MATH 1700

CALCULUS II

Lectures

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To the Student

These notes are an exact copy of the slides used in lectures. Of course there is a good deal of explanation, and annotation as we go along. But these notes should save you some writing during the class and give you more time to listen.

The ideal way to use these notes is to read ahead and be prepared for the lecture. The narrow format was chosen to give you room for your own notes. As you work through a lecture or a problem, you can look back at what was said earlier to clarify a point. If you think of a question, please ask it! Any question is valuable and good questions make a real contribution to the class.

Previous versions of these notes were hand written. This version was written using TEX, a type setting program for computers. Canvas, a graphics program, and Mathematica, a symbolic mathematics program, were used to create the diagrams.

Feedback about the notes is very important to me. I would like to thank the many students who have taken Calculus using these notes and who have made helpful comments and suggestions. In fact the idea of printed notes evolved from requests from students to make the hand written slides available. Naturally I want to correct any errors. Additional ideas are very welcome.

In the end, calculus is learned by doing calculus, and not by reading, or watching someone else do it. You can practice alone or in small groups (explaining calculus to each other is a good idea). All of you have at least a C in a first course in Calculus, so you know what is involved in mastering the subject. This course builds on the previous one and you may have to review old material. Keep up the good work!

Contents

Chapter 1. L'Hôpital's Rule

In which

We find that indeterminants can (sometimes) be determined.

1.1 Indeterminant forms

$\lim_{x\to 0} \frac{\sin x}{x}$	$\frac{0}{0}$
$\lim_{x\to\infty} \frac{x}{e^x}$	$\frac{\infty}{\infty}$
$\lim_{x\to\infty} x \sin \frac{1}{x}$	$\infty \times 0$
$\lim_{t\to(\pi/2)^-} (\sec t - \tan t)$	$\infty - \infty$
$\lim_{x\to 0^+} x^{\sqrt{x}}$	0^0
$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^x$	∞^0
$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$	1^∞

- when form is indeterminant
  - cannot tell what the limit is (or even if it exists)
  - next two examples show any outcome is possible

Example 1.1.1 For any constant k,

$$\lim_{x\to 0} \frac{kx}{x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} k = k$$

Example 1.1.2 $\lim_{x\to 0^+} \frac{x}{x^2} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0^+} \frac{1}{x} = \infty$$

1.2 The rule

$$\boxed{\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}}$$

- requires f, g diffble on (a,b); $g'(x) \neq 0$ on (a,b)

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} \text{ exists or is } \pm\infty$$

either $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$

!

or $\lim_{x\to a^+} f(x) = \pm\infty$ and $\lim_{x\to a^+} g(x) = \pm\infty$

! **The form must be indeterminant**

Example 1.2.1 $\lim_{x\to 0^+} \frac{x^2-1}{x+5} = -\frac{1}{5}$

- result is clear by quotient rule for limits
- applying l'Hôpital gives

$$\lim_{x\to 0^+} \frac{x^2-1}{x+5} = \lim_{x\to 0^+} \frac{2x}{1} = 0$$

- the form is *not* indeterminant

! – l'Hôpital cannot be used

• <u>variants</u>

$$\lim_{x \to b^-}$$

$$\lim_{x \to c}, \quad \text{for } a < c < b$$

$$a = -\infty \text{ and/or } b = \infty$$

• the proof is easy in the following situation

$$f(a) = g(a) = 0, \quad g'(a) \neq 0$$

f' and g' exist and continuous at a

• in this case

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

1.3 Examples of the various forms

Example 1.3.1

$$\lim_{x \to 0} \frac{\sin x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x}{1} \quad \text{(by H)}$$

$$= 1$$

[!] Question: Can the geometric proof of this limit given in 136.150 be avoided using l'Hôpital?

Example 1.3.2

$$\lim_{x\to\infty} \frac{x}{e^x} \qquad (\frac{\infty}{\infty})$$

$$= \lim_{x\to\infty} \frac{1}{e^x} \quad \text{(by H)}$$

$$= 0$$

Example 1.3.3

$$\lim_{x\to\infty} x \sin\frac{1}{x} \qquad (\infty \times 0)$$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \qquad \left(\text{or} = \lim_{t\to 0} \frac{\sin t}{t},\ \text{setting } t = \frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\left(\cos\frac{1}{x}\right)\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'} \quad \text{(by H)}$$

$$= \lim_{x\to\infty} \left(\cos\frac{1}{x}\right) = \cos 0 = 1$$

Example 1.3.4

$$\lim_{t\to(\pi/2)^-} (\sec t - \tan t) \qquad (\infty - \infty)$$

$$= \lim_{t\to(\pi/2)^-} \left(\frac{1}{\cos t} - \frac{\sin t}{\cos t}\right)$$

$$= \lim_{t\to(\pi/2)^-} \frac{1 - \sin t}{\cos t} \qquad \left(= \frac{0}{0}\right)$$

$$= \lim_{t\to(\pi/2)^-} \frac{-\cos t}{-\sin t} \quad \text{(by H)}$$

$$= 0$$

Example 1.3.5 Evaluate $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$ $\quad (0^0)$.

