(0,0)$, where $t = 0$ and $(0,3)$, where $t = \pm\sqrt{3}$, are on the curve.)

3 x 7 (i) Sketch the curve.
(Exploit information which can be obtained from derivatives, identify concavity.)

(ii) Express the area bounded by the curve as an integral. Do **not** evaluate the integral.

(iii) Express the circumference of the bounded region as an integral. Do **not** evaluate the integral.

## Final: April, 1998

Values

1. Let $R$ be the region of the first quadrant bounded between the $y$-axis, the parabola $y = 5 - x^2$ and the line $y = 4x$.

4 (a) Sketch a picture of $R$. On your sketch indicate the co-ordinates of the point in the first quadrant where the line and the parabola intersect.

6 (b) Find the area of $R$.

5 (c) Set up an integral giving the volume of the solid swept out by rotating $R$ about the x-axis. Do not evaluate this integral.

5 2. Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}(x^2)$. [Note: $\tan^{-1}(z)$ means the same as $\arctan(z)$].

3. Evaluate the following integrals:

6 (a) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

6 (b) $\displaystyle\int_0^{\pi/4} \sec^2(x)\tan(x)\,dx$

6 (c) $\displaystyle\int x\sin(x)\,dx$

6 (d) $\displaystyle\int \tan^{-1}(x)\,dx$ [Note: $\tan^{-1}(z)$ means the same as $\arctan(z)$].

6 (e) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$

8 (f) $\displaystyle\int \frac{3x^2+3x+6}{(x+3)(x^2+3)}\,dx.$

4. Evaluate the following limits:

5 (a) $\displaystyle\lim_{x\to\pi/2} \frac{\sin(x)-\cos(x)-1}{\sin(x)+\cos(x)-1}$

6 (b) $\displaystyle\lim_{x\to+\infty} x^{\left(\frac{1}{(x^2-1)}\right)}.$

5. Decide whether each of the following improper integrals converges or diverges. If it converges, find the value to which it converges. Explain briefly the reasons for your conclusion.

7 (a) $\displaystyle\int_e^\infty \frac{1}{x(\ln(x))^2}\,dx$

7 (b) $\displaystyle\int_0^1 \frac{2x}{(1-x^2)^{3/2}}\,dx.$

6. Let $f(x)=\cos(x)$.

4 (a) Write an integral giving the length of the curve $C$ with equation $y=f(x)$ between the points $(0,1)$ and $(\pi/2,0)$. Do not evaluate this integral.

4 (b) Write an integral giving the area of the surface swept out by rotating the curve $C$ about the $x$-axis. Do not evaluate this integral.

7. Consider the curve $C$ with parametric equations

$$x(t)=t^3-12t,\quad y(t)=t^2+1 \qquad (-4\le t\le 4).$$

5 (a) Find the equation of the tangent line to the curve at the point on the curve corresponding to $t=1$.

5 (b) Give the co-ordinates of the point(s) on $C$ at which $C$ has a vertical tangent line. Explain what you are doing.

4 (c) Set up an integral for the arc length of the portion of $C$ lying between the points on $C$ corresponding to $t=0$ and $t=1$. Do not evaluate this integral.

7 8. Find the area of the portion of the plane in the first quadrant bounded by the cardiod $r=1+\cos(\theta)$ and between $\theta=\pi/4$ and $\theta=\pi/2$.

**SPECIAL INSTRUCTION. Do <u>not</u> answer question 9 if you are in the computer section (L03).**

8 9. **DO PART A OR PART B, BUT NOT BOTH. (<u>Not</u> for L03)**

**PART A**

Let $F(x)$ be a function whose derivative function is continuous on $[a, b]$. Here is a table of values of $F(x)$ for various values of $x$.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $F(x)$ | 1 | 3 | 4 | 6 | 5 |

Use this information to find the value of the integral $\int_0^1 [2x + 3F'(2x)]\, dx$.

[Hint: find a suitable antiderivative.]

**PART B**

Let $f(x) = 2x + 1$ and let $r$ be a number greater than 1. If the average value of $f(x)$ on the interval $[1, r]$ is 6, find the numerical value of $r$.

**(Indicate whether you are answering Part A or Part B.)**

**END OF EXAM (non-L03)**

**SPECIAL INSTRUCTION. Question 10 is <u>only</u> for the students in the computer section (L03).**

8 10. **(L03 only)**
Consider the following sequence of Maple commands, which is intended to solve a standard problem in the style of our assignments.

```
[> int(x*exp(x^2),x);
[> changevar(u=x^2,");
[> value(")
[> changevar(u=x^2,");
```

Each command has exactly one error in it, either in the way it is printed or in the type of command used.

Rewrite each command correctly, and indicate after it (in ordinary mathematical notation) the output that you would expect Maple to give. (You are not expected to re-create the exact form of Maple's output).

**END OF EXAM (L03)**

## Final: December, 1998

Values

1. Simplify

3 (a) $\cos(\sin^{-1}(\sqrt{x}))$

3 (b) $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$

2. Evaluate the following integrals:

8 (a) $\displaystyle\int_0^{\pi/4} \frac{\sin^3(x)}{\cos(x)}\, dx$

8 (b) $\displaystyle\int \frac{dx}{x^3 + x}$

8 (c) $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 12x - 5}}$

8 (d) $\displaystyle\int x \sin x \, dx$

8 (e) $\displaystyle\int \frac{\sqrt{x^2 - 4}}{x}\, dx$

3. Evaluate the following limits

7 (a) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1}(x)}{\sin^{-1}(x)}$

7 (b) $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^x}$

7 4. Determine whether the following integral diverges or converges.

$$\int_1^{\infty} \frac{dx}{\sqrt{x + x^3}}$$

5. Consider the curve $y = x^2$ for $0 \le x \le 2$.

Find integrals which give the following. Do **not** evaluate these integrals.

5 (a) The length of the curve.

5 (b) The surface area of the solid formed when the curve is rotated about the $x$-axis.

5 (c) The volume of the solid in part (b).

5 (d) The surface area of the solid formed when the curve is rotated about the line $y = -1$.

5 (e) The volume of the solid in part (d).

6. The curve $r = 2\cos 3\theta$ has 3 "petals".

4 (a) Sketch the curve.

5 (b) Find an integral which represents the area of one of the petals. Do **not** evaluate the integral.

5 (c) Find an integral which represents the length of the arc enclosing one of the petals. Do **not** evaluate the integral.

7. Consider the parametric curve $x = t^2 + t, \quad y = t^2 + 1.$

6 (a) Sketch the curve. Incorporate what you can learn from derivatives. Discuss concavity.

4 (b) Represent the length of the curve for $-2 \le t \le 1$ as an integral. Do **not** evaluate the integral.

4 (c) Notice that when $t = -2$ and $t = 1$, the $x$-coordinates of the points on the curve are equal to 2. Represent the area enclosed by the curve for $-2 \le t \le 1$ and the line $x = 2$ as an integral. Do **not** evaluate the integral.

## Final: April, 1999

Values

1. Let $F(x) = \displaystyle\int_0^{\sin x} \sin^{-1} t \, \cos^{-1} t \, dt, \quad 0 \le x \le \frac{\pi}{2}$; here $\sin^{-1} x$ and $\cos^{-1} x$ are inverse functions to $\sin x$ and $\cos x$, respectively.

6 (a) Find $F'(x)$.

4 (b) Show that the answer in part (a) can be converted into a form not involving inverse trigonometric functions.

2. Calculate each of the following integrals

8 (a) $\displaystyle\int \sec^3 x \tan^3 x \, dx$

8 (b) $\displaystyle\int \sin^2 x \cos^2 x \, dx$

8 (c) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+9}} \, dx$

8 (d) $\displaystyle\int \sin(2x)\cos x\,dx$

5 3. (a) Write out the form of the partial fraction decomposition of

$$\frac{x^6+3x^5+2}{(x-2)^3(x^2+x+1)^2}$$

Do not calculate the numerical values of the coefficients.

8 (b) Evaluate $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+1)}\,dx.$

4. Let $R$ be the region bounded by the graphs of the functions $f(x)=e^x$ and $g(x)=-x$, the $y$-axis and the line $x=3$.

8 (a) Find the volume of the solid generated by revolving the region $R$ about the $y$-axis.

4 (b) Let $L(x)$ be the length of the line segment drawn perpendicular to the $x$-axis at a point $x$, with one end point lying on the graph of $f(x)=e^x$ and the other one on the graph of $g(x)=-x$. Find the average value of $L(x)$, over the interval $[0,3]$.

5. Calculate these limits:

5 (a) $\displaystyle\lim_{x\to 0}\frac{e^{2x}-2x-1}{1-\cos x}$

5 (b) $\displaystyle\lim_{x\to\infty}\left(1+\sin\left(\frac{3}{x}\right)\right)^x$

6. Evaluate these improper integrals or explain why they do not exist.

6 (a) $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$

6 (b) $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}}$

7. The curve $C$ is given parametrically by $x=t^3+1$, $y=(1-t^2)^{3/2}$ for $0\le t\le 1$.

5 (a) Find the slope of the tangent line at the point corresponding to $t=\dfrac{1}{2}$.

5 (b) Calculate the length of $C$.

5 (c) Evaluate the area of the surface obtained by rotating the curve $C$ about the $x$-axis.

8. The curve $C$ is given by polar equation $r=\sin(3\theta)$ for $0\le\theta\le\dfrac{\pi}{3}$.

4 (a) Sketch the graph of $C$.

4 (b) Find the area of the planar region $R$ enclosed by $C$.

3 (c) Write the integral expressing the length of $C$. Do not evaluate this integral.

5 (d) Find an equation of the tangent line to the curve $C$ at the point corresponding to $\theta = \dfrac{\pi}{6}$.

## Final: December, 1999

Values

12 1. The region $R$ is bounded by $y = x(2-x)$ and $y = 0$.
(a) Sketch $R$.
(b) Find the volume of the solid obtained by rotating $R$ around the $x$-axis.
(c) Find the volume of the solid obtained by rotating $R$ around the $y$-axis.

11 2. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$ and $x = 1$. Sketch the region.

8 3. At one moment the temperature on the surface of the earth along a fixed latitude is $\dfrac{x}{2} + \sqrt{x} - 25$ (in °C), where $x$ is the distance in hundreds of kilometres measured from the north pole toward the equator ($0 \le x \le 100$). Find the average temperature along that latitude.

18 4. Evaluate the following integrals

(a) $\displaystyle\int (x+1)\ln x\, dx.$

(b) $\displaystyle\int \sin^3 x \cos^6 x\, dx.$

(c) $\displaystyle\int \frac{dx}{x^2 - 3x + 2}.$

12 5. Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\, dx.$ Your final answer should be in terms of $x$ and simplified as much as possible.

12 6. Compute the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{2^x - 1}{\ln(x+1)}.$

(b) $\displaystyle\lim_{x\to\infty} x^{(1/x)}.$

12 7. (a) Evaluate $\displaystyle\int_{-\infty}^{0} \frac{e^x}{1+e^x}\, dx.$

(b) Use the Comparison Theorem to determine if $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt[5]{x}}$ converges or diverges.

11 8. Find the length of the arc cut from the curve $y = x^{3/2}$ by the line $y = x$.

12 9. Consider the curve defined by the parametric equations

$$x = 3t - t^3, \quad y = t^2 - t\ .$$

(a) Find the points where the tangent lines are horizontal and the points where the tangent lines are vertical.

(b) Sketch the curve, indicating precisely the points found in (a).

12 10. The curve $r = \dfrac{\theta}{\pi} - \dfrac{\theta^3}{\pi^3}$, $\quad (0 \le \theta \le \pi)$ is given in polar coordinates.

(a) Why are the points $A(0,0)$, $B\left(\dfrac{8}{27}, \dfrac{\pi}{3}\right)$, $C\left(\dfrac{3}{8}, \dfrac{\pi}{2}\right)$, $D\left(\dfrac{10}{27}, \dfrac{2\pi}{3}\right)$ and $E(0,\pi)$ points on the curve $r = \dfrac{\theta}{\pi} - \dfrac{\theta^3}{\pi^3}$? Plot all these points. Sketch the curve $r = \dfrac{\theta}{\pi} - \dfrac{\theta^3}{\pi^3} \quad (0 \le \theta \le \pi)$.

(b) Find the area of the region enclosed by the curve $r = \dfrac{\theta}{\pi} - \dfrac{\theta^3}{\pi^3} \quad (0 \le \theta \le \pi)$.

## Final: April, 2000

Values

1. Calculate each of the following integrals

7 (a) $\displaystyle\int_0^1 \frac{1}{(x^2+1)(1+\tan^{-1} x)}\, dx$

7 (b) $\displaystyle\int \sin(2x)\, e^x\, dx$

7 (c) $\displaystyle\int \frac{\sqrt{x^2+4}}{x^4}\, dx$

7 (d) $\displaystyle\int_{\pi/6}^{\pi/4} \sec^4 x \tan^2 x\, dx\ .$

5 2. (a) Write out the form of the partial fraction decomposition of $\dfrac{3x^4+1}{x^2(x^2+2)^2}$.

DO NOT CALCULATE THE NUMERICAL VALUES OF THE COEFFICIENTS.
JUST REPRESENT THEM BY $A$, $B$, $C$, etc.

7 (b) Calculate the following integral $\displaystyle\int \frac{3x^2+1}{(x-1)(x^2+x+2)}\,dx$.

3. Let $R$ be the region bounded by the graph of $y = \sin x$, the line $y = x$ and the lines

$x = \dfrac{\pi}{4}$, $x = \dfrac{\pi}{2}$. [Note: $\sin x \le x$, for $0 \le x \le \dfrac{\pi}{2}$].

7 (a) Set up a definite integral for the volume of the solid generated by revolving the above region $R$ about the $x$-axis.
[DO NOT EVALUATE THE INTEGRAL] [It would help if you make a sketch].

7 (b) Set up a definite integral for the solid generated by revolving the above region $R$ about the $y$-axis. [DO NOT EVALUATE THE INTEGRAL].

8 4. The base of a solid is an elliptical region in the $x \circ y$ plane (Cartesian plane), whose boundary is given by the equation $x^2 + \dfrac{y^2}{4} = 1$. All cross sections of the solid perpendicular to the $x$-axis are squares. Find the volume of the solid. [It would help if you make a sketch].

5. Let $C$ be the curve $y = \sin^{-1} x + x, \quad 0 \le x \le 1$.

6 (a) Set up a definite integral for the length of $C$. DO NOT EVALUATE THE INTEGRAL.

4 (b) Set up a definite integral for the surface area generated by rotating the above curve $C$ about the $x$-axis.

6. Calculate the following limits

5 (a) $\displaystyle\lim_{x\to\infty} \frac{\tan^{-1} x - \frac{\pi}{2}}{\sin \frac{1}{x}}$

7 (b) $\displaystyle\lim_{x\to 0^+} (1 + \sin^{-1} x)^{(1/x)}$.

8 7. (a) Show that the following improper integral converges and determine its value $\displaystyle\int_0^\infty x\, e^{-2x}\,dx$.

4 (b) Show that the following improper integral converges $\displaystyle\int_1^\infty \frac{x^{17}}{x^{19}+1}\,dx$.

8. Let $C$ be the curve given by parametric equations

$$x = t + \sin t \qquad y = \cos t \qquad 0 \le t \le \frac{\pi}{2}$$

6 (a) Give the equation of the tangent line to the graph of $C$ at the point corresponding to $t = \dfrac{\pi}{4}$.

4 (b) Give the definite integral for the length of arc $C$. DO NOT EVALUATE THIS INTEGRAL.

6 9. (a) Sketch the graph of a curve $C$ whose equation in polar coordinates is given by
$r = 1 - \sin\theta, \quad 0 \le \theta \le 2\pi$.

4 (b) Set up a definite integral for the area of the region enclosed by the above curve $C$.
DO NOT EVALUATE THE INTEGRAL.

4 (c) Set up a definite integral for the length of the above curve $C$.
DO NOT EVALUATE THE INTEGRAL.

## Final: April 2004

Values

1. Evaluate.

3 (a) $\displaystyle\int \tan^2 x\, dx$

5 (b) $\displaystyle\int \tan^3 x\, dx$

5 (c) $\displaystyle\int x\left(e^{x2} + 1\sin\left(x^2\right)\right)\, dx$

9 (d) $\displaystyle\int \frac{x^2}{1-x^2}\, dx$

8 (e) $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$

9 (f) $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$

12 (g) $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-1}}$

3 2. Write the general form (in terms of unknown coefficients) of the partial fractions expansion of the expression $\dfrac{2x+1}{(x+1)^2(x^2+1)}$. Do NOT determine the numerical values of these coefficients.

3 Find the limit, if it exists.

8 (a) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{e^x - x - 1}$

10 b) $\displaystyle\lim_{x\to 0^+} \left(e^x + x\right)^{\frac{5}{x}}$

6 4. Use the comparison test to determine whether the improper integral converges or diverges.

$$\int_1^{\infty} \frac{e^{-x}}{x+3}\, dx$$

7 5. Write an integral which represents the volume created when the region enclosed by $y = 0$, $x = 1$, $x = 2$, and $y = \dfrac{6}{x}$ is rotated about the line with equation $y = -3$.
Justify your work. A sketch is sufficient. Do NOT evaluate this integral.

9 6. Write an integral which represents the length of the arc enclosed by one petal of the curve $r = 5\sin 3\theta$.
Justify your work. A sketch is sufficient. Do NOT evaluate this integral.