• let $y = x^{\sqrt{x}}$, then $\ln y = \sqrt{x}\ln x = \dfrac{\ln x}{(1/\sqrt{x})}$

$$\begin{aligned}\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln x}{(1/\sqrt{x})} \qquad \left(\frac{-\infty}{\infty}\right)\\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}} \quad \text{(by H)}\\ &= -\lim_{x\to 0^+} 2x^{1/2} = 0\end{aligned}$$

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} y = e^0 = 1$$

Example 1.3.6 Evaluate $\lim\limits_{x\to 0^+} \left(\frac{1}{x}\right)^x$ $\quad (\infty^0)$.

• let $y = \left(\dfrac{1}{x}\right)^x$

$$\begin{aligned}\ln y &= x\ln\frac{1}{x}\\ &= -x\ln x\\ &= -\frac{\ln x}{\left(\frac{1}{x}\right)}\end{aligned}$$

$$\begin{aligned}-\lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} &\qquad \left(-\frac{-\infty}{\infty}\right)\\ &= -\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad \text{(by H)}\\ &= \lim_{x\to 0^+} x = 0\end{aligned}$$

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^x = e^0 = 1$$

Example 1.3.7 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \qquad (1^\infty)$

- let $y = \left(1+\frac{1}{x}\right)^x$

$$\ln y = x\ln\left(1+\frac{1}{x}\right)$$

$$= \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

- "substitution" gives $\frac{0}{0}$ so apply L'Hôpital

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+(1/x)}\cdot\left(\frac{1}{x}\right)'}{\left(\frac{1}{x}\right)'}$$

$$= \lim_{x\to\infty}\frac{1}{1+(1/x)} = 1$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$$

1.4 Further examples

Example 1.4.1 Evaluate $\lim_{x\to 2}\frac{\ln(2x-3)}{x^2-4} \quad \left(\frac{\ln 1 = 0}{0}\right).$

$$\lim_{x\to 2}\frac{\ln(2x-3)}{x^2-4} = \lim_{x\to 2}\frac{\left(\frac{2}{2x-3}\right)}{2x} = \frac{2}{4}$$

Example 1.4.2 Evaluate $\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1} \quad \left(\frac{0}{0}\right)$.

$$\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1} = \lim_{x\to 1} \frac{\frac{1}{3}x^{-2/3}}{\frac{2}{3}x^{-1/3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Example 1.4.3 Evaluate $\lim_{x\to 0} \frac{x-\sin x}{x^3} \quad \left(\frac{0-0}{0}\right)$.

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} \qquad \left(\frac{1-1}{0}\right)$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cos x}{6} \quad \text{(by H)}$$

$$= \frac{1}{6}$$

Example 1.4.4 Evaluate $\lim_{x\to 2^+} \frac{8}{x^2-4} - \frac{x}{x-2}$.

$$\lim_{x\to 2^+} \frac{8}{x^2-4} - \frac{x}{x-2} = \lim_{x\to 2^+} \frac{8-x(x+2)}{x^2-4} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 2^+} \frac{-2x-2}{2x} = -\frac{6}{4} = -\frac{3}{2}$$

- note $\frac{8-x(x+2)}{x^2-4} = \frac{(x-2)(-x-4)}{(x-2)(x+2)} = \frac{-x-4}{x+2}$
  - so $\lim_{x\to 2^+} \frac{8}{x^2-4} - \frac{x}{x-2} = \lim_{x\to 2^+} \frac{-x-4}{x+2} = -\frac{3}{2}$

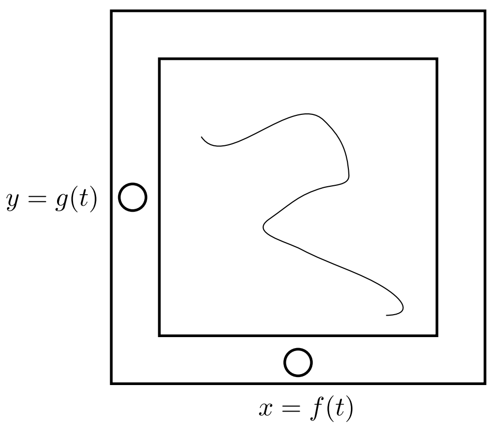
    (without L'Hôpital)

Chapter 2. Parametric Equations and Polars

In which

We study other ways to describe curves.