8 7. Write an integral which represents the surface area of the solid formed when the curve with equation $y = x^3 + 1$ for $0 \le x \le 2$ is rotated about the $x$-axis.
Do NOT evaluate this integral.

8 8. The base of a solid is the region enclosed by the parabola $y = -(x-2)^2 + 9$ and the $x$-axis. Every cross-section is a semicircle perpendicular to the $x$-axis. Write an integral which represents the volume of this solid.
Justify your work. A sketch is sufficient. Do NOT evaluate this integral.

10 9. The curve, given parametrically by $x = t^2$, $y = t^3 - 4t$, crosses itself at the point with coordinates $(4,\ 0)$. Write an integral which represents the area of the loop formed by this curve.
Justify your work. A sketch is sufficient. Do NOT evaluate this integral.

## Final: December 2004

Values

46 1. Integrate, using any appropriate method:

a) $\displaystyle\int \sin^3 x \cos^3 x\, dx$

b) $\displaystyle\int \sec^3 x \tan x\, dx$

c) $\displaystyle\int x \ln x\, dx$

d) $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-1}}$

e) $\displaystyle\int_8^{27} e^{\sqrt[3]{x}}\, dx$

f) $\int e^{3x}\ cos\, x\ dx$

g) $\displaystyle\int_{-1}^{0} x^2(x+1)^{10}\, dx$

h) $\displaystyle\int \frac{1}{x^2+2x-3}\,dx$

i) $\displaystyle\int \frac{1}{x^2+2x+3}\,dx$

6 2. Find $\displaystyle\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x$.

8 3. A solid concrete monument, similar in shape to the CN tower but on a smaller sclae, is to be constructed as follows. It's height is to be 15 meters, and a horizontal cross-section at height $h$ (meters) is to be an equilateral triangle with side length $\dfrac{1}{\sqrt{h+1}}$ (meters). Calculate the volume of contrete required for this project.

15 4. Make a rough sketch of the curve given by the parametric equations
$x = t^2 - 2; \quad y = t^3 - 2t.$
Let us call the enclosed region $R$.

a) Set up and simplify, BUT DO NOT EVALUATE, an integral for the area of $R$.
b) Set up and simplify, BUT DO NOT EVALUATE, an integral for the perimeter of $R$.
c) Observe that the curve is symmetric relative to the $x$-axis. Set up and simplify, BUT DO NOT EVALUATE, an integral for the surface area obtained by rotating $R$ around the $x$-axis.

12 5. Improper integrals.

a) Evauate $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$.

b) Determine whether $\displaystyle\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$ converges or diverges.

10 6. Find the partial fractions decomposition for the rational function

$$\frac{x^6 - x^5 + x^4 + x - 2}{x^2(x^2+1)}.$$

DO NOT INTEGRATE.

10 7. The region under the curve $y = \dfrac{1}{x}, x \geq 1$, is rotated about the $x$ axis.

a) Find the volume of the resulting solid.
b) Find the area of the surface of revolution.

13 8. Consider the curve $r = 1 - 2\cos\ \theta, 0 \leq \theta \leq 2\pi$.

a) Sketch the graph of this curve. Be sure to indicate the values of all intercepts on your graph.
b) Find the values of $\theta$ at which the curve passes through the pole. Give a formula, using an integral for the area of the inner loop. DO NOT evaluate this integral.
c) Express the length of the outer loop of this curve as an integral. DO NOT evaluate this integral.

## Final: April 2005

Values

5 1. Calculate $\lim_{x\to\infty} \dfrac{x \ln x}{x^2+1}$ . Justify your calculations.

8 2. A curve $C$ has parametric equations

$$y = 2t^3 - 6t$$
$$x = 2t^3 + 3t^2$$

Find the co-ordinates of the points on $C$ at which the tangent line to $C$ has slope $1/2$ .

8 3. Suppose that $g(x)$ is a differentiable function and that

$$\int_0^8 (\sqrt{x+1} + 3g'(x))\, dx = 24$$

If $g(0) = 7/9$, calculate the value of $g(8)$.

8 4. Let $C$ be a curve with polar equation $r = \sqrt{\sin(\theta)}$ for $0 \le \theta \le \pi$. Find the area of the region bounded by the positive $y$-axis, the curve $C$, and the line $\theta = \dfrac{2\pi}{3}$.

8 5. Find the area of the portion of the first quadrant bounded between the curves $y = 8\sqrt{x}$ and $y = x^2$.

8 6. Let $R$ be the region of the first quadrant bounded below by the $x$-axis, above by the parabola $y = x^2$, and on the right by the vertical line $x = k$ (where $k > 0$). The volume of the solid swept out by rotating $R$ around the $x$-axis equals the volume of the solid swept out by rotating $R$ about the $y$-axis. What is the value of $k$? What is this common volume?

9 7. Evaluate $\displaystyle\int_{-\frac{1}{2}}^{\frac{\pi-2}{4}} \sqrt{\sin(2x+1)}\cos(2x+1)\, dx$. Is your answer bigger than 0.25?

8. Evaluate the following integrals:

8 (a) $\displaystyle\int \tan(2x)\sec^4(2x)\, dx$

8 (b) $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\, dx$

9 (c) $\displaystyle\int \frac{2x^2-x+2}{x^3+2x}\, dx$

9. Do the following improper integrals converge? If so, to what? Explain.

8 (a) $\displaystyle\int_0^\infty \frac{dx}{(2x+1)(x+1)}$

8 (b) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^x-1}}$

8 10. The curve $C$ has parametric equations

$$x = \frac{t^3}{3} + \frac{t^2}{2}$$
$$y = \frac{t^3}{3} - \frac{t^2}{2}$$

for $0 \le t \le 1$. What is the length of $C$?

8 11. Let $S$ be the surface swept out by rotating the curve $y = \dfrac{x^3}{3}$, $(0 \le x \le 1)$. about the $x$-axis. Compute the surface area of $S$.

9 12. Evaluate $$\lim_{x \to 0} \left( \frac{\displaystyle\int_{2x}^{3x} \sin(t^2)\,dt}{\displaystyle\int_0^x \sin(t^2)\,dt} \right).$$

## Final: December 2005

Values

18 1. (a) Find the area of the region bounded by the curves $y = x^2$ and $y = 8 - x^2$.
(b) Find the volume of the solid generated by revolving the region in part (a) about the $x$-axis.

34 2. Calculate the following integrals:

(a) $$\int \frac{e^x}{1 + e^{2x}}\,dx$$

(b) $$\int e^x \sin 2x\,dx$$

(c) $$\int_1^{\sqrt{3}} \frac{x^3}{\sqrt{4 - x^2}}\,dx$$

(d) $$\int \frac{x^2 + 2x + 1}{x^3 + x^2 + x}\,dx$$

14 3. (a) Calculate the length of the arc $C$ given by the equation $y = e^x + \frac{1}{4}e^{-x} \quad (0 \le x \le 1)$.
(b) Set up a definite integral for the area of the surface generated by rotating the arc $C$ about the $x$-axis. Do not evaluate the integral.

12 4. A curve $C$ has the equations $x = t^3 + 3t^2$ and $y = t^3 - 3t^2$.
(a) Find the co-ordinates of the point(s) on the curve where the tangent line is horizontal and point(s) where the tangent line is vertical.
(b) Set up a definite integral for the length of an arc of the above curve corresponding to $0 \le t \le 1$.

14 5. (a) Find the area of the region bounded by the curve whose equation in polar co-ordinates is given by $r = \cos\theta - 1$.

(b) Find the length of an arc of the curve in part (a) corresponding to $0 \le \theta \le \pi$.

8 6. Calculate the following limit: $\lim\limits_{x \to 0^+} (1 + \sin x)^{\frac{1}{x}}$.

6 7. Find the value of the improper integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos x}}\, dx$.

14 8. (a) By using the comparison test, determine whether the improper integral

$\displaystyle\int_1^{\infty} \frac{1 + |\sin x|}{x}\, dx$ converges or diverges.

(b) Find the following limit: $\displaystyle\lim_{x \to \infty} \frac{1}{x} \int_1^x \frac{1 + |\sin t|}{t}\, dt$.

## Final: April, 2006

Values

1. Evaluate the following limits. Justify your calculations.

8 a) $$\lim_{x \to 2} \frac{(x-2)^2 - (x-2) + \sin(x-2)}{x^3 - 3x^2 + 4}$$

4 b) $$\lim_{x \to 1} \frac{(e^x - 1)(\cos(\pi x))}{\sin\left(\frac{\pi x}{2}\right)}$$

9 2. A curve $C$ has parametric equations

$$y = 2e^t + 3t$$
$$x = e^t - t$$

Let $P$ be a point on $C$. Suppose that the tangent line to $C$ at the point $P$ has slope 7. Find the co-ordinates of $P$.

8 3. If $g(x)$ is a continuous function and if

$$\int_0^1 \left(4g(x) + \frac{3}{x^2+1}\right) dx = \frac{15\pi}{4},$$

find the value of $\displaystyle\int_0^1 \left(g(x) + \frac{2}{x+1}\right) dx$.

8 4. The curve $C$ has polar equation $r + \theta = \pi$ $(0 \le \theta \le \pi)$. Find the area of the region $R$ of the first quadrant bounded by the $x$-axis, the $y$-axis and $C$.

5. Consider the region $R$ of the first quadrant bounded between the $y$-axis, the parabola $y = x^2$ and the line $y = x + 6$.

9 a) Compute the volume of the solid swept out by rotating $R$ about the $x$-axis. (Leave you answer as a sum or difference of fractions.)

9 b) Compute the volume of the solid swept out by rotating $R$ about the $y$-axis. (Leave your answer as a sum or difference of fractions.)

6. Evaluate the following integrals. Do not simplify your answers.

9 a) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x^3}{\sqrt{4-x^2}}\,dx$

9 b) $\displaystyle\int \frac{4x+1}{(x^2+2)(x+3)}\,dx$

8 c) $\displaystyle\int_0^{\frac{\pi}{8}} \sec^4(2x)\tan^3(2x)\,dx$

7. Do the following improper integrals converge? Justify your answers, and evaluate any of them that do.

9 a) $\displaystyle\int_1^{\infty} (x-1)e^{-2x}\,dx$

9 b) $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

4 8. Does $\displaystyle\int_1^{\infty} \left(\sqrt{x}-1\right)e^{-2x}dx$ converge or diverge? Justify your answer.
[Hint: look at the previous question.]

9 9. Let the curve $C$ have equation $y = \frac{4}{3}x^{\frac{3}{2}}$. Let $a > 2$. The portion of $C$ between the points $\left(2,\ \frac{4}{3}(\sqrt{2})^3\right)$ and $\left(a,\ \frac{4}{3}(\sqrt{a})^3\right)$ has length $\frac{49}{3}$. Find the value of $a$.

8 10. Find the area of the surface swept by rotating the curve with equation

$$y = \frac{x^5}{20} + \frac{1}{3x^3} \quad (1 \le x \le 2)$$

about the $x$-axis.
[Hint: there is a perfect square where you want it to be.]
Do not simplify your answer.

8 11. Let $k > 0$. Let $A(k)$ be the area of the region bounded by the curve $y = \sin(x^2)$, the $x$-axis, and the line $x = k$. Let $B(k)$ be the area of the triangle with vertices $(0,0)$, $(k,0)$, and $(k, \sin(k^2))$. Compute $\displaystyle\lim_{k\to 0^+} \frac{A(k)}{B(k)}$.

## Final: December, 2006

1. Calculate the following integrals:

(a) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$ (b) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$

(c) $\displaystyle\int (\ln x)^2\,dx$ (d) $\displaystyle\int \sec^4 x \tan^{10} x\,dx$

2. Find the area inside the ellipse $x^2+\dfrac{y^2}{4}=1$ and above the line $y=1$.

3. For each of the following improper integrals, determine whether it converges or diverges. If it converges, determine its value.

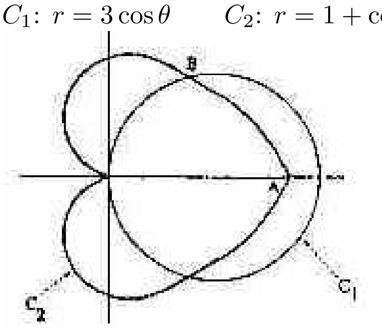
(a) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\cos^3 x}\,dx$

4. (a) Find the length of the arc $C$ given by the equation $y=\dfrac{x^3}{12}+\dfrac{1}{x}$, $\quad 1\le x\le 2$.

(b) Set up a definite integral for the area of the surface generated by rotating the arc $C$ in part (a) about the $x$-axis. Do not evaluate the integral.

5. A curve $C$ has parametric equations $x=-t+\cos t$, $\quad y=2+\sin t$.
(a) Determine all the point(s) on the curve where the tangent line is vertical or horizontal.
(b) Find the area of the region bounded by the graph of the curve, the $x$-axis, the lines $x=1$ and $x=\pi/2$. Hint: the equation $-t+\cos t=\pi/2$ has the solution $t=-\pi/2$. You do not need to sketch the graph, but note that the curve is above the $x$-axis and does not cross itself.
(c) Set up a definite integral for the length of the arc of this curve $C$ corresponding to $0\le t\le\pi$. You'll receive 2 bonus points if you can evaluate the integral.

6. The curves $C_1$ and $C_2$ with polar equations and graphs are as given:
$C_1$: $r=3\cos\theta$ $\qquad C_2$: $r=1+\cos\theta$.




(a) Find the polar co-ordinates of the intersection points of the two curves.
(b) Find the area of the region inside $C_1$ and outside $C_2$.
(c) Set up a definite integral for [the length of] an arc of $C_2$ with initial point $A$ and terminal point $B$. Do not evaluate the integral.

7. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\int_0^x \cos(t^2)\,dt}{e^{2x}-1}$ (b) $\displaystyle\lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+e^x}+\tan^{-1} x\right)$

## Final: April, 2007

1. Evaluate the following integrals:

(a) $\displaystyle\int_0^{\pi/2} \cos^3(x)\,dx$ (b) $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$

(c) $\displaystyle\int \sqrt{2x-x^2}\,dx$ (d) $\displaystyle\int \frac{dx}{x^4+x^2}$

2. Let $R$ be the region of the first quadrant bounded between the $x$-axis, the $y$-axis, the curve $y=\sqrt{x^2+9}$, and the line $x=4$. Write down an integral giving the volume of the solid swept out by rotating $R$ about the $y$-axis. Do not evaluate this integral.

3. Write down an integral giving the area of the region bounded by the $x$-axis and the curve $C$, whose polar equation is $r=\dfrac{1}{1+\theta}$. Do not evaluate this integral.

4. Evaluate the following improper integrals: (a) $\displaystyle\int_0^{\infty} xe^{-2x}\,dx$, (b) $\displaystyle\int_{-1}^{2} \frac{x+1}{x^{1/3}}\,dx$.

5. Use the comparison test to decide whether $\displaystyle\int_2^{\infty} \frac{1}{x(1+\sin^2(x))}\,dx$ converges or diverges.

6. Let $C$ be the curve with parametric equations $x=2t^2$, $y=t^3$ $(0\le t\le 1)$. Write down an integral giving the length of $C$. Do not evaluate this integral.

7. Let $C$ be the curve with equation $y=x^3$ $(0\le x\le 8)$.
(a) Set up an integral that gives the area of the surface obtained by rotating $C$ about the $x$-axis. Do not evaluate the integral.
(b) Set up an integral that gives the area of the surface obtained by rotating $C$ about the $y$-axis. Do not evaluate the integral.

8. Let $R$ be the region of the first quadrant bounded by the $x$-axis, the $y$-axis, and the parabola with the equation $y=1-x^2$. Let $k$ be a positive constant, and suppose that the parabola $y=kx^2$ divides $R$ into two regions of equal area. Find the value of $k$.

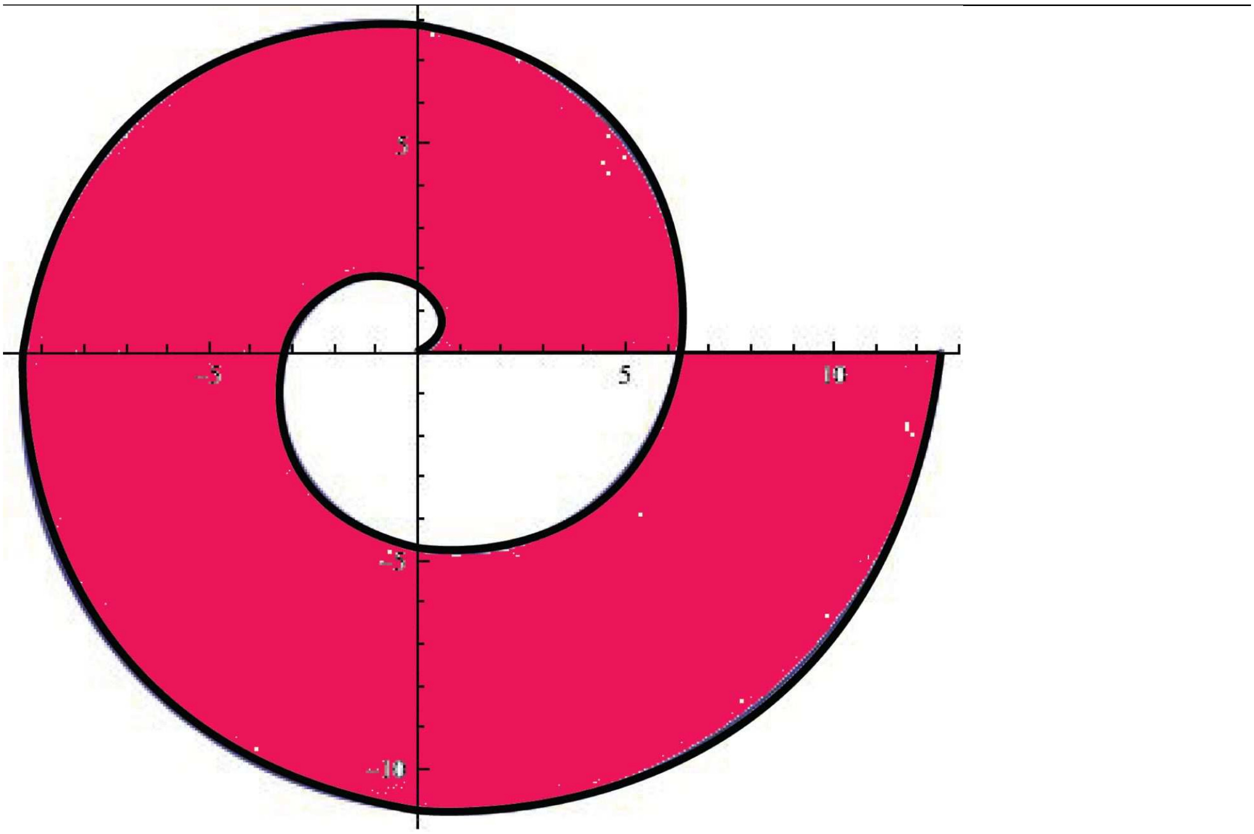
9. (Bonus question) Evaluate $\displaystyle\lim_{x\to 1^+}\left[\frac{\int_2^{2x}\cos\left(\frac{\pi t^2}{16}\right)dt}{x^3-1}\right]$.

## Final: December, 2007

1. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 1}\frac{1-x}{x-\sin\frac{\pi x}{2}}$ (b) $\displaystyle\lim_{x\to 0^+} x^{\frac{3}{4+\ln x}}$

2. (a) Sketch the region $R$ bounded by the curve $x=y^3$ and the lines $y=1$ and $x=8$; then find the area of $R$.
(b) Set up the integral for the area of the shaded region (see the figure below). The spiral is defined by $r=\theta$ in polar co-ordinates. Do not evaluate the integral.




(c) Write the equation of the tangent line of the curve $r = \theta$ at the point $(\pi/2, \pi/2)$.
3. (a) Sketch the region $R$ bounded by $y = \sin^2 x$ and the $x$-axis from $x = 0$ to $x = \pi$.
(b) Find the volume of the solid obtained by rotating the region $R$ around the $x$-axis.
(c) Set up but do not evaluate the integral for the volume of the solid obtained by rotating the region $R$ around the $y$-axis.
4. (a) Write the general form (in terms of unknown coefficients) of the partial-fraction decomposition (expansion) of each of the following expressions. Do not solve for the coefficients.

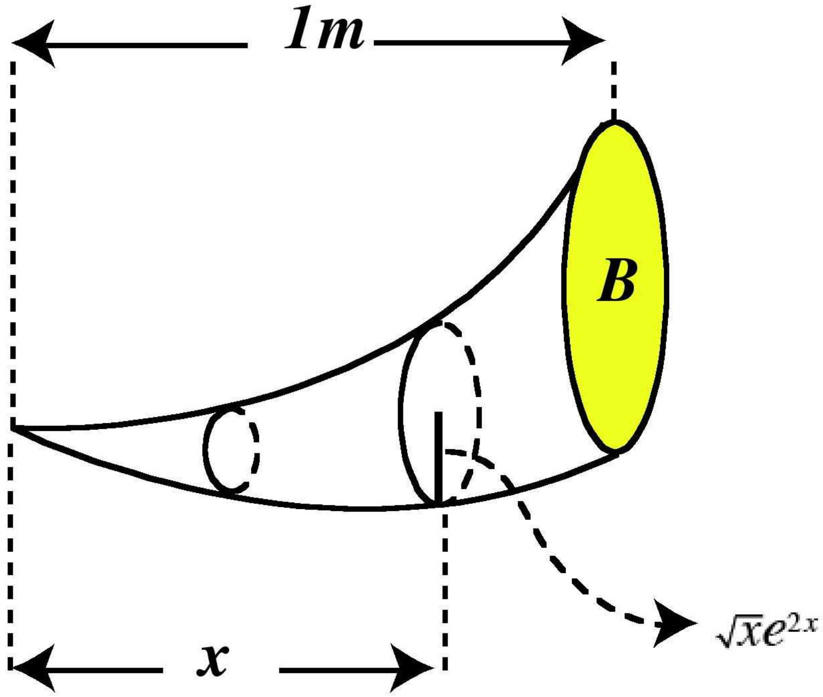
(i) $\dfrac{1}{(x-1)^2(x^2+1)}$ (ii) $\dfrac{x+1}{(x^2-1)(x^2+1)^2}$

(b) Evaluate $\displaystyle\int \frac{dx}{x^2-5x+6}$.
5. Evaluate each of the following integrals.

(a) $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$ (b) $\displaystyle\int \tan^{-1} x\,dx$

(c) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$ (d) $\displaystyle\int xe^{2x}\,dx$

6. A one-meter-long mammoth tusk is shown in the figure below. The intersection of the tusk and a plane parallel to the base $B$ (see the figure) is a circular disk of radius $\sqrt{x}e^x$, where $x$ is the distance from the tip of the tusk to the disk as shown in the figure.
(a) Set up the integral for the volume of the tusk.
(b) Evaluate the integral in (a).




7. (a) Evaluate the improper integral $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x(\ln x)^2}$ or show it diverges.

## Final: April, 2008

1. Calculate each of the following integrals.

(a) $\displaystyle\int \cos^3 x\, dx$

(b) $\displaystyle\int_0^{\pi} \sin^2 x\, dx$

(c) $\displaystyle\int x^2 \tan^{-1} x\, dx$

(d) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$

(e) $\displaystyle\int_0^1 x^3 \sqrt[3]{x^2+1}\, dx$

(f) $\displaystyle\int \frac{dx}{\sqrt{5+2x-x^2}}$

(g) $\displaystyle\int_2^5 e^{\sqrt{x-1}}\, dx$

(h) $\displaystyle\int \frac{x}{1+x^4}\, dx$

2. Write the general form (in terms of unknown coefficients) of the partial-fractions expansion of the expression $\dfrac{x+3}{(x^2-4x+4)(x^2+4)}$. Do not determine the numerical values of these coefficients.

3. Find the value of $\cos[\tan^{-1}(-\frac{2}{5})]$.

4. Use the comparison test to determine whether the improper integral $\displaystyle\int_0^1 \frac{e^x}{x^2}\, dx$ converges or diverges.

5. Write an integral which represents the length of the arc enclosing one petal of the curve $r = 2\cos(5\theta)$. Justify your work. A sketch is sufficient. Do not evaluate this integral.

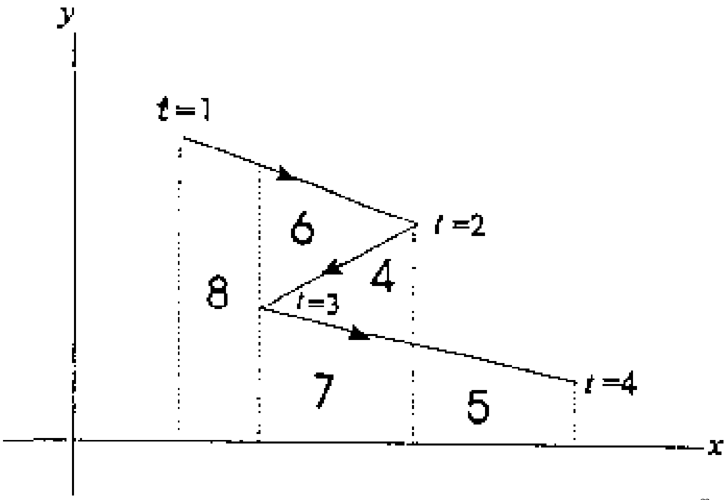
6. Write an integral which represents the surface area of the surface formed when the curve with equation $y = \sin^{-1}(x^2)$ for $0 \le x \le 1$ is rotated about the $x$-axis. Do not evaluate this integral.

7. The base of a solid is the region enclosed by the curve $y = e^x - 3$, the $y$-axis, and the $x$-axis. Cross-sections perpendicular to the $y$-axis are squares. Write an integral which represents the volume of this solid. Be sure to draw a sketch. Do not evaluate this integral.

8. A parametric curve, shown below, is traced once by the parametric equations

$$x = f(t), \qquad y = g(t), \qquad 1 \le t \le 4.$$

The area of each of the five regions is given in the diagram. State the value of $\displaystyle\int_1^4 y\frac{dx}{dt}\,dt$, showing how the five indicated areas were used to arrive at your final answer.




9. Find the intervals in which the curve defined by $f(x) = \displaystyle\int_{-10}^{x} e^{2t^3-t^2}\,dt$ is concave up and in which intervals it is concave down.

## Final: December, 2008

1. Consider the curve $x = \dfrac{t^3}{3} - \dfrac{t^2}{2}, \quad y = e^t - t.$
(a) Find the equation of one vertical tangent line to this curve.
(b) Compute $\displaystyle\lim_{t\to 0} \frac{dy}{dx}$.

2. (a) Sketch the region $R$ bounded by the curve $x = -y^2 + 2$ and the line $y = x$.
(b) Find the area of $R$.
(c) Set up, but do not evaluate, the integral for the volume of the solid obtained by rotating the region $R$ around the line $y = -2$.

3. (a) Sketch the curves $r = 2$ and $r = 4\cos\theta$ given in polar co-ordinates.
(b) Set up but do not evaluate the integral for the area of the region inside both of the curves $r = 2$ and $r = 4\cos\theta$.

4. (a) Write the general form (in terms of unknown coefficients) of the partial-fraction decomposition (expansion) of the following expression. Do not solve for the coefficients.
$\dfrac{1}{(x-2)(x+1)^2(x^2+3)^2} =$

(b) Evaluate $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$.

5. Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$ (b) $\displaystyle\int x\ln(x+1)\,dx$

(c) $\displaystyle\int \sqrt{4-x^2}\,dx$ (d) $\displaystyle\int_0^{\pi/2} (1-\cos x)\sin^3 x\,dx$

6. (a) Evaluate the following improper integral or show it diverges: $\displaystyle\int_0^2 \frac{dx}{\sqrt{x}-1}$.
(b) Determine whether the following improper integral converges or diverges. Do not try to evaluate the given integral directly. $\displaystyle\int_3^\infty \frac{dx}{x-(1+\sin x)}$.

7. Find the arc length of the arc cut from the curve $y = x^{3/2}$ by the line $y = x$.

8. Set up the following integrals. Do not evaluate the integrals.
(a) The integral for the arc length of the cardioid $r = 3 + 3\cos\theta$ given in polar co-ordinates.
(b) Consider the part of the curve $y = (2x-4)(4-x)$ that is above the $x$-axis. Set up but do not evaluate the integral for the surface area of the surface obtained by rotating that part of the curve around the $x$-axis.

## Final: April, 2009

1. Find $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^3-1}$.

2. Suppose that

$$\int_0^1 f(x)\,dx = \lim_{x\to 0^+}\left(-\frac{\sin^{-1} x}{x}\right) + \frac{\sin^{-1} 1}{1}.$$

What is the value of $\int_0^1 f(x)\,dx$? Determine $f(x)$.

3. Set up an integral or integrals to find the area bounded by the curves

$$x = -(y-5)(y+1), \quad 3y = x+1.$$

You **need not integrate** for full marks; 2 bonus marks if a correct expression is successfully integrated. No partial credit for bonus marks. No credit for integrating something incorrect.

4. Write the form for the partial-fraction decomposition of

$$f(x) = \frac{x^3+2x^2-7x+1}{(x^3+x^2-6x)(x^4-2x^2+1)},$$

**but do not find the constants or integrate.** No bonus marks available.

5. Set up an integral or integrals to find the volume of the solid of revolution obtained by rotating the area bounded by the curves

$$y = x, \quad y = 4-x, \quad y = 0,$$

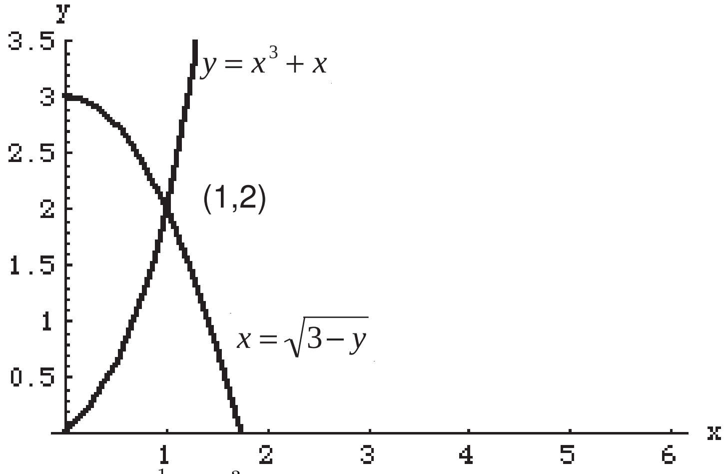
about the $y$-axis. You **need not integrate** for full marks; 2 bonus marks if a correct expression is successfully integrated. No partial credit for bonus marks. No credit for integrating something incorrect.

6. Set up an integral for the exact length of the polar curve $r = \theta$ for $0 \le \theta \le \pi$. You **need not integrate** for full marks; 3 bonus marks if a correct expression is successfully integrated. No partial credit for bonus marks. No credit for integrating something incorrect.

7. Set up an integral or integrals to find the volume of the solid of revolution obtained by rotating the area bounded by the illustrated curves

$$y = x^3 + x, \quad x = \sqrt{3-y}, \quad x = 0,$$

about the line $x = 3$. You **need not integrate** for full marks; 2 bonus marks if a correct expression is successfully integrated. No partial credit for bonus marks. No credit for integrating something incorrect.




8. Evaluate $\displaystyle\int_0^1 \frac{x^3}{(x^2+2)^3}\,dx$.

9. Evaluate $\int \tan^5 x \sec^7 x\,dx$.

10. Evaluate $\displaystyle\int \frac{\sqrt{3+2x-x^2}}{(x-1)^2}\,dx$.

11. Determine whether each integral is convergent or divergent. Evaluate any that is convergent.
(a) $\displaystyle\int_0^{1/2} \frac{1}{2x-1}\,dx$, (b) $\displaystyle\int_{-\infty}^{2} xe^{x/3}\,dx$.

12. Find the exact length of the curve $x = 1 + 2t^3$, $y = 4 + 3t^2$, for $0 \le t \le 1$.

13. Set up an integral or integrals to find the area of the surface of revolution obtained by rotating the curve $y = \cos 2x$ for $0 \le x \le \pi/6$ about the $x$-axis. You **need not integrate** for full marks; 3 bonus marks if a correct expression is successfully integrated. No partial credit for bonus marks. No credit for integrating something incorrect.

# Term Tests

## Term Test: October 2003

Values

1. Calculate the following integrals.

5 (a) $\displaystyle\int \frac{6x^2}{\sqrt{4+x^3}}\,dx$

4 (b) $\displaystyle\int \frac{6x^2}{4+x^3}\,dx$

4 (c) $\displaystyle\int \sec^5 x \tan x\,dx$

5 2. Evaluate $\displaystyle\int_0^3 15x\sqrt{x+1}\,dx$

6 3. Find the area between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

4. Set up but DO NOT EVALUATE, an integral for the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and the $x$-axis about:

3 (a) the $x$-axis

4 (b) the line $x = 2$.

5 5. Set up but DO NOT EVALUATE, an integral for the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = 2\sqrt{x}$ about the $x$-axis.

6 6. The base of a solid $S$ is a semi-circular disc $\{(x, y) \mid x^2 + y^2 \leq 4,\ x \geq 0\}$. Cross-sections of $S$ perpendicular to the $x$-axis are squares. Find the volume of $S$.

3 7. Express the following limit as a definite integral. DO NOT EVALUATE this integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\cos\left(\frac{2i}{n}\right) + \frac{2i}{n}\right]$$

## Term Test: November 2003

Values

1. Evaluate the following integrals:

6 (a) $\displaystyle\int_0^{\frac{\pi}{3}} \tan^3 x \sec x \, dx$

5 (b) $\displaystyle\int \sin 5x \sin 2x \, dx$

7 (c) $\displaystyle\int \frac{1}{x^2 - 2x + 5} \, dx$

12 (d) $\displaystyle\int \frac{x^2}{\sqrt{4 - x^2}} \, dx$

6 (e) $\displaystyle\int x \ln x \, dx$

4 2. Find $\dfrac{dy}{dx}$ if $y = 3x \cos^{-1}(3x)$. DO NOT SIMPLIFY YOUR ANSWER.

4 3. Write the general form (in terms of unknown coefficients) of the partial fractions expansion of the expression $\dfrac{2x+1}{(x^3-1)(x+1)^2}$.

DO NOT DETERMINE THE NUMERICAL VALUE OF THESE COEFFICIENTS.

11 4. Find the average value of the function $f(x) = e^{\sqrt{x}}$ over the inteval $[0, 4]$.