2.1 Etch-a-sketch



$$x = f(t)$$
$$y = g(t)$$

- parametric equations
- t is the "parameter"
- as t changes the point $(x, y) = (f(t), g(t))$ moves
  - traces out a curve

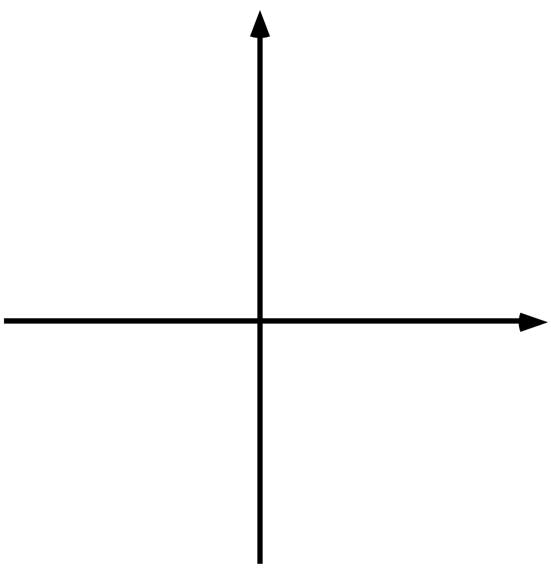
Example 2.1.1 What curve has the parametric equations:

$$x = t + 1$$
$$y = 2t - 5?$$

- in a simple case can eliminate the parameter

$$t = x - 1$$
$$y = 2t - 5 = 2(x - 1) - 5$$
$$y = 2x - 7$$

- the point moves on the line $y = 2x - 7$
- visualize the movement as t changes



Example 2.1.2 Find parametric equations for the line through the points $(1, 2)$ and $(7, 16)$.

- slope $\frac{7}{3}$
- parametric equations $x = 3t + c$; $y = 7t + d$
  - if x increases by 3, then y increases by 7
- for $t = 0$, (c, d) is on the line e.g. $(c, d) = (1, 2)$
  - line: $x = 3t + 1$; $y = 7t + 2$
- $t = 2$ puts $(7, 16)$ on the line

Example 2.1.3 Identify the curve $x = t^2 - 1$, $y = t + 1$.

• eliminate the parameter

$t = y - 1$

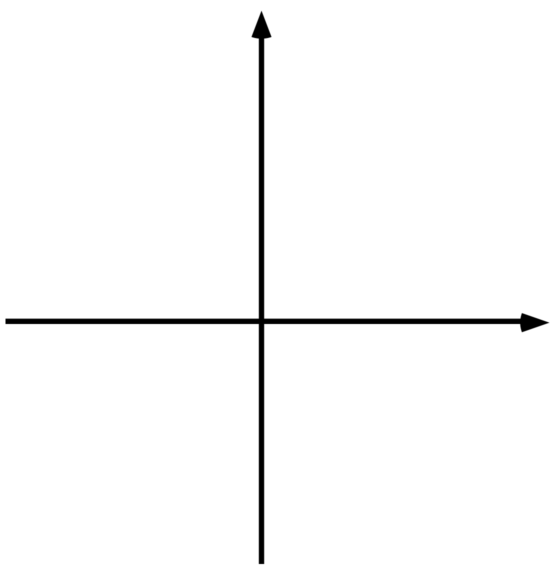
$x = t^2 - 1 = (y - 1)^2 - 1 = y^2 - 2y$

• this is a parabola

• do we get the entire parabola?

– yes if $-\infty < t < \infty$ then $-\infty < y < \infty$

• visualize how the point moves as the parameter changes



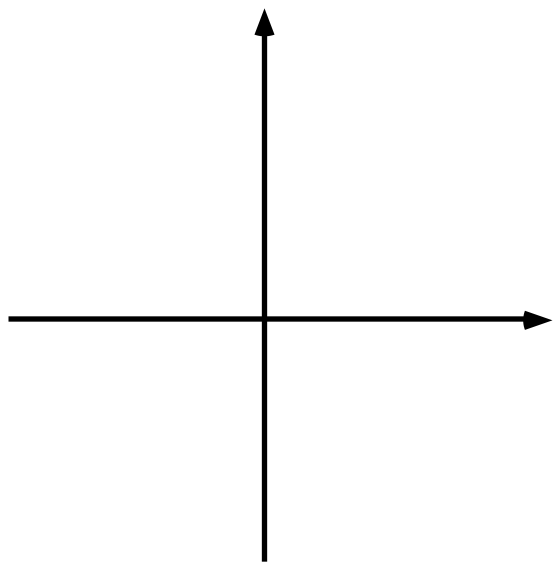
Example 2.1.4 Identify the curve $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$.

• $x^2 + y^2 = 1$

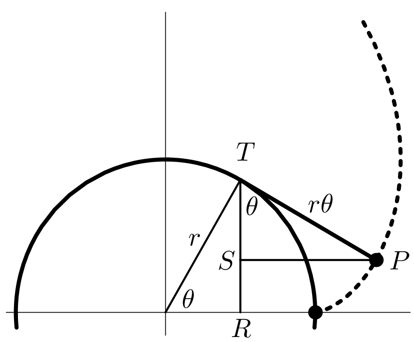
• this is a circle

• do we get the entire circle?