5 5. By proving that $f(x) = \tan^{-1} + \tan^{-1}\left(\dfrac{1}{x}\right)$, $x > 0$, is a constant function find the value of the constant $C$ such that $\tan^{-1} + \tan^{-1}\left(\dfrac{1}{x}\right) = C$.

## Term Test: February 2004

Values

15 1. Evaluate.

(a) $\displaystyle\int \cos x \sin^2 x dx$

(b) $\displaystyle\int (x+2)^{20}\,(20x)\,dx$

(c) $\displaystyle\int_1^3 x\left(x^2-1\right)^{\frac{2}{3}}\,dx$

23 2. Consider the parametric equation $x = t^2 + t, \quad y = t^2 + 2t$.

(a) Find the $x$ and $y$ intercepts.
(b) Determine the four limits $\lim\limits_{t\to\pm\infty} x$ and $\lim\limits_{t\to\pm\infty} y$. If the limit does not exist indicate if the trend is to $+\infty$ or $-\infty$.
(c) Find the coordinates of the points where tha curve has horizontal and/or vertical tangent lines.
(d) Determine the intervals of $t$ in which the curve is concave up and in which it is concave down.
(e) Sketch the parametric curve. Incorporate what you discovered in parts (a) through (d) and what you can learn from derivatives.

3 3. Sketch the graph of the curve given in polar coordinates by $r = 4 + 3\cos\theta,\ 0 \le \theta \le 2\pi$. Indicate the values of all the intercepts on your graph.

6 4. Let $C$ be the curve given by parametric equations $x = 2\sin(2t)$ and $y = 3\sin t$. Find an equation of the tangent line to the graph $C$ at the point corresponding to $t = \dfrac{2\pi}{3}$.

5 5. Set up a definite integral whose value is the area of the region enclosed by the parabola $y = x^2 + 1$ and the line $y = x + 7$.

4 6. Find the interval(s) in which the function $F(x) = \displaystyle\int_2^{x^3} e^{t^2}\,dt$ is increasing.

4 7. Find the value of $\displaystyle\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]$ by writing this limit as a definite integral and evaluating the resulting definite integral.

## Term Test: March 2004

Values

22 1. Evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/2} \sin^2(x)\,dx$

(b) $\displaystyle\int e^x \sin(2x)\,dx$

(c) $\displaystyle\int \cos^5(x)\sin^2(x)\,dx$

(d) $\displaystyle\int \cos(\sqrt{x})\,dx$

10 2. (a) Write the definition of $y = \sin^{-1}(x)$.
(b) Show that if $y = \sin^{-1}(x)$, then $\dfrac{dy}{dx} = 1\sqrt{1-x^2}$.

8 3. Compute:

(a) $\sin(2\tan^{-1}(x))$

(b) $\dfrac{d}{dx}\tan^{-1}(\sqrt{x})$.

8 4. Set up (DO NOT SOLVE) the definite integral required to solve the following problem. Show your work; a (rough) sketch is essential.

Find the area of the region inside $r = 1 + \cos(\theta)$ and outside $r = 1$.

12 5 Set up (DO NOT SOLVE) the definite integrals required to solve the following two problems. Show your work; (rough) sketches are essential.

In both cases you are asked to consider the region$\mathcal{R}$ betweeen the curves $y = 2$ and $y = \dfrac{x^2}{2}$.

(a) Give a rough sketch of the region $\mathcal{R}$.
(b) The region $\mathcal{R}$ is rotated around the $x$-axis. Find the volume.
(c) The region$\mathcal{R}$ is rotated around the line $x = 5$. Find the volume.

## Term Test: October 2004

Values

12 1. Evaluate.

(a) $\displaystyle\lim_{x\to 0} \frac{1-e^{5x}}{x}$

(b) $\displaystyle\lim_{x\to 0^+} (\sin x)^x$

4 2. Sketch the graph of the curve given in polar coordinates by

$$r = 2 - 2\cos\theta,\ 0 \le \theta \le 2\pi$$

Indicate the values of all the intercepts on your graph.

24 3. Consider the parametric equations $x = 9 - 3t^2$, $y = t^3 - 3t$.

(a) Find the $x$ and $y$ intercepts.
(b) Determine the four limits $\displaystyle\lim_{t\to\pm\infty} x$ and $\displaystyle\lim_{t\to\pm\infty} y$.
If the limit does not exist indicate if the trend is to $+\infty$ or $-\infty$.
(c) Find the coordinates of the points where the curve has horizontal and/or vertical tangent lines.
(d) Determine the intervals of $t$ in which the curve is concave up and in which it is concave down. Set up a derivative table indicating the signs of the derivatives and concluding with the direction of the curve in appropriate regions of this sign diagram.
(e) Sketch the parametric curve. Incorporate what you discovered in part (a) through (d) and what you can learn from derivatives.

5 4. Write an equation of the tangent line to the curve $r = 2 + \cos\theta$ at the point where $\theta = \dfrac{\pi}{2}$.

8 5. Evaluate the following integrals.

(a) $\displaystyle\int 3x(x^2+3)^3\,dx$

(b) $\displaystyle\int (\sec^2 x)(\tan^2 x + 1)\,dx$

3 6. Find the value of $\displaystyle\lim_{n\to\infty} \frac{2}{n^4}[2^3 + 4^3 + 6^3 + \ldots + n^3]$ by writing this limit as a definite integral. Find the value of the resulting definite integral.

4 7. Find $\dfrac{dy}{dx}$ if $y = \displaystyle\int_{x^2}^{\sin x} \frac{\cos\sqrt{t}}{t^2}\,dt$.

## Term Test: November 2004

Values

24 1. Integrate, using any appropriate method:

(a) $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$

(b) $\displaystyle\int \frac{x^2}{(2x-5)^{20}}\,dx$

(c) $\displaystyle\int \sin 2x \sin 3x\,dx$

(d) $\displaystyle\int \sin\sqrt{x}\,dx$

6 2. Find the average value of $f(x) = xe^x$ over the interval $0 \le x \le 3$.

8 3. A machine part is constructed as follows. Its base is the region under the curve $y = \sin x$, from $x = 0$ to $x = \pi$ (units are in cm.). Vertical cross sections taken parallel to the $y$-axis are squares. What is the volume of the part?

4. Let $Z$ be the region in the plane bounded by the graphs of

$$y = \frac{1}{\sqrt{1+x^2}},\ x = 0,\ x = 1 \text{ and } y = 0.$$

2 (a) Give a rough sketch of the region $Z$.
5 (b) Find the volume obtained by rotating $Z$ around the $x$-axis.
5 (c) Find the volume obtained by rotating $Z$ around the $y$-axis.

5. Let $T$ be the region bounded by the graphs of $x = 6y + 6$ and $x = 2y^2 - 2y + 12$.

4 (a) Find the points of intersection of the two graphs.
6 (b) Find the area of $T$.

## Term Test: February 2005

Values

1. Use L'Hôpital's rule to evaluate

4 a) $\lim_{x \to 0} \left( \dfrac{\ln(1+x)}{1-e^{2x}} \right)$

5 b) $\lim_{x \to 0} x^{\frac{1}{\sqrt{x}}}$

3 2. a) For a curve a with parametric equations $x = 2t - 1,\ y = 4t^2 - 4t$, find the equation of the tangent line to the curve at $t = 2$.

4 b) For the curve $x = \dfrac{t^3}{3} - 2t^2 + 3t - 7,\ y = t^2 - 4t;\ -\infty < t < \infty$, find all values of $t$ such that the curve has a verticle or horizontal tangent line.

5 3. a) Sketch the curve $r = 2 - \cos\theta$

2 b) A curve has equation $r^3 \sin\theta = 2$. Write its equation in cartesian coordinates.

6 4. a) Evaluate $\lim_{n \to \infty} \sum_{i=1}^{n} \dfrac{1}{n} \left( \dfrac{i^2}{n^2} \right)$ by converting to a definite inegral from 0 to 1 and evaluating.

3 b) Find $\dfrac{d}{dx} \left( \int_1^{x^2} \dfrac{1}{t^3+3} dt \right)$ at $x = 3$.

5 5. a) Evaluate the definite integral if it exists. If it doesn't exist, explain why not.

$$\int_{-2}^{1} \frac{1}{x^2}\, dx$$

b) Find $\int (x-1)^2\, dx$.

## Term Test: March, 2005

Values

8 1. a) Find the points of the intersection of the curves $y = 2x^2$ and $y = 3 - x^2$.

b) Find the area of the region enclosed by the curves $y = 2x^2$, $y = 3 - x^2$, $x = -2$, and $x = 0$.

8 2. Find the area of the region that lies inside the curve $r = 1 + \sin\theta$ and outside the curve $r = 2 - \sin\theta$.

12 3. Let $S$ be the region in the plane enclosed by curves $y = x^2 + 1$ and $y = 1 - x$.

a) Find the volume obtained by rotating $S$ around the $x$-axis.
b) Find the volume obtained by rotating $S$ around the $y$-axis.

4 4. Evalute $\dfrac{d}{dx}\tan^{-1}(x^4)$ when $x = 2$. Is your answer larger than $\frac{1}{8}$?

8 5. Integrate, using any appropriate method.

a) $\displaystyle\int_1^2 \frac{1}{x^2}\sin\left(\pi + \frac{\pi}{x}\right)dx$

b) $\displaystyle\int (2x-1)e^{-x}\,dx$

## Term Test: October, 2005

Values

12 1. Evaluate.

(a) $\displaystyle\lim_{x\to\pi}\frac{(x-\pi)^2}{1+\cos x}$

(b) $\displaystyle\lim_{x\to\infty}(e^x + x)^{\frac{1}{x}}$

20 2. Evaluate the following integrals.

(a) $\displaystyle\int 3x(x+3)^{20}\,dx$

(b) $\displaystyle\int_0^2 x^2(\sqrt{9-x^3})\,dx$

(c) $\displaystyle\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

(d) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$

12 3. Sketch each of the curves $y = x + 2$ and $y = x^2 + 2x$. Find the area of the region bounded by these curves.

8 4. Set up a definite integral but DO NOT EVALUATE it, for the solid generated by revolving the region in the first quadrant whose boundary consists of the curves $y = x^3$, $y = 1$ and $x = 0$ about the $x$-axis.

4. 5. Express $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{10}}{n^{11}}$ as a definite integral over $[0, 1]$. Find a numerical value of the resulting definte integral.

4 6. Find the derivative of the function $f(x) = \displaystyle\int_{-x^2}^{x^2+1} \sin(e^t)\,dt$

## Term Test: November, 2005

Values

5 1. a) Simplify the following expression.
(Your answer should include no trigonometric or inverse trigonometric functions).

$$\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) \qquad (0 \le x \le 2)\,.$$

3 b) Calculate the following derivative:

$$\frac{d}{dx}\left(\tan^{-1}\left(x^2\right)\right).$$

2. Calculate the following integrals:

7 a) $\displaystyle\int_1^{e^3} x^{-\frac{2}{3}} \ln x\; dx.$

7 b) $\displaystyle\int \frac{\sin^3 x}{\cos^8 x}\,dx.$

10 c) $\displaystyle\int_0^3 \frac{1}{(9+x^2)^{\frac{5}{2}}}\,dx.$

6 3. Write the following rational function in its most decomposed form as a sum of partial fractions with unknown coefficients. DO NOT CALCULATE THE COEFFICIENTS.

$$\frac{3x^2+1}{(x-1)^2(x+2)^2(x^2+x+2)^2}.$$

10 4. Calcuate the following integral:

$$\int \frac{5x-3}{x^2-3x+2}\,dx.$$

12 5. Each cross section of an object with a plane perpendicular to the $x$-axis at the point $x$ is a square having the length of each side equal to $\sin x$, when $0 \le x \le \frac{\pi}{2}$. Find the volume of the object when $x$ varies between 0 and $\frac{\pi}{2}$.

## Term Test: February, 2006

Values

1. Use l'Hôpital's rule to evaluate

3 a) $\lim\limits_{x\to 0} \dfrac{\sin^2(x)}{1-cos(x)}$

5 b) $\lim\limits_{x\to 0^+} (1-x)^{1/x}$

7 2. Consider the curve $C$ with parametric equations

$$x(t) = t^3 - 12t, \quad y(t) = t^2 + 1 \qquad (-4 \le t \le 4)$$

a) Find the equation of the tangent line to the curve at the point on the curve corresponding to $t = 1$.

b) Give the coordinates of the point(s) on $C$ at which $C$ has a vertical tangent line.

4 3 a) Sketch the curve $r = 1 + \cos\theta$ for $0 \le \theta \le 2\pi$.

3 b) Find a polar equation for the curve whose equation in Cartesian coordinates is $x^2 - y^2 = 1$. Find the Cartesian coordinate(s) of the point(s) on this curve whose $\theta$- coordinate is 0.

5 4. Find $\dfrac{d}{dx}\left(\displaystyle\int_{3x}^{5x} \frac{dt}{15+t^4}\right)$ at $x = 1$. Do not simplfy your answer.

5 5. Evaluate the definite integral $\displaystyle\int_0^{\pi} \left(1 + \sec^2\left(\frac{x}{4}\right)\right) dx.$

3 6. Evaluate the indefinite integral $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx.$

5 7. If $f(x) = x^3 + 1$, evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n}\right) f\left(\frac{2i}{n}\right)$ by writing it as a definite integral with lower limit of integration 0, and then evaluating that integral. Briefly explain what you are doing.

## Term Test: March, 2006

Values

1. Evaluate the following integrals.

3 a) $\displaystyle\int \frac{2e^x}{e^x+2}\,dx$

4 b) $\displaystyle\int_0^8 \frac{cos(\sqrt{x+1})}{\sqrt{x+1}}\,dx$

6 2. Find the area of the region bounded by the curves $x = y^2 - 12$ and $x = y$.

6 3. Find the area of the region inside the curve $r = 2\sin(2\theta)$ (given in polar coordinates) and outside the curve $r = \sin(2\theta)$ (in polar coordinates).

7 4. The region $R$ is bounded by the curves $y = \cos(x)$, $y = 1$ and $x = 1$.
Find the volume of the solid obtained by revolving $R$ around the $x$-axis. [Hint: washer method!]

7 5. The region $R$ is bounded by the curves $y = sin(x^2)$, $y = 1$ and $x = 0$.
Find the volume of the solid obtained by revolving $R$ around the $y$-axis. [Hint: Cylindrical shells!]

6. Evaluate the following integrals

3 a) $\displaystyle\int_0^{\frac{1}{2}} \frac{2+3x}{\sqrt{1-x^2}}\,dx$

4 b) $\displaystyle\int 2x\ln(x)\,dx$

## Term Test: October 2006

Values

1. Evaluate the following limits:

2 a) $\displaystyle\lim_{x\to 1} \frac{x \ln x}{x^2 - 1}$

3 b) $\displaystyle\lim_{x\to 0^+} (\sin x)^x$

3 2. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{3i}{n}\right)^4 \frac{1}{n}$.

4 3. Find $\displaystyle\frac{d}{dx} \int_{-x^2+1}^{x^2+1} \ln(2 + \cos t)\, dt$.

2 4. a) Simplify $\tan\cos^{-1} x$ for $x \in (0, 1]$.

2 b) Calculate the derivative of the function $x \sin^{-1}\sqrt{x}$.

1 5. a) Find all the points where the line $x + y = 1$ and the parabola $y = (x - 1)^2$ intersect.

4 b) Calculate the area of the bounded region enclosed by the above curves.

4 6. Calculate the following integral

$$\int \sin^6 x \cos^3 x\, dx.$$

7. A region $R$ is bounded by $y = x^x$, $y = (x - 1)^2$ and $x = 2$.

4 a) Find the volume of the solid generated by rotating $R$ about the $x$–axis.

3 b) Set up a definite integral for the volume of the solid generated by rotating $R$ about the $y$–axis. DO NOT EVALUATE THIS INTEGRAL.

3 8. Evaluate the following definite integral

$$\int_1^e x^3 \ln x\, dx.$$

## Term Test: Feb 2007

Values

12 1. Evaluate.

(a) $\displaystyle\lim_{x\to 0}\frac{e^x - x - 1}{x^2}$

(b) $\displaystyle\lim_{x\to 0^+}(2x+1)^{3/x}$

8 2. Evaluate the following integrals.

(a) $\displaystyle\int_1^{64}\frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[6]{x}}\,dx$

(b) $\displaystyle\int\left(e^x + \frac{2}{x}\right)dx$

26 3. Consider the curve whose parametric equations are: $x = 3t - t^3$, $y = 4 - t^2$.

(a) Find the $x$ and $y$ intercepts.

(b) Evaluate $\displaystyle\lim_{t\to\infty} x$, $\displaystyle\lim_{t\to-\infty} x$, $\displaystyle\lim_{t\to\infty} y$ and $\displaystyle\lim_{t\to-\infty} y$.

(c) Find the values of $t$ for which the graph has horizontal and/or vertical tangent lines. State the coordinates, (x,y), of these points of tangency.

(d) Set up a derivative table indicating the signs of the first derivatives and concluding with the direction of the curve in appropriate regions of this sign diagram.

(e) Find the intervals of values of $t$ in which the curve is concave up and where it is concave down.

(f) Use the above information, and any other information you may deem useful, to sketch the graph of this curve. Indicate the direction of the curve with an increasing value of $t$.

(g) State the coordinates of any local maxima or minima which exist.

10 4. (a) Sketch the graph of the curve $r = \sin(3\theta)$, $0 \le \theta \le \pi$.
State the polar coordinates of all pertinent points on this graph.

(b) Find an equation of the tangent line to the above curve at the point where $\theta = \pi/6$.

4 5. Find an equation of the tangent line at the point with coordinates $(1/2, \pi/4)$ to the curve

$$y = f(x) \text{ if } f(x) = \int_0^2 x\frac{1}{1+t^2}\,dt.$$

BONUS [ Maximum value is 3 marks ]

Sketch the graph of the curve $r = 1 + 2\cos(2\theta)$, $0 \le \theta \le 2\pi$.
State the polar coordinates $(r, \theta)$ of all pertinent points on this graph.

## Term Test: March 2007

Values

1. Evaluate the following integrals:

5 (a) $\displaystyle\int \frac{dx}{\sqrt{1-x}}$

6 (b) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)(1+\tan^{-1}x)}$

6 (c) $\displaystyle\int x(1-x)^{20}\,dx$

5 (d) $\displaystyle\int (\cos(\sin\theta))\cos\theta\,d\theta\,dx$

5 2. Find the area of region bounded by the parametric curve with equations $y = t^2 + 1$ and $x = -t^3$, the $x$-axis and the vertical lines $x = 0$ (when $t = 0$) and $x = -1$ (when $t = 1$).

8 3. Find the area between the curves $y = x^3 + x^2$ and $y = x^2 + x$.

3 4. Show that $\sin^{-1}(x) + \sin^{-1}(-x) = 0$.
It might be useful to recall that a function whose derivative is 0 is a constant function.

5. In each case write an integral which represents the requested quantity.
DO NOT EVALUATE THE INTEGRAL.

5 (a) Find the area outside the circle $r = 1$ which is enclosed by one leaf of the polar curve $r = 2\sin 2\theta$.

5 (b) The base of a region is the circle $x^2 + y^2 = 4$. Each cross-section perpendicular to the $x$-axis is a square. Find the volume of the region.

5 (c) Find the volume of the region formed when the area bounded by the curve $y = -(x-2)^2 + 1 = -(x-1)(x-3)$ and the $x$-axis is revolved about the $x$-axis.

5 (d) Find the volume of the region formed when the area bounded by the curve $y = -(x-2)^2 + 1 = -(x-1)(x-3)$ and the $x$-axis is revolved about the $y$-axis.

## Term Test: February 2009

1. (a) Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^3}{3^x}$. (b) Evaluate $\displaystyle\lim_{x\to-\infty} xe^x$.

2. Consider the curve given parametrically as follows: $x = t^3 - 3t$, $y = t^3 - 3t^2$. The intercepts have been found to be as follows.

| | | | | |
|---|---|---|---|---|
| $t$ | $-\sqrt{3}$ | $0$ | $\sqrt{3}$ | $3$ |
| $x$ | $0$ | $0$ | $0$ | $18$ |
| $y$ | $-3\sqrt{3} - 9$ | $0$ | $3\sqrt{3} - 9$ | $0$ |

Note that $-3\sqrt{3} - 9 \approx -14$ and $3\sqrt{3} - 9$ lies between $-3$ and $-4$.
Because the trends are $\lim_{t\to\pm\infty} x = \pm\infty$ and $\lim_{t\to\pm\infty} y = \pm\infty$, the curve begins in the third quadrant and ends in the first quadrant.
Because $\frac{dy}{dt} = 3t^2 - 6t$, horizontal tangents occur when $t = 0$ or 2, that is at $(0, 0)$ and $(2, -4)$, respectively. And because $\frac{dx}{dt} = 3t^2 - 3$, vertical tangents occur when $t = -1$ or 1, that is at $(2, -4)$ and $(-2, -2)$, respectively. This is true because those parameter values don't overlap.

(a) Fill in a direction table with columns

| $t$ | $dx/dt$ | $dy/dt$ | $x$ | $y$ | Curve |
|---|---|---|---|---|---|

and rows

$(-\infty, -1)$

$(-1, 0)$

$(0, 1)$

$(1, 2)$

$(2, \infty)$

(b) Find when the curve is concave up and concave down.

(c) Sketch the curve, showing agreement with given and deduced facts about its location and direction.

3. (a) Identify the curve $(x - 1)^2 + (y - 2)^2 = 5$.
(b)Convert its equation to polar co-ordinates by expressing $r$ as a function of $\theta$.

4. (a) $\displaystyle\int \frac{\sec^2 x - 1}{\sin x}\, dx$. (b) $\displaystyle\int_0^1 \frac{4x^3}{\sqrt{2 + 2x^4}}\, dx$.

# Exam Answers

*In which*

We provide answers to some questions.

These answers are deliberately brief. You should work through the questions on your own, then use the answers to check your work. Answers to theory questions are not given. The topics in the course change from time to time so there may be questions which are not relevant for you. If in doubt, ask. By the time you are doing these old exams for review, you should be becoming confident in your work. As always seek help as early as possible with difficulties. It is (almost) guaranteed that if you can do these old exams you will be able to do the exam you will face.

## Final: April, 1992

1. (a) domain, range, $(-\infty, \infty)$

   (b) $f'(x) > 0$ except at $x = 0$ implies $f$ strictly increasing so $f$ is 1-1

   (c) $(-\infty, \infty)$

   (d) $f(1) = \frac{1}{2}$, $f^{-1}(\frac{1}{2}) = 1$

   (e) $\frac{1}{2}$

2. (a) see text

   (b) $\dfrac{\left(\dfrac{1}{\sqrt{1-x^2}}\right)\cos^{-1} x - \left(\dfrac{-1}{\sqrt{1-x^2}}\right)\sin^{-1} x}{(\cos^{-1} x)^2}$

   (c) Hint: $(\tan^{-1} x + \cot^{-1} x)' = 0$

3. (a) $\infty$

   (b) $0$

   (c) $\dfrac{1}{e}$

4. (a) $\ln(\ln(x)) + C$

   (b) $-\dfrac{\cos^3\theta}{3} + \dfrac{\cos^5\theta}{5} + C$

   (c) $\dfrac{1}{2^{5/2}} \arctan\left(\dfrac{2+x}{\sqrt{2}}\right) + C$

(d) $\ln|x + \sqrt{x^2+1}| + C$

(e) $\dfrac{e^x}{2}(\cos x + \sin x) + C$

(f) $\dfrac{3}{2}\ln|x| - \dfrac{1}{2}\ln|2+x| + C$

5. (a) converges for $p < -1$, diverges for $p \geq -1$

   (b) converges

6. $$\frac{2\sin x^2}{x}$$

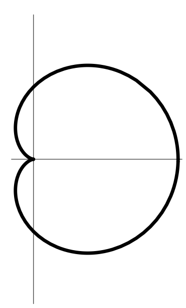
7. (a) $\displaystyle\int_0^1 (x^2 - x^3)\, dx$

   (b) $\displaystyle\int_0^1 \sqrt{1+(2x)^2}\, dx + \int_0^1 \sqrt{1+(3x^2)^2}\, dx$

   (c) $\displaystyle\pi\int_0^1 (x^4 - x^6)\, dx = 2\pi\int_0^1 y(y^{1/3} - y^{1/2})\, dy$

   (d) $\displaystyle 2\pi\int_0^1 (x^2\sqrt{1+(2x)^2} + x^3\sqrt{1+(3x^2)^2})\, dx$

8. (a)



   (b) $\displaystyle\int_0^{2\pi} \sqrt{(-\sin\theta)^2) + (1+\cos\theta)^2}\, d\theta$

   (c) $\displaystyle\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta$

9. (a) \$32,000

   (b) \$500

## Final: December, 1992

1. $\sqrt{\dfrac{7-x^2}{6-x^2}}\left(-\dfrac{1}{2}(7-x^2)^{-3/2}(-2x)\right)$

2. $\dfrac{1}{\sqrt{3}}\left(\dfrac{\pi}{4}-\dfrac{\pi}{6}\right)$

3. $\ln(\ln x)+C$

4. $\dfrac{x}{2}+\dfrac{\sin 6x}{12}+C$

5. $\dfrac{\sec^5 x}{5}-\dfrac{\sec^3 x}{3}+C$

6. $\dfrac{9}{13}(e^{2x})\left(\dfrac{-\cos 3x}{3}+\dfrac{2\sin 3x}{9}\right)+C$

7. $-\dfrac{\sqrt{9-x^2}}{x}-\sin^{-1}\left(\dfrac{x}{3}\right)+C$

8. $\dfrac{9}{10}\ln|x-3|+\dfrac{1}{20}\ln(x^2+1)+\dfrac{3}{10}\tan^{-1}x+C$

9. $\dfrac{1}{3}\tan^{-1}\left(\dfrac{x+1}{3}\right)+C$

10. $-\dfrac{1}{25}\ln|x|-\left(\dfrac{1}{5x}\right)+\dfrac{1}{25}\ln|x+5|+C$

11. $-\dfrac{1}{5}\ln|x+2|+\dfrac{1}{5}\ln|x-3|+C$

12. 2

13. 1

14. $\displaystyle\int_1^3 (4x-3-x^2)\,dx$

15. (a) $\displaystyle\pi\int_0^1 y^4\,dy$

    (b) $\displaystyle 2\pi\int_0^1 (x-x^{3/2})\,dx$

16. 1

17. $\displaystyle\int_0^3 (x^2+1)\,dx$

## Final: April, 1993

1. (a) $e^{\sqrt{x}}(-2+2\sqrt{x})+C$

   (b) $\sqrt{1-x^2}+x\sin^{-1}x+C$

   (c) $\sin x-\dfrac{2}{3}\sin^3 x+\dfrac{1}{5}\sin^5 x+C$

   (d) $\arctan x+\ln|x|-\dfrac{\ln(1+x^2)}{2}+C$

2. (a) 1

   (b) 1

3. (a) 1

   (b) diverges

4. (a) $\displaystyle\int_{-2}^{4}(y+1)-\left(\frac{y^2-6}{2}\right)\,dy=\int_{-3}^{-1}2\sqrt{2x+6}\,dx+\int_{-1}^{5}\sqrt{2x+6}-(x-1)\,dx$

   (b) (i) $\displaystyle\int_0^1 \pi(x^2-x^4)\,dx=\int_0^1 2\pi y(\sqrt{y}-y)\,dy$

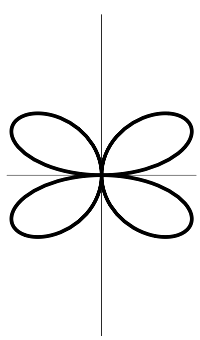
   (ii) $\displaystyle\int_0^1 \pi(y-y^2)\,dy=\int_0^1 2\pi x(x-x^2)\,dx$

   (c) $\displaystyle 2\pi\int_1^2(2+x)\sqrt{1+x^4}\,dx$

5. (a) $\displaystyle\int_1^2\sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$

   (b) $\dfrac{17}{12}$

6. (a)



   (b) $\displaystyle\int_0^{2\pi}\sqrt{4\cos^2 2\theta+\sin^2 2\theta}\,d\theta$

7. (a) $1 + \frac{5}{2}(x+1) + \frac{15}{8}(x+1)^2$

(b) $\frac{15}{8 \cdot 3!}(c+2)^{-1/2}(x+1)^3$ for $c$ between $-1$ and $x$

(c) 0.575

(d) $|R_2(x)| \le \frac{15}{6} \cdot \frac{1}{2^3}(0.2)^3 \frac{1}{(0.8)^{1/2}}$

## Final: December, 1993

1. $\left(1 + \left(\frac{\cos x}{1+\sin x}\right)^2\right)^{-1} \cdot \frac{(1+\sin x)(-\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2}$

2. $\frac{3}{100}(1+5x^2)^{10/3} + C$

3. $\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

4. $\frac{(9-x^2)^{5/2}}{5} - 3(9-x^2)^{3/2} + C$

5. $-\frac{16}{\sqrt{3}}\arctan\left(\frac{3+x}{\sqrt{3}}\right) + \frac{7}{2}\ln(12+6x+x^2) + C$

6. $\frac{e^{\pi/2}}{2} - \frac{1}{2}$

7. $-\frac{1}{(x-1)^2} + \ln|x-1| - \ln|x+1| + C$

8. $\frac{7}{3}\arctan\left(\frac{3}{x}\right) + \frac{2}{3}\ln|x| + \frac{7}{6}\ln(9+x^2) + C$

9. 0

10. 1

11. $\frac{\pi}{4}$

12. diverges

13. $\int_0^{1/2} (x^2 - 2x^3)\, dx$

14. (a) $\pi \int_1^2 (3-x)^2 - \left(\frac{2}{x}\right)^2 \, dx$

    (b) $2\pi \int_1^2 y\left(3 - y - \frac{2}{y}\right) \, dy$

15. $\int_{\pi/6}^{\pi/3} \sqrt{1+\cot^2 x} \, dx$

16. $2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\left(\frac{-2x}{2\sqrt{4-x^2}}\right)^2} \, dx$

## Final: April, 1994

1. (a) $\frac{1}{\sqrt{1-x}}$

   (b) $\frac{x^2}{\sqrt{1+x^4}}$

   (c) $\sin^{-1} x^2 + \frac{2x^2}{\sqrt{1-x^4}}$

2. (a) $\frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$

   (b) $\ln|\sqrt{(x-3)^2+1} + x - 3| + C$

   (c) $\frac{3\pi}{16}$

   (d) $\frac{\ln|x-5|}{2} - \frac{\ln|x-3|}{2}$

3. (a) 1

   (b) 0

4. (a) 1

   (b) 1

5. (a) $\left|\int_{-1}^{3} (x^2 - 2x - 3) \, dx\right|$

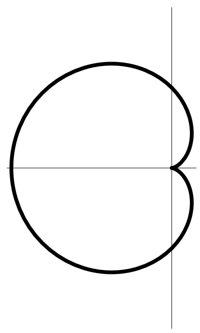
   (b) (i) $\pi \int_0^3 (5x)^2 - x^4 \, dx$

(ii) $2\pi \int_0^3 x(5x - x^2)\, dx$

(c) $2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\, dx$

6. (a) $\int_0^{4/3} \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\, dx$
   (b) $56/27$

7. (a)



(b) $\int_0^{2\pi} \sqrt{\sin^2\theta + (1 - \cos\theta)^2}\, d\theta$

## Final: December, 1994

1. $3(\cos^{-1}(x^2 - 3x))^2 \cdot \dfrac{-1}{\sqrt{1 - (x^2 - 3x)^2}} \cdot (2x - 3)$

2. $\dfrac{2\pi}{3}$

3. $-\dfrac{1}{3} \dfrac{1}{e^{3x} - 1} + C$

4. $\dfrac{1}{4}\left(2x - \dfrac{1}{4}\sin 8x\right) + C$

5. $-\dfrac{1}{2}\left(\dfrac{\csc^5 2x}{5} - \dfrac{\csc^3 2x}{3}\right) + C$

6. $\dfrac{4}{17} e^x \left(\sin 4x + \dfrac{\cos 4x}{4}\right) + C$

7. $\sqrt{x^2 - 5} - \sqrt{5} \tan^{-1} \dfrac{\sqrt{x^2 - 5}}{\sqrt{5}} + C$

8. $\ln|x| - \dfrac{1}{2}\ln(x^2+1) + C$

9. $\sqrt{x^2+2x+5} - \ln|x+1+\sqrt{x^2+2x+5}| + C$

10. $\dfrac{-1}{4}\ln|x| + \dfrac{1}{4}\ln|x-2| - \dfrac{1}{2}(x-2)^{-1} + C$

11. 1

12. 1

13. $1/e$

14. diverges

15. $\displaystyle\int_0^9 (3\sqrt{x} - x)\,dx = \int_0^9 (y - \frac{y^2}{9})\,dy$

16. $\displaystyle\int_1^2 \sqrt{1 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2}\,dx$

17. (a) $\displaystyle\pi\int_0^9 (9x - x^2)\,dx$

    (b) $\displaystyle 2\pi\int_0^9 y\left(y - \frac{y^2}{9}\right)\,dy$

## Final: April, 1995

1. 1

2. (ii) $\dfrac{\pi}{2}$

3. 1

4. 4

5. a/b

6. $22\frac{1}{2}$

7. $\dfrac{17}{10}\pi$

8. 9/2

9. (a) $\dfrac{2}{27}(10^{3/2} - 1)$

10. $2\pi \int_1^2 (x-1)(x-2)\sqrt{1+(2x-3)^2}\, dx$

11. $\tan^{-1} e^x + C$

$\sqrt{x^2-4} - 2\tan^{-1}\left(\frac{\sqrt{x^2-4}}{2}\right) + C$

$\frac{1}{10}(e^{3x}\sin x + 3e^{3x}\cos x) + C$

$-\ln\frac{1}{\sqrt{2}} - \frac{1}{4} + C$

12. (i) $\pi/4$

(ii) converges

## Final: December, 1995

1. (a) $-\frac{\pi}{4}$

(b) $\frac{1}{\sqrt{1-\left(\frac{\cos x}{1+\sin x}\right)^2}} \cdot \left(\frac{(1+\sin x)(-\sin x) - \cos^2 x}{(1+\sin x)^2}\right)$

(c)

2. (a) $\sin x - x\cos x + C$

(b) $\frac{x^2}{2} + x + 2\ln|x-1| + C$

(c) $\ln|x| - \frac{1}{2}\ln(x^2+1) + C$

(d) $-\cot x - x + C$

(e) $\frac{1}{3}\ln\left|\frac{3x+2}{3} + \frac{\sqrt{9x^2+12x-5}}{3}\right| + C$