• visualize how the point moves as the parameter changes



Example 2.1.5 Involute: path traced by a point at the end of a string which has been wound around a circle and is then unwound while being held taut (so that the string is extended as a tangent to the circle).



- to find the point P
- coordinates of T: $(r\cos\theta, r\sin\theta)$
- drop perpendicular from T to axis
- label the point where it meets the axis R
- draw horizontal line from P to the perpendicular
- label the point where they meet S
- angle $PTS = \theta$. Why?
- $|TP| = r\theta$. Why?

- $|SP| = r\theta \sin\theta$
- $|ST| = r\theta \cos\theta$

$$x = |OR| + |SP| = r\cos\theta + r\theta\sin\theta$$

$$y = |RT| - |ST| = r\sin\theta - r\theta\cos\theta$$

2.2 Tangents to Parametric Curves

- parametric curve

$$x = f(t);\ y = g(t)$$

- assume the curve is

$$y = F(x)$$

- examples of this; eliminated the parameter t
- general conditions where possible

$$f' \text{ cont., } f'(t) \neq 0 \text{ on a } t\text{-interval}$$

$$x = f(t);\ y = g(t);\ y = F(x)$$

$$g(t) = F(f(t))$$

$$g'(t) = F'(f(t))f'(t) \qquad \text{(Chain Rule)}$$

$$F'(f(t)) = F'(x) = \frac{g'(t)}{f'(t)} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Example 2.2.1 What happens if $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$?

$$y = t^2, \quad x = t \quad \text{at} \quad t = 0$$

- curve is $y = x^2$
- describe the situation in words

Example 2.2.2 What happens if $\dfrac{dy}{dt} \neq 0$ and $\dfrac{dx}{dt} = 0$?

$$y = t, \quad x = t^2 \quad \text{at} \quad t = 0$$

- curve is $x = y^2$
- describe the situation in words

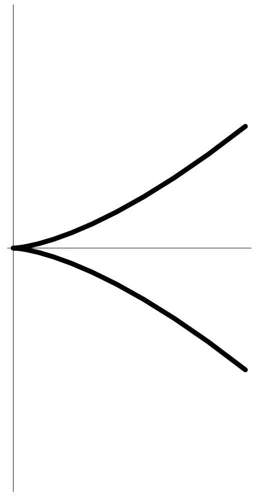
Example 2.2.3 An example of what can happen if both x' and y' are 0.

- both derivatives are 0; the curve stops
  - may start again in a different direction

$$x = t^2, \quad y = t^3$$

$$x' = 2t, \quad y' = 3t^2$$

- both derivatives are 0 at $t = 0$
- curve is $x = y^{2/3}$



Example 2.2.4 Find the slope of the tangent to the parametric curve $x = t^2 - t;\ y = t^2 + t$ when $t = 2$.

$$\frac{dx}{dt} = 2t - 1$$

$$\frac{dy}{dt} = 2t + 1$$

$$\frac{dy}{dx} = \frac{2t + 1}{2t - 1}$$

$$\text{Slope } (t = 2) = \left.\frac{dy}{dx}\right|_{t=2} = \frac{5}{3}.$$

Example 2.2.5 Find $\dfrac{d^2y}{dx^2}$ for the above curve.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{2t + 1}{2t - 1}\right) \cdot \frac{1}{(2t - 1)}$$

$$= \frac{(2t - 1)2 - (2t + 1)2}{(2t - 1)^3}$$

$$= \frac{-4}{(2t - 1)^3}$$

Example 2.2.6 Sketch the curve, incorporating information from the derivatives discussed in the above examples.

	$t < -1/2$	$-1/2 < t < 1/2$	$1/2 < t$
$dx/dt = 2t - 1$			
$dy/dt = 2t + 1$			
x			
y			
curve			

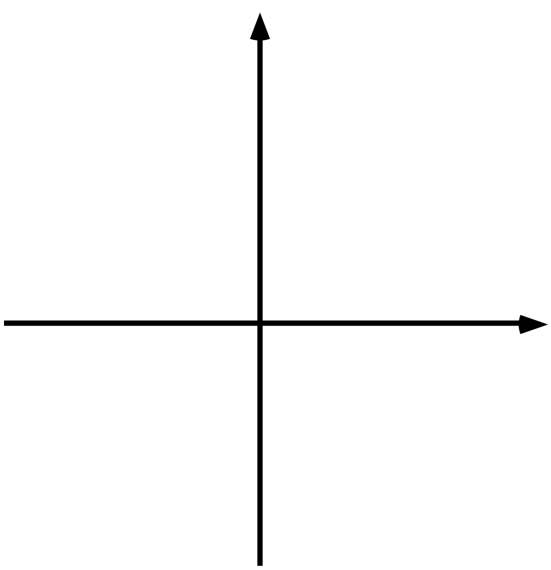
• some points

t	x	y
$-1/2$	$3/4$	$-1/4$
0	0	0
$1/2$	$-1/4$	$3/4$

• concavity

shape y

sign $\dfrac{d^2y}{dx^2}$ $\left(= \dfrac{-4}{(2t-1)^3}\right)$



Example 2.2.7 Sketch $x = 4\cos^3 t$, $y = 4\sin^3 t$. (This is called an astroid and has equation $x^{2/3} + y^{2/3} = 4^{2/3}$.

$$\frac{dx}{dt} = 12\cos^2 t(-\sin t)$$

$$\frac{dy}{dt} = 12\sin^2 t(\cos t)$$

$$\frac{dy}{dx} = \frac{12\sin^2 t(\cos t)}{12\cos^2 t(-\sin t)} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{12\cos^2 t(-\sin t)} = \frac{1}{12\cos^4 t(\sin t)}$$

	$0 < t < \frac{\pi}{2}$	$\frac{\pi}{2} < t < \pi$	$\pi < t < \frac{3\pi}{2}$	$\frac{3\pi}{2} < t < 2\pi$
$\frac{dx}{dt}$				
$\frac{dy}{dt}$				
x				
y				
curve				

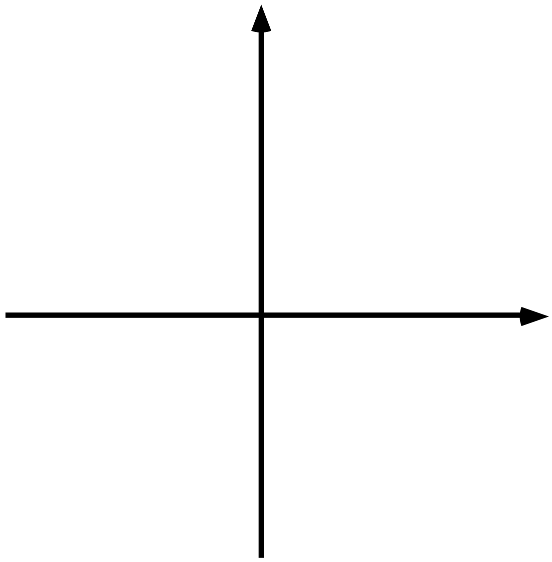
- some points

t	x	y
0	4	0
$\pi/2$	0	4
π	-4	0
$3\pi/2$	0	-4

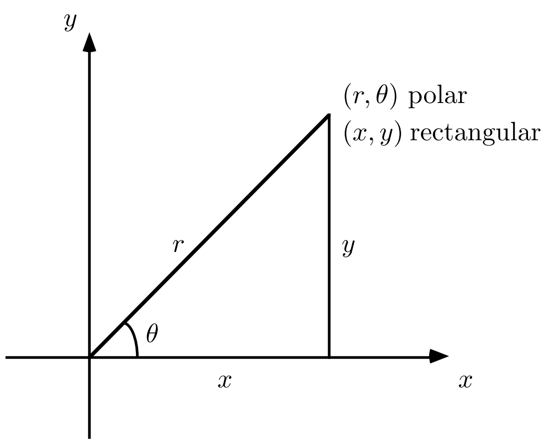
- concavity

shape y

sign $\frac{d^2y}{dx^2}$



2.3 Polar Coordinates



$$x = r\cos\theta$$

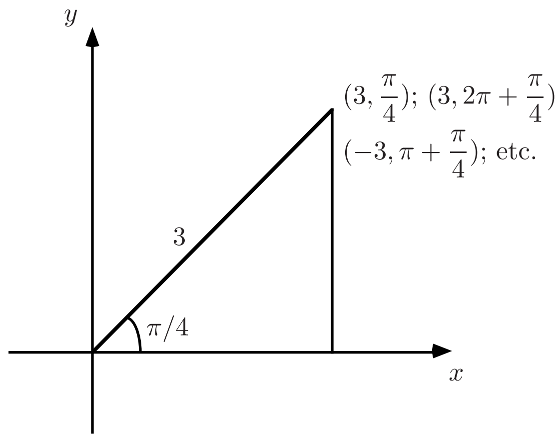
$$y = r\sin\theta$$

- each (r, θ) determines a unique (x, y)

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

- each (x, y) corresponds to many (r, θ)



2.4 Curves in Polar Coordinates

Example 2.4.1 Give the polar equation for a circle centred at the origin, with radius 3.