3. (a) 1

(b) $\infty$

4. $-1$

5. (a) $\int_0^1 (\sqrt{x} - x^2)\, dx$

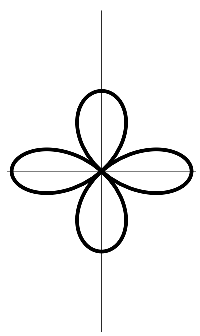
(b) $\pi\int_0^1 (x - x^4)\, dx = 2\pi\int_0^1 y(\sqrt{y} - y^2)\, dy$

(c) $\pi \int_0^1 (y - y^4)\, dy = 2\pi \int_0^1 x(\sqrt{x} - x^2)\, dx$

(d) $\int_0^1 \sqrt{1+(2x)^2} + \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$

(e) $2\pi \int_0^1 x^2\sqrt{1+(2x)^2} + \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$

6. (a)



(b) $\frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$

(c) $\int_{-\pi/4}^{\pi/4} \sqrt{(-2\sin 2\theta)^2 + \cos^2 2\theta}\, d\theta$

7. (a) $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

(b) $1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6}$

(c) $\frac{3(0.1)^4}{4!}$

## Final: April, 1996

1. (a) $-3x^2\sin(x^6)$

   (b) $\frac{-3}{16}$

   (c) $\frac{-1}{4\pi}$

2. (a) $x\sin^{-1}(x) + \sqrt{1-x^2} + C$

(b) $-\frac{1}{4}\ln|1-x|+\frac{1}{4}\ln|1+x|+\frac{1}{2}\tan^{-1}(x)+C$

(c) $\frac{\pi}{4}$

3. (a) 1

(b) $\frac{1}{2}$

(c) 1

4. (a) 1

(c) $\frac{1}{4}$

5.A. (a) $4\int_0^{1/2}\sqrt{\frac{1+12x^2}{1-4x^2}}\,dx$

(b) $4\pi\int_0^{1/2}\sqrt{1+12x^2}\,dx$

(c) $\frac{2\pi}{3}$

5.B. (a) $4\pi\int_0^{1/2}\sqrt{1+12x^2}\,dx$

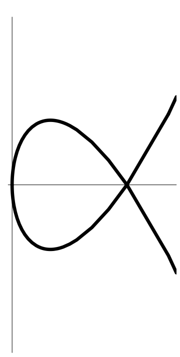
(b) 4*Pi*Integrate[Sqrt[1+12*x^2],{x,0,1/2}] (or NIntegrate)

(d) FindRoot[Cos[x]==E^x-1,{x,Pi/2}]

6. (b) $1, -1$

(c) $(0,0)$

(d)



(e) $\int_{-1}^{1}(4t^2+(3t^2-1)^2)^{1/2}\,dt$

## Final: December, 1996

1. (a) $-\frac{2}{3}\left(\frac{1}{x-1}\right)+\frac{4}{9}\ln|x-1|+\frac{5}{9}\ln|x+2|+C$

   (b) $\ln|(x+2)^2+1|-3\tan^{-1}(x+2)+C$

   (c) $\frac{8}{15}$

   (d) $\frac{2}{5}e^{2x}\sin x-\frac{1}{5}e^{2x}\cos x+C$

2. (a) $\frac{1}{2}$

   (b) 0

3. (a) $-\infty$, diverges

   (b) converges by comparison

4. (a) $2\int_{-2}^{2}\frac{1}{2}\sqrt{4-x^2}\,dx$

   (b) $2\int_{-2}^{2}\sqrt{1+\frac{x^2}{4(4-x^2)}}\,dx$

   (c) $\int_{-2}^{2}\pi\left(\frac{1}{2}\sqrt{4-x^2}\right)^2\,dx$

   (d) $2\pi\int_{0}^{2}x\sqrt{4-x^2}\,dx$

   (e) $\int_{-2}^{2}2\pi\left(\frac{1}{2}\sqrt{4-x^2}\right)\sqrt{1+\frac{x^2}{4(4-x^2)}}\,dx$

5. (a) $y-\left(4+\frac{3\sqrt{2}}{2}\right)=\frac{3}{4}\left(x-\left(3-\frac{4\sqrt{2}}{2}\right)\right)$

   (b) $(3,7)$, $(3,1)$

   (c) $(-1,4)$, $(7,4)$

   (d) $\int_{0}^{2\pi}\sqrt{(4\cos t)^2+(3\sin t)^2}\,dt$

## Final: April, 1997

1. (a) $2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx = \ldots = \pi r^2$
2. $e^{-1}$
3. (a) $\dfrac{2(1+\ln(x))^{3/2}}{3}+C$

   (b) $\dfrac{1}{2}\ln|1+x| - \dfrac{1}{4}\ln(1+x^2)+\dfrac{1}{2}\tan^{-1}(x)+C$

   (c) $\dfrac{x^2\ln(x)}{2}-\dfrac{x^2}{4}+C$
4. (a) 0

   (b) 0

   (c) $e^2$
5. (a) $\dfrac{1}{e}$

   (b) does not converge

   (c) converges by comparison

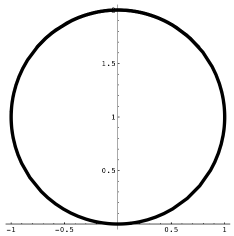
   (d) diverges
6. (a) $\displaystyle\int_0^2 \sqrt{1+\left(\frac{x}{\sqrt{x^2+4}}\right)^2}\,dx$

   (b) $\displaystyle\int_0^2 2\pi(x^2+4)^{1/2}\sqrt{1+\left(\frac{x}{\sqrt{x^2+4}}\right)^2}\,dx$

   (c) $\dfrac{32\pi}{3}$
7. (a) $\displaystyle\int_0^5 \sqrt{(2t)^2+(3t^2)^2}\,dt$

   (b) $\displaystyle\int_0^5 2\pi t^3\sqrt{(2t)^2+(3t^2)^2}\,dt$

8. (a)

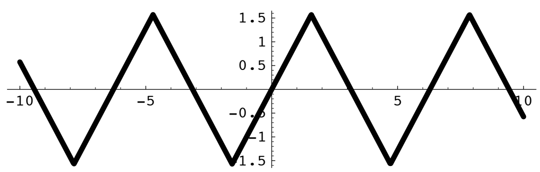



(b) $\dfrac{\sin 2\theta}{\cos 2\theta}$

(c) horizontal $(1, \dfrac{\pi}{2})$; vertical $(\sqrt{2}, \dfrac{\pi}{4})$

## Final: December, 1997

1.




2. (a) $\dfrac{1}{6\cos^6 x} - \dfrac{1}{4\cos^4 x} + C$

(b) $\tan^{-1}(\sin x) + C$

(c) $\sin^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \left(\dfrac{x}{\sqrt{2}}\right) \cdot \dfrac{\sqrt{2-x^2}}{\sqrt{2}} + C$

(d) $x + \dfrac{1}{3}(\ln|x| - 5\ln|x+3| + 4\ln|x-3|) + C$

(e) 1

3. (a) $\dfrac{a^2}{b^2}$

(b) 1

4. converges by comparison

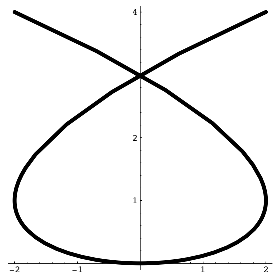
5. (a) $\displaystyle\int_{-1}^{1} [(4-y^2) - (y^2 - 2y)]\, dy$

(b) $\displaystyle\int_{10}^{30} \pi(900 - x^2)\, dx$

(c) $4\int_0^a \sqrt{1+\left(\frac{-bx}{a\sqrt{a^2-x^2}}\right)^2}\,dx$

(d) $\int_0^1 2\pi\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$

6. (i)



(ii) $2\int_0^{\sqrt{3}} t^2(3t^2-3)\,dt$

(iii) $2\int_0^{\sqrt{3}} \sqrt{(3t^2-3)^2+(2t)^2}\,dt$

## Final: April, 1998

1. (a) $(1,4)$

   (b) $\frac{8}{3}$

   (c) $\pi\int_0^1 (5-x^2)^2-(4x)^2\,dx$

2. $$\frac{2x}{1+(x^2)^2}$$

3. (a) $\frac{2}{5}+\frac{2}{3}$

   (b) $\frac{1}{2}$

   (c) $-x\cos x+\sin x+C$

   (d) $x\tan^{-1}x-\frac{1}{2}\ln|1+x^2|+C$

   (e) $\frac{\sqrt{x^2-4}}{4x}$

   (f) $2\ln|x+3|+\frac{1}{2}\ln|x^2+3|+C$

4. (a) $-1$

   (b) $1$

5. (a) $1$

   (b) $\infty$

6. (a) $\displaystyle\int_0^{\pi/2} \sqrt{1 + (-\sin x)^2}\, dx$

   (b) $\displaystyle 2\pi \int_0^{\pi/2} \cos x \sqrt{1 + (-\sin x)^2}\, dx$

7. (a) $-\dfrac{2}{9}$

   (b) $(\pm 16, 5)$

   (c) $\displaystyle\int_0^1 \sqrt{(3t^2 - 12)^2 + (2t)^2}\, dt$

8. $\displaystyle\frac{1}{2}\int_{\pi/4}^{\pi/2} (1 + \cos\theta)^2\, d\theta$

9. (A) $\dfrac{11}{2}$

9. (B) $6$

## Final: December, 1998

1. (a) $\sqrt{1 - x}$

   (b) $\dfrac{\pi}{4}$

2. (a) $-\dfrac{1}{4} - \ln\left(\dfrac{1}{\sqrt{2}}\right)$

   (b) $\ln x - \dfrac{1}{2}\ln(1 + x^2) + C$

   (c) $\dfrac{1}{3}\ln(3x + 2 + \sqrt{9x^2 + 12x - 5}) + C$

   (d) $\sin x - x\cos x + C$

(e) $\sqrt{x^2-4} + 2\arctan\dfrac{2}{\sqrt{x^2-4}} + C$

3. (a) 1

(b) 0

4. Converges

5. (a) $\displaystyle\int_0^2 \sqrt{1+(2x)^2}\,dx$

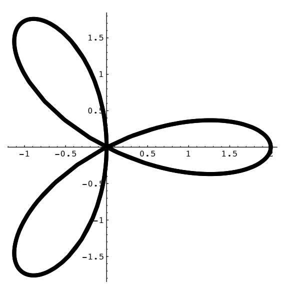
(b) $\displaystyle 2\pi\int_0^2 x^2\sqrt{1+(2x)^2}\,dx$

(c) $\displaystyle \pi\int_0^2 (x^2)^2\,dx$

(d) $\displaystyle 2\pi\int_0^2 (x^2+1)\sqrt{1+(2x)^2}\,dx$

(e) $\displaystyle \pi\int_0^2 (x^2+1)^2\,dx$

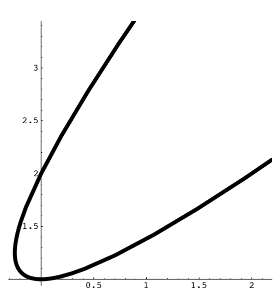
6. (a)



(b) $\displaystyle\int_0^{\pi/6} (2\cos 3\theta)^2\,d\theta$

(c) $\displaystyle 2\int_0^{\pi/6} \sqrt{(-6\sin 3\theta)^2 + (2\cos 3\theta)^2}\,d\theta$

7. (a)



(b) $\int_{-2}^{1} \sqrt{(2t+1)^2 + (2t)^2}$

(c) $\int_{-2}^{1} (t^2+1)(2t+1)\,dt$

## Final: April, 1999

1. (a) $[\sin^{-1}(\sin x)\cos^{-1}(\sin x)]\cos x$

   (b) $x(\frac{\pi}{2} - x)\cos x$

2. (a) $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

   (b) $-\frac{1}{4}\sin x\cos^3 x + \frac{1}{8}\cos x\sin x + \frac{x}{8} + C$

   (c) $-\frac{\sqrt{x^2+9}}{9x} + C$

   (d) $-\frac{2}{3}\cos^3 x + C$

3. (a) $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+x+1} + \frac{Fx+G}{(x^2+x+1)^2}$

   (b) $\ln(x+1) + \frac{1}{2}\ln(x^2+1) + C$

4. (a) $2\pi \int_0^3 x(e^x + x)\,dx$

   (b) $\frac{1}{3}\int_0^3 (e^x + x)\,dx$

5. (a) 4

   (b) 3

6. (a) $\frac{1}{4}$

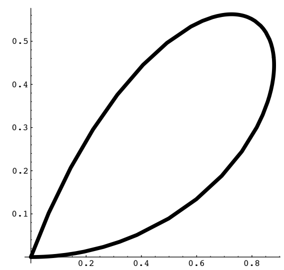
   (b) 3

7. (a) $-2\left(\dfrac{3}{4}\right)^{1/2}$

   (b) $\displaystyle\int_0^1 \sqrt{(-3t(1-t^2)^{1/2})^2 + (3t^2)^2}\, dt = \frac{3}{2}$

   (c) $\displaystyle 2\pi \int_0^1 (1-t^2)^{3/2} \sqrt{9t^2(1-t^2) + 9t^4}\, dt = \frac{6\pi}{5}$

8. (a)




   (b) $\dfrac{\pi}{12}$

   (c) $\displaystyle\int_0^{\pi/3} \sqrt{9\cos^2(3\theta) + (\sin^2 3\theta)}\, d\theta$

   (d) $\dfrac{y - (1/2)}{x - (\sqrt{3}/2)} = -\sqrt{3}$

## Final: December, 1999

1. (a)

   (b) $\displaystyle V = \pi \int_0^2 [x(2-x)]^2 \; dx = \frac{4\pi}{3}$

   (c) $\displaystyle V = 2\pi \int_0^2 x[x(2-x)] \; dx = \frac{8\pi}{3}$

2. $\displaystyle A = \int_{-2}^0 (e^{-x} - e^x) \; dx + \int_0^1 (e^x - e^{-x}) \; dx = e^2 + e^{-2} - 2 + e + e^{-1} - 2$

3. $\displaystyle \text{Average} = \frac{1}{100} \int_0^{100} \left(\frac{x}{2} + \sqrt{x} - 25\right) \; dx = \frac{20}{3} \;{}^\circ\text{C}$

4. (a) $\displaystyle \left(\frac{x^2}{2} + x\right)\ln x - \left(\frac{x^2}{4} + x\right) + C$

(b) $-\dfrac{\cos^7 x}{7} + \dfrac{\cos^9 x}{9} + C$

(c) $\ln\left|\dfrac{x-2}{x-1}\right| + C$

5. $$-\frac{\sqrt{4-x^2}}{x} - \arcsin\left(\frac{x}{2}\right) + C$$

6. (a) $\ln 2$

   (b) 1

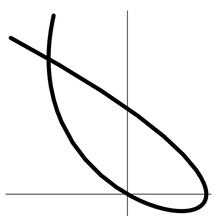
7. (a) $\ln 2$

   (b) Convergent, compare with $\dfrac{1}{\sqrt[5]{x}}$.

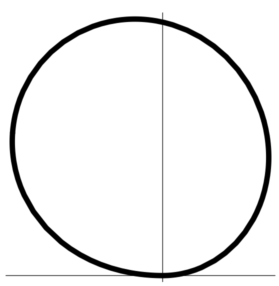
8. $$\int_0^1 \sqrt{1+\frac{9}{4}x}\,dx = \frac{13}{27}\sqrt{13} - \frac{8}{27}$$

9. (a) horizontal tangent when $t = 1/2$, vertical tangents when $t = \pm 1$

   (b)



10 (a)



   (b) $A = \dfrac{4}{105}\pi$

## Final: April, 2000

1. (a) $\ln\left(1 + \dfrac{\pi}{4}\right)$

(b) $\dfrac{e^x}{5}[\sin(2x) - 2\cos(2x)] + C$

(c) $-\dfrac{1}{12}\dfrac{(4+x^2)^{3/2}}{x^3} + C$

(d) $\dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{3}\left(\dfrac{\sqrt{3}}{3}\right)^3 - \dfrac{1}{5}\left(\dfrac{\sqrt{3}}{3}\right)^5$

2. (a) $\dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{Cx+D}{(x^2+2)^2} + \dfrac{Ex+F}{x^2+2}$

(b) $\ln|x-1| + \ln(x^2+x+2) + C$

3. (a) $\pi \displaystyle\int_{\pi/4}^{\pi/2} (x^2 - \sin^2 x)\, dx$

(b) $2\pi \displaystyle\int_{\pi/4}^{\pi/2} x(x - \sin x)\, dx$

4. $\dfrac{64}{3}$

5. (a) $\displaystyle\int_0^1 \sqrt{1 + \left(\frac{1}{\sqrt{1-x^2}} + 1\right)^2}\, dx$

(b) $2\pi \displaystyle\int_0^1 (\sin^{-1} x + x)\sqrt{1 + \left(\frac{1}{\sqrt{1-x^2}} + 1\right)^2}\, dx$

6. (a) $-1$

(b) $e$

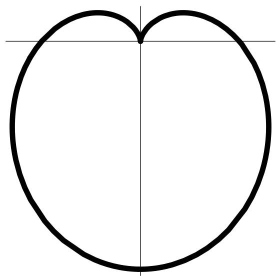
7. (a) $\dfrac{1}{4}$

(b) covergent, compare with $\dfrac{1}{x^2}$

8. (a) $y - \dfrac{1}{\sqrt{2}} = \dfrac{-1}{\sqrt{2}+1}\left(x - \dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}\right)$

(b) $\displaystyle\int_0^{\pi/4} \sqrt{(1+\cos t)^2 + (-\sin t)^2}\, dt$

9. (a)



(b) $\displaystyle\int_0^{2\pi} \frac{1}{2}(1-\sin\theta)^2\, d\theta$

(c) $\displaystyle\int_0^{2\pi} \sqrt{(1-\sin\theta)^2 + (-\cos\theta)^2}\, d\theta$

## Final: April, 2004 (Solutions)

1. (a) $\displaystyle = \int \left(\sec^2 x - 1\right) dx = \tan x - x + C$

(b) $\displaystyle = \int \tan x \tan^2 x\; dx = \int \tan x \left(\sec^2 x - 1\right)\; dx = \int \tan x \sec^2 x\; dx - \int \tan x\; dx = \frac{\tan^2 x}{2} + \ln|\cos x| + C$

OR $\displaystyle = \frac{\tan^2 x}{2} - \ln|\sec x| + C$

(c) Let $u = x^2$ then $du = 2xdx$

$$= \frac{1}{2}\int \left(e^{x2} + \sin x^2\right)(2xdx) = \frac{1}{2}\int \left(e^u + \sin u\right) du = \frac{1}{2}\, e^u - \frac{\cos u}{2} + C = \frac{1}{2}\, e^{x2} - \frac{\cos x^2}{2} + C$$

OR

$$= \frac{1}{2}\int \left(e^{x2} + \sin x^2\right)(2x\; dx) = \frac{1}{2}e^{x2} - \frac{1}{2}\cos\left(x^2\right) + C$$

(d) $\displaystyle = \int \left(-1 + \frac{1}{1-x^2}\right)\; dx = \int \left(-1 + \frac{A}{1-x} + \frac{B}{1+x}\right)\; dx$

$1 = A\,(1+x) + B\,(1-x)$
Let $x = 1$ $\quad 1 = 2A \quad \frac{1}{2} = A$
Let $x = -1$ $\quad 1 = 2B \quad \frac{1}{2} = B$

$$= \int \left(-1 + \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right)\; dx = -x - \frac{1}{2}\ln\;|1-x| + \frac{1}{2}\;|1+x| + C$$

OR

Let $x = \sin\theta$ then $dx = \cos\theta d\theta$

$$\int \frac{x^2}{1-x^2}\; dx = \int \frac{\sin^2\theta\;\cos\theta\; d\theta}{1-\sin^2\theta} = \int \frac{\sin^2\theta\;\cos\theta\; d\theta}{\cos^2\theta} = \int \frac{1-\cos^2\theta\; d\theta}{\cos\theta} = \int \sec\theta\; d\theta - \int \cos\theta\; d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C = \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| - x + C$$

(e) $= x\sin^{-1}x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\frac{\pi}{6}\right) - 0 + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x\,dx}{\sqrt{1-x^2}} = \frac{\pi}{12} + \frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^{\frac{1}{2}}$

$= \frac{\pi}{12} + \sqrt{1-x^2}\,\Big|_0^{\frac{1}{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1$

OR

$= x\sin^{-1}x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\frac{\pi}{6}\right) - 0 + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x\,dx}{\sqrt{1-x^2}}$

Let $u = 1-x^2 \qquad du = -2xdx$

$= \frac{\pi}{12} - \int_1^{\frac{3}{4}} \frac{du}{-2u^{\frac{1}{2}}} = \frac{\pi}{12} + \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_1^{\frac{3}{4}} = \frac{\pi}{12} + \sqrt{u}\,\Big|_1^{\frac{3}{4}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

(f) $= \int \frac{dx}{\sqrt{3-(x^2-2x+1-1)}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$

Let $(x-1) = 2\sin\theta \qquad dx = 2\cos\theta d\theta$

$= \int \frac{2\cos\theta\;d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\;d\theta}{2\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + C = \sin^{-}1\frac{x-1}{2} + C$

(g) Let $x = \sec\theta \qquad dx = \sec\theta\;\tan\theta\;d\theta$

$= \int \frac{\sec\theta\;\tan\theta\;d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\;\tan\theta\;d\theta}{\sec^3\theta\tan\theta} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2}$

$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + C = \frac{1}{2}\theta + \frac{2\sin\theta\cos\theta}{4} + C = \frac{1}{2}\sec^{-1}\theta + \frac{1}{2}\cdot\frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{x} + C$

2. $\frac{2x+1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$

3. a) $\lim_{x\to 0} \frac{\cos x - 1}{e^x - x - 1} \quad \left(\frac{0}{0}\right) \quad =[\text{H}] \quad \lim_{x\to 0} \frac{-\sin x}{e^x-1} \quad \left(\frac{0}{0}\right) \quad =[\text{H}] \quad \lim_{x\to 0}\frac{\cos x}{e^x} = \frac{1}{1} = 1$

3. b) Let $y = (e^x+x)^{\frac{5}{x}} \qquad (1^\infty) \qquad \ln y = \frac{5}{x}\ln(e^x+x)$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{5\ln(e^x+x)}{x} \quad \left(\frac{0}{0}\right) \quad =[\text{H}] \quad \lim_{x\to 0^+} \frac{5\cdot\frac{1}{e^x+x}(e^x+1)}{1} = \frac{5\left(\frac{1}{1+0}\right)(1+1)}{1} = 10$

Therefore $\lim_{x\to 0^+}(e^x+x)^{\frac{5}{x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{10}$

4. $\int_1^\infty \frac{e^{-x}}{x+3}\,dx = \int_1^\infty \frac{1}{(x+3)e^x}\,dx$

For $x \geq 1 \qquad 0 < \frac{1}{(x+3)e^x} < \frac{1}{e^x}$

But $\int_1^\infty \frac{1}{e^x}\,dx$ converges. Therefore by comparison test $\int_1^\infty \frac{dx}{(x+3)e^x}$ converges.

5. $V = \int \pi\left(r_2^2 - r_1^2\right)\,dx$ $V = \int_1^2 \pi\left[\left(\frac{6}{x}+3\right)^2 - 3^2\right]\,dx$

6. $\frac{dr}{d\theta} = 15\cos 3\theta$ Arc Length$= \int_0^{\frac{\pi}{3}} \sqrt{(5\sin 3\theta)^2 + (15\cos 3\theta)^2}\,d\theta$

7. $\frac{dy}{dx} = 3x^2$ Surface Area $= \int 2\pi y\,ds$

$= \int_0^2 2\pi\left(x^3+1\right)\sqrt{1+(3x^2)^2}\,dx$ OR $\int_1^9 2\pi y\sqrt{1+\left[\frac{1}{3}(y-1)^{-\frac{2}{3}}\right]^2}\,dy$

8. $0 = -(x-2)^2+9$; $(x-2)^2 = 9$; $x-2 = \pm 3$; $x = 5,\ -1$

radius$=\frac{y}{2}$

$A(y) = \frac{\pi r^2}{2} = \frac{\pi\left(\frac{y}{2}\right)^2}{2} = \frac{\pi y^2}{8}$

But $y = -(x-2)^2+9$ therefore $y^2 = \left[-(x-2)^2+9\right]^2$ therefore $A(x) = \frac{\pi\left[-(x-2)^2+9\right]^2}{8}$

$V = \int_{-1}^5 \pi 8\left[-x(x-2)^2+9\right]^2\,dx$

9. $0 = t(t^2-4)$; $0 = t(t-2)(t+2)$; $t = 0,\ 2,\ -2$

OR

$4 = t^2$; $\pm 2 = t$;

since counterclockwise $A = \int y dx$

$A = -\int_{-2}^2 (t^3-4t)(2t)\,dt$

## Final: April, 2005

1. 0
2. $(\pm 6\sqrt{3}+9, 0)$
3. 3
4. 1/4
5. 64/3
6. $k = 5/2$, $V = \frac{625}{32}\pi$
7. 1/3

8. (a) $\dfrac{\sec^4(2x)}{8} + C$

   (b) $-27\left(\dfrac{\sqrt{9-x^2}}{3}\right) + 9\left(\dfrac{\sqrt{9-x^2}}{3}\right)^3 + C$

   (c) $\ln|x| + \dfrac{1}{2}\ln|x^2+2| - \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{x}{\sqrt{2}}\right) + C$

9. (a) $\ln 2$

   (b) $2\sqrt{e-1}$

10. $4/5$

11. $\dfrac{\pi}{9}(2\sqrt{2}-1)$

12. 19

## Final: December, 2007

1. (a) $-1$ (b) $e^3$

2. (a) $5 - 3/4$ (b) $\frac{1}{2}\int_{2\pi}^{4\pi} \theta^2\, d\theta - \frac{1}{2}\int_{0}^{2\pi} \theta^2\, d\theta$

(c) $y - \frac{\pi}{2} = -\frac{2}{\pi}(x - 0)$

3. (b) $\frac{3\pi^2}{8}$ (c) $\int_0^{\pi} 2\pi x \sin^2 x\, dx$

4. (a) (i)$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$ (ii) $\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$

(b) $\ln|x-3| - \ln|x-2| + C$

5. (a) $\frac{2}{3}2^{2/3} - 2\cdot 2^{1/2} - (\frac{2}{3} - 2)$ (b) $x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

(c) $-\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$ (d) $\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

6. (a) $\int_0^1 xe^{2x}\pi\, dx$ (b) $\pi\left(\frac{e^2}{4} + \frac{1}{4}\right)$

7. (a) $-\frac{1}{\ln\frac{1}{2}}$ (b) converges by comparison with $\int_1^{\infty} \frac{dx}{\sqrt{x^3}} = \int_1^{\infty}\frac{dx}{x^{3/2}}$

8. (a) $\int_{-1}^{2} \sqrt{1 + (-2x+1)^2}\, dx$ (b) $\int_0^{\pi/2} \sqrt{(-\sin t)^2 + (1+\cos t)^2}\, dt$

9. $\int_0^3 2\pi\frac{1}{3}(3-x)\sqrt{x}\sqrt{1 + \left(\frac{-1}{3}\sqrt{x} + \frac{1}{3}\frac{3-x}{2\sqrt{x}}\right)^2}\, dx$

## Final: April, 2008

1. (a) $\sin x - \frac{\sin^3 x}{3} + C$ (b) $\frac{\pi}{2}$

(c) $\frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln|x^2+1| + C$ (d) $\frac{\sqrt{x^2-1}}{x} + C$

(e) $\frac{1}{2}\left[4\sqrt[3]{2}\left(\frac{3}{7}\right) - 2\sqrt[3]{2}\left(\frac{3}{4}\right)\right]$ (f) $\sin^{-1}\left(\frac{x-1}{\sqrt{6}}\right) + C$

(g) $2e^2$ (h) $\frac{1}{2}\tan^{-1}(x^2) + C$

2. $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4}$ 3. $\frac{5}{\sqrt{29}}$

4. diverges by comparison with $\int_0^1 \frac{1}{x^2}\, dx$ 5. $2\int_0^{\pi/10} \sqrt{(-10\sin 5\theta)^2 + (2\cos 5\theta)^2}$

6. $\int_0^1 2\pi \sin^{-1}(x^2)\sqrt{1 + \left(\frac{2x}{\sqrt{1-x^4}}\right)^2}\, dx$ 7. $\int_{-2}^{0} [\ln(y+3)]^2\, dy$

8. 26
9. concave upward on $(-\infty, 0)$ and $(1/3, \infty)$, downward on $(0, 1/3)$

## Final: December, 2008

1. (a) When $t = 1$ the tangent line is $x = \dfrac{1}{6}$ (b) $-1$

2.(b) $-\dfrac{1}{3} + 2 - \dfrac{1}{2} - \left(\dfrac{8}{3} - 4 - 2\right)$ (c) $\displaystyle\int_{-2}^{1} 2\pi(y+2)(2-y^2-y)\,dy$

3. (a) circles with radius 2 and centres $(0,0)$ and $(2,0)$.

(b) $2\displaystyle\int_0^{\pi/3} \frac{1}{2}2^2\,d\theta + 2\int_{\pi/3}^{\pi/2} \frac{1}{2}(4\cos\theta)^2\,d\theta$

4. (a) $\dfrac{A}{x-2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2} + \dfrac{Dx+E}{(x^2+3)} + \dfrac{Fx+G}{(x^2+3)^2}$

(b) $-2\ln|x-2| + 3\ln|x-1| + C$

5. (a) $\sin^{-1}(x+1) + C$ (b) $\dfrac{x^2}{2}\ln(x+1) - \dfrac{1}{4}(x+1)^2 + x + 1 - \ln(x+1) + C$

(c) $\sin\left[2\left(\sin^{-1}\dfrac{x}{2}\right)\right] + 2\sin^{-1}\dfrac{x}{2} + C$ (d) $\dfrac{5}{12}$

6. (a) diverges with $\displaystyle\lim_{t\to 1^-} 2\ln|\sqrt{t}-1|$ (b) diverges by comparison with $\displaystyle\int_3^{\infty} \frac{1}{x}\,dx$

7. $\dfrac{8}{27}\left(1+\dfrac{9}{4}\right)^{3/2} - \dfrac{8}{27}$

8. (a) $\displaystyle\int_0^{2\pi} \sqrt{(3+3\cos\theta)^2 + (-3\sin\theta)^2}\,d\theta$ (b) $\displaystyle\int_2^4 2\pi(2x-4)(x-4)\sqrt{1+(-4x+12)^2}\,dx$

## Final: April, 2009

1. 2/3

2. $-1 + \dfrac{\pi}{2}$ $f(x) = \dfrac{1}{x\sqrt{1-x^2}} + \dfrac{-1}{x^2}\sin^{-1} x$

3. $\displaystyle\int_{-2}^{3} -(y-5)(y+1) - (3y-1)\,dy = \frac{125}{6}$ square units

4. $\dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3} + \dfrac{Dx+E}{x^2-1} + \dfrac{Fx+G}{(x^2-1)^2}$

5. Using shells or washers, $2\pi\displaystyle\int_0^2 x\cdot x\,dx + 2\pi\int_2^4 x(4-x)\,dx = \pi\int_0^2 (4-y)^2 - y^2\,dy = 16\pi$

6. $\displaystyle\int_0^{\pi} \sqrt{1+\theta^2}\,d\theta = \frac{\pi}{2}\sqrt{1+\pi^2} + \frac{1}{2}\ln(\pi + \sqrt{1+\pi^2})$

7. $2\pi\displaystyle\int_0^1 (3-x)[3-x^2-(x^3+x)]\,dx = \frac{151\pi}{15}$

8. $\dfrac{1}{72}$ 9. $\dfrac{\sec^{11} x}{11} - \dfrac{2}{9}\sec^9 x + \dfrac{\sec^7}{7} + C$

10.$-\dfrac{\sqrt{3+2x-x^2}}{x-1} - \sin^{-1}\dfrac{x-1}{2} + C$ or $-\dfrac{\sqrt{3+2x-x^2}}{x-1} + \cos^{-1}\dfrac{x-1}{2} + D$

11. (a) diverges with $\displaystyle\lim_{x\to\frac{1}{2}^-} \frac{1}{2}\ln|2x-1|$ (b) $-3e^{2/3}$

12. $2(2\sqrt{2}-1)$

13. $\displaystyle\int_0^{\pi/6} 2\pi \cos 2x\sqrt{1+4\sin^2 2x}\,dx = \frac{\pi}{4}[2\sqrt{3}+\ln(2+\sqrt{3})]$

# Term Test Answers

## Term Test: February, 2004 (Solutions)

1. (a) Let $\quad u = \sin x \qquad du = \cos x dx$

   $$= \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

   (b) Let $\quad u = x + 2 \qquad du = dx$

   $$=\int u^{20} 20\,(u-2)\,du$$
   $$=20\int u^{21} - 2u^{20} du$$
   $$=20\left[\frac{u^{22}}{22} - \frac{2u^{21}}{21}\right] + C$$
   $$=20\left[\frac{(x+2)^{22}}{22} - \frac{2\,(x+2)^{21}}{21}\right] + C$$

   (c) Let $\quad u = x^2 - 1 \qquad du = 2x dx$

   $$= \frac{1}{2}\int_1^3 \left(x^2-1\right)^{\frac{2}{3}} (2x dx) = \frac{1}{2}\int_0^8 u^{\frac{2}{3}} du$$
   $$= \frac{1}{2}\, u^{\frac{5}{3}} \cdot \frac{3}{5}\Big|_0^8 = \frac{3}{10}\,[32 - 0] = \frac{96}{10}$$

   OR

   $$\int_1^3 x\left(x^2-1\right)^{\frac{2}{3}} dx = \frac{1}{2}\int_1^3 \left(x^2-1\right)^{\frac{2}{3}} (2x dx)$$
   $$= \frac{1}{2}\left(x^2-1\right)^{\frac{5}{3}} \cdot \frac{3}{5}\Big|_1^3 = \frac{3}{10}\left[8^{\frac{5}{3}} - 0\right] = \frac{3}{10}\,[32] = \frac{96}{10}$$

2. (a) $0 = t(t+1) \quad t = 0 \quad t = -1$

   $y$-intercepts are: $0, -1$ or at $(0,0)$ and $(0,1)$

   $0 = t(t+2) \quad t = 0 \quad t = -2$

   $x$-intercepts are: $0, 2$ or at $(0,0)$ and $(2,0)$

   (b) $\lim_{t\to\infty} x = \lim_{t\to\infty} t^2 + t = \infty$

   $\lim_{t\to-\infty} x = \lim_{t\to-\infty} t^2 + t = \lim_{t\to-\infty} t\,(t+1) = \infty$

   $\lim_{t\to\infty} y = \lim_{t\to\infty} t^2 + 2t = \infty$

   $\lim_{t\to-\infty} y = \lim_{t\to-\infty} t^2 + 2t = \lim_{t\to-\infty} t\,(t+2) = \infty$

   (c) $\dfrac{dy}{dt} = 2t + 2$

   $0 = 2\,(t+1); \quad t = -1$

   $\dfrac{dx}{dt} \neq 0$ therefore horizontal tangent at $(0, -1)$.