- obviously $r = 3$ is the equation
- converting between polar and rectangular

  $r = 3 \Rightarrow \sqrt{x^2 + y^2} = 3,\ \text{or}\ x^2 + y^2 = 9$

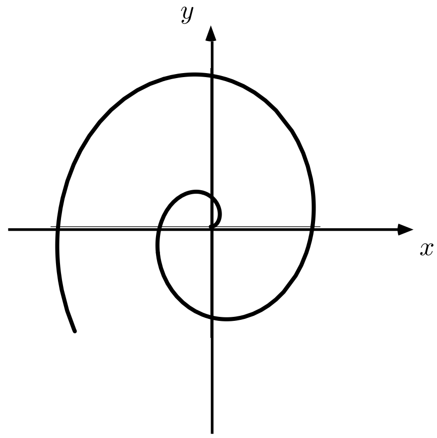
  $x^2 + y^2 = 9 \Rightarrow r^2 = 9,\ \text{or}\ r = \pm 3$
- $r = -3$ is also the equation of this circle!

Example 2.4.2 Describe the curve $r = \theta$.

- conversion to rectangular is no help

$$\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{y}{x}\right)$$

- think what happens
  - r increases with θ
- this is a spiral
  - the "spiral of Archimedes"



Example 2.4.3 Identify the polar equation $r = -6\cos\theta$.

- multiply by r

$$r^2 = -6r\cos\theta$$

- replace r, θ by x, y

$$x^2 + y^2 = -6x; \qquad x^2 + 6x + y^2 = 0$$

- complete the square

$$(x+3)^2 + y^2 = 9$$

- a circle, centre $(-3, 0)$, radius 3

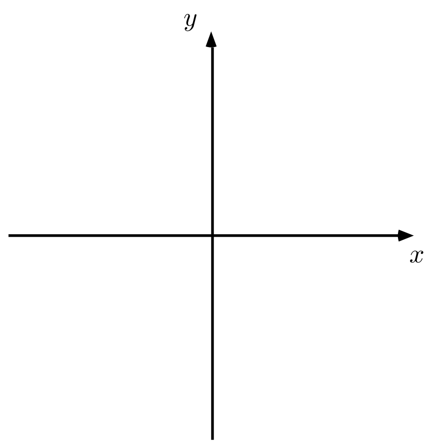
- how to sketch this from the polar form?

shape r

sign $\dfrac{dr}{d\theta} = 6\sin\theta$

- some values

θ	$r = -6\cos\theta$
0	-6
$\pi/2$	0
π	6
$3\pi/2$	0



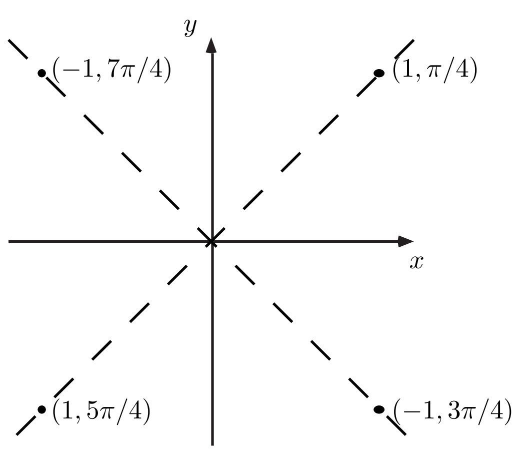
Example 2.4.4 Sketch $r = \sin 2\theta$.

shape r

sign $\frac{dr}{d\theta} = 2\cos 2\theta$

• some values

θ	$r = \sin 2\theta$
0	0
$\pi/4$	1
$3\pi/4$	-1
$5\pi/4$	1
$7\pi/4$	-1



Example 2.4.5 Sketch $r^2 = \sin 2\theta$.

shape r

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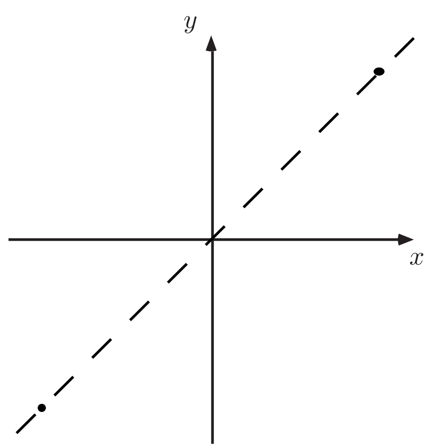
sign $\dfrac{dr}{d\theta} = \dfrac{2\cos 2\theta}{2r}$

• some values

θ	$r = \pm\sqrt{\sin 2\theta}$
0	0
$\pi/4$	± 1
$5\pi/4$	± 1



Example 2.4.6 Sketch $r = 1 - \sin\theta$.

shape r

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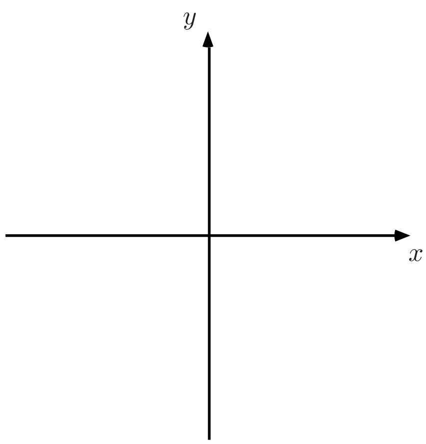
sign $\dfrac{dr}{d\theta} = -\cos\theta$