   $\dfrac{dx}{dt} = 2t + 1$

$0 = 2t + 1; \quad t = -\frac{1}{2}$

$\frac{dy}{dt} \neq 0$ therefore vertical tangent at $\left(-\frac{1}{4},\ -\frac{3}{4}\right)$.

(d) $\frac{dy}{dx} = \frac{2t+2}{2t+1}$

$\frac{d^2y}{dx^2} = 2\,(2t+1) - 2\,(2t+2)2t + 1^3 = \frac{-2}{(2t+1)^3}$

When $t$ is $< -\frac{1}{2}$, $\frac{d^2y}{dx^2}$ is positive therefore the curve is concave up.

When $t$ is $> -\frac{1}{2}$, $\frac{d^2y}{dx^2}$ is negative therefore the curve is concave down.

(e)

| $t$ | | $-1$ | | $-\frac{1}{2}$ | |
|---|---|---|---|---|---|
| $\frac{dy}{dt}$ | $-$ | | $+$ | | $+$ |
| $\frac{dx}{dt}$ | $-$ | | $-$ | | $+$ |
| $x$ | $\leftarrow$ | | $\leftarrow$ | | $\rightarrow$ |
| $y$ | $\downarrow$ | | $\uparrow$ | | $\uparrow$ |
| curve | $\swarrow$ | | $\nwarrow$ | | $\nearrow$ |

3.

4. $\frac{dx}{dt} = 4\cos 2t \qquad \frac{dy}{dt} = 3\cos t,$ therefore $\frac{dy}{dx} = 3\cos t 4\cos 2t$

at $\quad t = \frac{2\pi}{3} \qquad m = \frac{3\left(-\frac{1}{2}\right)}{4\left(-\frac{1}{2}\right)} = \frac{3}{4}$

Point of tangency has coordinates: $\left(-\sqrt{3},\ \frac{3\sqrt{3}}{2}\right)$

Equation of tangent line is: $y - \frac{3\sqrt{3}}{2} = \frac{3}{4}\left(x + \sqrt{3}\right)$

5. Intersection at:
$x^2 + 1 = x + 7\ ; \quad x^2 - x - 6 = 0\ ; \quad (x-3)(x+2) = 0\ ; \quad x = 3,\ -2$
Area $= \int_{-2}^{3} (x+7) - \left(x^2+1\right)\ dx$

6. $F'(x) = e^{(x^3)^2} = e^{x^6}(3x^2)\ ; \qquad 0 = e^{x^6}(3x^2); \qquad x = 0$

| $x$ | | $0$ | |
|---|---|---|---|
| $F'(x)$ | $-$ | | $+$ |
| $F$ | $\searrow$ | | $\nearrow$ |

Therefore $F$ is increasing on $[0,\ \infty)$

7. $= \int_0^1 x^2\ dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

## Term Test: March, 2004

1. (a) $\dfrac{\pi}{4}$

   (b) $\dfrac{1}{5}e^x \sin 2x - \dfrac{2}{5}e^x \cos 2x + C$

   (c) $\dfrac{\sin^3 x}{3} - \dfrac{2}{5}\sin^5 x + \dfrac{1}{7}\sin^7 x + C$

   (d) $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

2.

3. (a) $\dfrac{2x}{x^2+1}$

   (b) $\dfrac{1}{1+(\sqrt{x})^2}\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right)$

4. $\dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(1+\cos\theta)^2 - 1]\, d\theta$

5. (a)

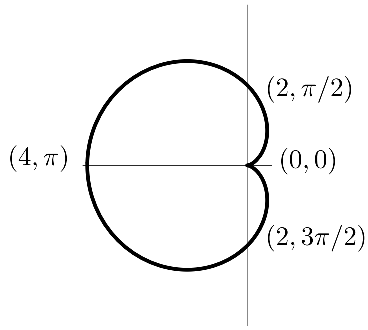
   (b) $V = 2\displaystyle\int_0^2 \pi\left(2^2 - \left(\frac{x^2}{2}\right)^2\right) dx$

   (c ) $V = \displaystyle\int_{-2}^2 2\pi(5-x)\left(2 - \frac{x^2}{2}\right) dx$

## Term Test: October, 2004

1. (a) $-5$

   (b) $1$

2. 


3. (a) $x$-intercepts 0,9; $y$-intercept 0

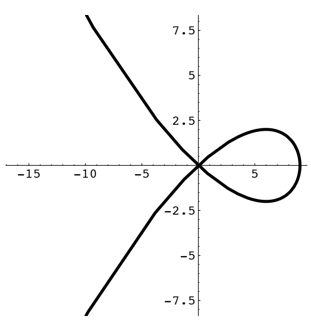
   (b) $\lim\limits_{t\to\pm\infty} = -\infty$; $\quad \lim\limits_{t\to\pm\infty} = \pm\infty$

   (c) horizontal tangent at $(6,-2)$ and $(6,2)$; vertical tangent at $(9,0)$.

   (d) concave up on $(0,\infty)$; concave down on $(-\infty,0)$

$-1 \qquad 0 \qquad 1 \qquad t$

| | | | | |
|---|---|---|---|---|
| $dx/dt$ | $+$ | $+$ | $-$ | $-$ |
| $dy/dt$ | $+$ | $-$ | $-$ | $+$ |
| $x$ | $\rightarrow$ | $\rightarrow$ | $\leftarrow$ | $\leftarrow$ |
| $y$ | $\uparrow$ | $\downarrow$ | $\downarrow$ | $\uparrow$ |
| curve | $\nearrow$ | $\searrow$ | $\swarrow$ | $\nwarrow$ |

   (e) 


4. $$y - 2 = \frac{1}{2}x$$

5. (a) $\dfrac{3}{8}(x^2+3)^4 + C$

(b) $\dfrac{\tan^3 x}{3} + \tan x + C$

6. $\displaystyle\int_0^2 x^3\,dx = 4$

7. $\dfrac{\cos(\sqrt{\sin x})\cos x}{\sin^2 x} - \dfrac{\cos\sqrt{x^2}}{x^4}(2x)$

## Term Test: November, 2004

1. (a) 2

   (b) $-\dfrac{1}{136}(2x-5)^{-17} - \dfrac{5}{72}(2x-5)^{-18} - \dfrac{25}{152}(2x-5)^{-19} + C$

   (c) $\dfrac{-1}{10}\sin 5x + \dfrac{1}{2}\sin x + C$

   (d) $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

2. $\dfrac{1}{3}(2e^3 + 1)$

3. $\dfrac{\pi}{2}$

4. (a)

   (b) $\dfrac{\pi^2}{4}$

   (c) $2\pi(\sqrt{2} - 1)$

5. (a) $(12, 1)$ and $(24, 3)$

   (b) $\dfrac{8}{3}$

## Term Test: March, 2005 (Solutions)

1. (a) They intersect at points whose y-coordinates satisfy
   $2x^2 = 3 - x^2;\quad 3x^2 = 3;\quad x^2 = 1;\quad x = \pm 1$
   If $x = 1$, $y = 2(1)^2 = 2$; If $x = -1$, $y = 2(-1)^2 = 2$

   Therefore the points of intersection are $(1, 2)$ and $(-1, 2)$.

   (b) If $-2 \le x \le -1$ then $2x^2 \ge 3 - x^2$.
   If $-1 \le x \le 0$ then $3 - x^2 \ge 2x^2$.

$$A = \int_{-2}^{-1} [2x^2 - (3 - x^2)]\,dx + \int_{-1}^{0} [(3 - x^2) - (2x^2)]\,dx = \int_{-2}^{-1} (3x^2 - 3)]\,dx + \int_{-1}^{0} (3 - 3x^2)\,dx$$

$$= (x^3 - 3x)\Big|_{-2}^{-1} \quad + \quad (3x - x^3)\Big|_{-1}^{0} = \Big[(-1+3) - (-8+6)\Big] + \Big[(0-0) - (-3-(-1))\Big]$$

$= (2 + 2) + (2) = 6$ square units.

2. Intersections $1 + \sin\theta = 2 - \sin\theta$; $2\sin\theta = 1$; $\sin\theta = \frac{1}{2}$; $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

$$A = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1 + \sin\theta)^2\,d\theta - \int_{\pi 6}^{5\pi 6} \frac{1}{2}(2 - \sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \Big[(1 + 2\sin\theta + \sin^2\theta) - (4 - 4\sin\theta + \sin^2\theta)\Big]\,d\theta$$

$$= \frac{1}{2}\int_{\pi 6}^{5\pi 6} (6\sin\theta - 3)\,d\theta$$

$$= \tfrac{1}{2}(-6\cos\theta - 3\theta)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \tfrac{1}{2}\left[\left((-6)\left(\tfrac{-\sqrt{3}}{2}\right) - \tfrac{5\pi}{2}\right) - \left((-6)\left(\tfrac{\sqrt{3}}{2}\right) - \tfrac{\pi}{2}\right)\right]$$

$$= \tfrac{1}{2}(3\sqrt{3} + 3\sqrt{3} - 2\pi)$$

$= 3\sqrt{3} - \pi$ square units.

3 (a) The $x$-coordinates of the points of intersection satisfy

$x^2 + 1 = 1 - x$; $x^2 + x = 0$; $x(x+1) = 0$; $x = 0$ or $x = -1$.

The curves intersect at $(-1, 2)$ and $(0, 1)$. The line lies above the parabola if $-1 \le x \le 0$.

Using "washers,"

$$V = \int_{-1}^{0} \pi(1 - x)^2\,dx - \int_{-1}^{0} \pi(x^2 + 1)^2\,dx = \pi\int_{-1}^{0} \Big[(1 - 2x + x^2) - (x^4 + 2x^2 + 1)\Big]\,dx$$

$$= \pi\int_{-1}^{0} (-2x - x^2 - x^4)\,dx = \pi\left[-x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{0}$$

$$= \pi\left[0 - \left(-1 + \frac{1}{3} + \frac{1}{5}\right)\right] = \pi\left(\frac{15 - 5 - 3}{15}\right) = \frac{7\pi}{15} \text{ cubic units.}$$

(b) Using "shells,"

$$V = \int_{-1}^{0} 2\pi|x|\Big[(1 - x) - (x^2 + 1)\Big]\,dx$$

($|x|$ is the radius $(1 - x) - (x^2 + 1)$ is the height, and $dx$ is the thickness of the shell)

As $x < 0$ if $-1 < x < 0$, we see that $|x| = -x$ for the values of $x$ we are using. Thus

$$V = 2\pi\int_{-1}^{0} (-x)[2 - x - x^2]\,dx = -2\pi\int_{-1}^{0} (2x - x^2 - x^3)\,dx = -2\pi\left(x^2 - \frac{x^3}{3} - \frac{x^4}{4}\Big|_{-1}^{0}\right)$$

$$= -2\pi\left(0 - \left(1 + \frac{1}{3} - \frac{1}{4}\right)\right) = 2\pi\left(1 + \frac{1}{3} - \frac{1}{4}\right) = 2\pi\left(\frac{13}{12}\right) = \frac{13\pi}{6} \text{ cubic units.}$$

4. $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ so $\frac{d}{dx}(\tan^{-1}(x^4)) = \frac{1}{1+(x^4)^2} \cdot \frac{d}{dx}(x^4) = \frac{4x^3}{1+x^8}$

When $x = 2$, this equals $\frac{4(2^3)}{1+2^8} = \frac{2^5}{1+2^8} < \frac{2^5}{2^8} = \frac{1}{2^3} = \frac{1}{8}$, so the value is less than $\frac{1}{8}$.

5 (a) Let $u = \pi + \frac{\pi}{x}$, therefore $\frac{du}{dx} = -\frac{\pi}{x^2}$, so $\frac{dx}{x^2} = -\frac{1}{\pi}\,du$

When $x = 1$, $u = \pi + \frac{\pi}{1} = 2\pi$. When $x = 2$, $u = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$.

$$= \int_{2\pi}^{\frac{3\pi}{2}} \sin(u)\left(-\frac{1}{\pi}\,du\right) = \left(-\frac{1}{\pi}\right)\left(-\cos(u)\right)\Big|_{2\pi}^{\frac{3\pi}{2}} = \frac{1}{\pi}\left[\cos\left(\frac{3\pi}{2}\right) - \cos(2\pi)\right] = \frac{1}{\pi}[0-1] = -\frac{1}{\pi}$$

(b) Let $u = 2x - 1$ and $dv = e^{-x}dx$, therefore $\frac{du}{dx} = 2$, so $du = 2dx$, and $v = \int e^{-x}\,dx = -e^{-x}$

$$\int u\,dv = uv - \int v\,du = (2x-1)(-e^{-x}) - \int(-e^{-x})(2\,dx) = -e^{x}(2x-1) + 2\int e^{-x}\,dx$$

$$= -e^{x}(2x-1) - 2e^{-x} + C$$

## Term Test: March, 2007 (Solutions)

1. (a) $\int \frac{dx}{\sqrt{1-x}} = \int \frac{d(1-x)}{\sqrt{1-x}} = -2(\sqrt{1-x}) + C$

(b) $u = \tan^{-1}x$ $\quad du = \frac{dx}{x^2+1}$ $\qquad x = 0, u = 0;\ x = 1, u = \frac{\pi}{4}$

$$\int_0^1 \frac{dx}{(x^2+1)(1+\tan^{-1}x)} = \int_0^{\pi/4} \frac{du}{1+u} = \ln(1+u)\Big|_0^{\pi/4} = \ln(1+\pi/4) - \ln 1$$

(c) $u = 1 - x;\ x = 1 - u;\ du = -dx$

$$\int x(1-x)^{20}\,dx = -\int(1-u)u^{20}\,du = \int u^{21} - u^{20}\,du = \frac{u^{22}}{22} - \frac{u^{21}}{21} = \frac{(1-x)^{22}}{22} - \frac{(1-x)^{21}}{21} + C$$

(c) $\int(\cos(\sin\theta))\cos\theta\,d\theta = \sin(\sin\theta) + C$

2. $$\left|\int_0^1 (t^2+1)(-3t^2)\,dt\right| = \int_0^1 3t^4 + 3t^2\,dt = \left[\frac{3t^5}{5} + \frac{3t^3}{3}\right]_0^1 = \frac{3}{5} + 1$$

3. $x^3 + x^2 = x^2 + x;$ $\qquad x = 0, \pm 1$

$$\left|\int_{-1}^0 x^3 - x\,dx\right| + \left|\int_0^1 x^3 - x\,dx\right| = \left|\left[\frac{x^4}{4} - \frac{x^2}{2}\right]\right|_{-1}^0 + \left|\left[\frac{x^4}{4} - \frac{x^2}{2}\right]\right|_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

4. $$(\sin^{-1}(x) + \sin^{-1}(-x))' = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$$\sin^{-1}(0) + \sin^{-1}(-0) = 0$$

5. (a) $\displaystyle \frac{1}{2}\int_{\pi/12}^{5\pi/12} 4\sin^2 2\theta - 1\, d\theta$

(b) $\displaystyle 4\int_{-2}^{2} 4 - x^2\, dx$

(c) $\displaystyle \int_{1}^{3} \pi[(x-1)(x-3)]^2\, dx$

(d) $\displaystyle -\int_{1}^{3} 2\pi x[(x-1)(x-3)]\, dx$

## Term Test: February, 2009

1. (a) 0 (b) 0

2. (a)

| $t$ | $dx/dt$ | $dy/dt$ | $x$ | $y$ | Curve |
|---|---|---|---|---|---|
| $(-\infty,-1)$ | $+$ | $+$ | $\rightarrow$ | $\uparrow$ | $\nearrow$ |
| $(-1,0)$ | $-$ | $+$ | $\leftarrow$ | $\uparrow$ | $\nwarrow$ |
| $(0,1)$ | $-$ | $-$ | $\leftarrow$ | $\downarrow$ | $\swarrow$ |
| $(1,2)$ | $+$ | $-$ | $\rightarrow$ | $\downarrow$ | $\searrow$ |
| $(2,\infty)$ | $+$ | $+$ | $\rightarrow$ | $\uparrow$ | $\nearrow$ |

(b) concave upward on $(-\infty,-1)$ and $(1,\infty)$, downward on $(-1,1)$

(c) As the parameter $t$ increases from $-\infty$, the curve begins in the third quadrant and crosses the $y$-axis at the $y$-intercept $-9-3\sqrt{3}$ then through $(2,-4)$ with a vertical tangent, through the origin tangent to the $x$-axis, through $(-2,-2)$ with a vertical tangent, and through $(2,-4)$ again this time with a horizontal tangent, with the second $y$-intercept $-9+3\sqrt{3}$ and then runs into the first quadrant with the $x$-intercept 18.

3. (a) circle with centre $(1,2)$ and radius $\sqrt{5}$

(b) $r = 2\cos\theta + 4\sin\theta$

4. (a) $\sec x + C$ (b) $2-\sqrt{2}$