• some values

θ	$r = 1 - \sin\theta$
0	1
$\pi/2$	0
π	1
$3\pi/2$	2
2π	1



2.5 Tangents to Polar Curves

- parametric equations, parameter θ

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r = r(\theta), \qquad r' = \frac{dr}{d\theta}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

$$\text{(provided } \frac{dx}{d\theta} \neq 0\text{)}$$

Example 2.5.1 Find the equation of the line tangent to the curve $r = \sin 2\theta$ at $\theta = \pi/4$.

$$x = \sin 2\theta\cos\theta, \quad y = \sin 2\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\sin 2\theta\cos\theta + 2\cos 2\theta\sin\theta}{-\sin 2\theta\sin\theta + 2\cos 2\theta\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{1(1/\sqrt{2}) + 0}{-1(1/\sqrt{2}) + 0} = -1$$

$$x = y = 1/\sqrt{2}$$

- tangent line $\dfrac{y - (1/\sqrt{2})}{x - (1/\sqrt{2})} = -1$

Example 2.5.2 Find where the curve $r = \sin 2\theta$ has horizontal tangents and where it has vertical tangents.

$$\begin{aligned}
y &= \sin 2\theta \sin\theta,\ x = \sin 2\theta \cos\theta \\
\frac{dy}{d\theta} &= \sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta \\
&= 2\sin\theta\cos\theta\cos\theta + 2(\cos^2\theta - \sin^2\theta)\sin\theta \\
&= 2\sin\theta(2\cos^2\theta + -\sin^2\theta) \\
\frac{dx}{d\theta} &= -\sin 2\theta\sin\theta + 2\cos 2\theta\cos\theta \\
&= -2\sin\theta\cos\theta\sin\theta + 2(\cos^2\theta - \sin^2\theta)\cos\theta \\
&= 2\cos\theta(-2\sin^2\theta + \cos^2\theta)
\end{aligned}$$

- horizontal tangents
  - where $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} \neq 0$
- $dy/d\theta = 0$ on $0 \leq \theta < 2\pi$ (enough in this case)
  - when $\sin\theta = 0 \Longrightarrow \theta = 0, \pi$
  - when $2\cos^2\theta = \sin^2\theta \Longrightarrow \tan\theta = \pm\sqrt{2}$
  - $dx/d\theta \neq 0$ at these points
- horizontal tangents at

  $\theta = 0,\ \pi$ and where $\tan\theta = \pm\sqrt{2}$

- vertical tangents
  - where $\dfrac{dx}{d\theta} = 0$ and $\dfrac{dy}{d\theta} \neq 0$
- $\dfrac{dx}{d\theta} = 0$
  - when $\cos\theta = 0 \Longrightarrow \theta = \pi/2, 3\pi/2$
  - when $2\sin^2\theta = \cos^2\theta \Longrightarrow \tan\theta = \pm 1/\sqrt{2}$
  - $dy/d\theta \neq 0$ at these points
- vertical tangents at

  $\theta = \pi/2,\ 3\pi/2$ and where $\tan\theta = \pm 1/\sqrt{2}$

2.6 Tangents at the Origin (Pole)

- $r = 0$ at the origin

- assume α is angle where $r = 0$ and $r' \neq 0$

$$\frac{dy}{dx} = \frac{0 \cos\theta + r' \sin\theta}{-0 \sin\theta + r' \cos\theta} = \tan\alpha$$

- line through origin, slope $\tan\alpha$ is

$$\boxed{\theta = \alpha}$$

Example 2.6.1 Find tangents to the curve $r = \sin 2\theta$ at the origin (pole).

- curve has horizontal and vertical tangents at the origin

  – saw this before

- $r = 0$ when $\theta = 0, \pi/2, \pi, 3\pi/2$ on $0 \leq \theta < 2\pi$

  $r' \neq 0$ at these points

- tangents at the origin are

  $\theta = 0$, $\theta = \pi/2$, $\theta = \pi$, $\theta = 3\pi/2$

Example 2.6.2 Find where the curve $r = 1 - \cos\theta$ has horizontal and vertical tangents and describe the situation at the origin.

$$y = (1 - \cos\theta)\sin\theta, \quad x = (1 - \cos\theta)\cos\theta$$

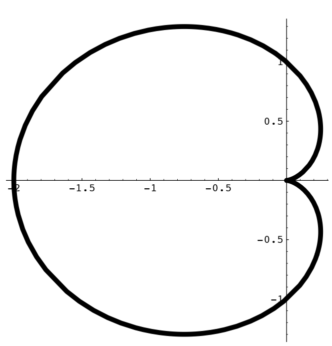
$$\begin{aligned}
\frac{dy}{d\theta} &= \sin\theta \sin\theta + (1 - \cos\theta)\cos\theta \\
&= (1 - \cos^2\theta) + (1 - \cos\theta)\cos\theta \\
&= (1 - \cos\theta)(1 + \cos\theta) + (1 - \cos\theta)\cos\theta \\
&= (1 - \cos\theta)(1 + 2\cos\theta) \\
\frac{dx}{d\theta} &= \sin\theta\cos\theta - (1 - \cos\theta)\sin\theta \\
&= \sin\theta(2\cos\theta - 1)
\end{aligned}$$

- $dy/d\theta = 0$ when $\cos\theta = 1$ or $\cos\theta = -1/2$
  - when $\theta = 0, 2\pi/3, 4\pi/3$
- $dx/d\theta = 0$ when $\sin\theta = 0$ or $\cos\theta = 1/2$
  - when $\theta = 0, \pi, \pi/3, 5\pi/3$
- horizontal tangents $\quad (dy/d\theta = 0,\ dx/d\theta \neq 0)$
  - when $\theta = 2\pi/3, 4\pi/3$
- vertical tangents $\quad (dy/d\theta \neq 0,\ dx/d\theta = 0)$
  - when $\theta = \pi, \pi/3, 5\pi/3$

- at the origin, $r = 1 - \cos\theta = 0$

  $\cos\theta = 1, \quad$ so $\theta = 0$
- when $\theta = 0$, $r = r' = dy/d\theta = dx/d\theta = 0$

$$\begin{aligned}\lim_{\theta\to 0}\frac{dy}{dx} &= \lim_{\theta\to 0}\frac{(1-\cos\theta)(1+2\cos\theta)}{\sin\theta(2\cos\theta-1)}\\ &= 3\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\sin\theta}\\ &= 3\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta} \qquad \text{(l'Hôpital)}\\ &= 0\end{aligned}$$



Chapter 3. Area and Integration

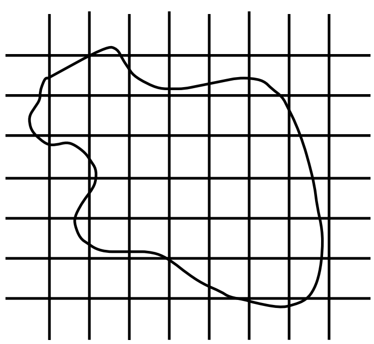
In which

We meet the other kind of calculus and learn the connection between areas and tangents.

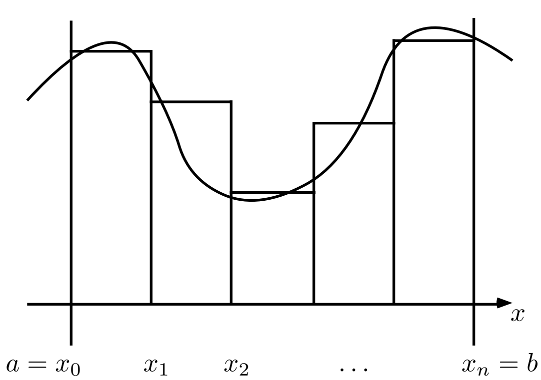
3.1 Area

• start with the area of a rectangle as $l \times w$

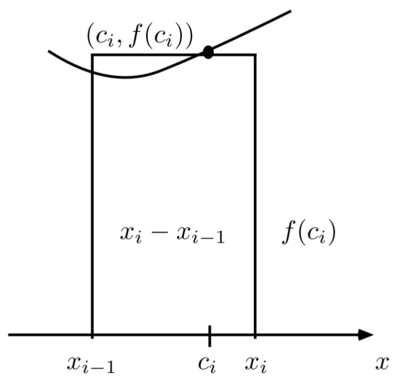
• approximate other areas with rectangles



• for functions $f \geq 0$



- typical approximating rectangle



- area $= f(c_i)(x_i - x_{i-1})$

- area under the curve $\approx \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$
- simpler if $x_i - x_{i-1}$ is always the same $(= \Delta x)$
- area $\approx \sum_{i=1}^{n} f(c_i)\Delta x$
- area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$

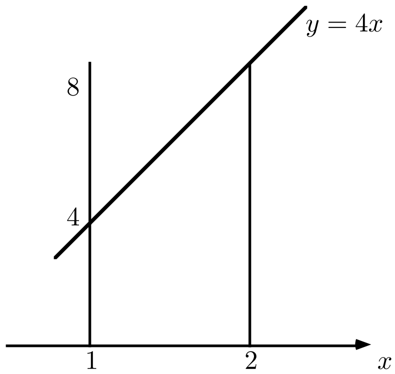
3.2 The definite integral

DEFINITION

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

- this is area if $f \geq 0$
- in case $f \not\geq 0$, $f(c_i)$ could be < 0 for some of the c_i
- definite integral gives algebraic sums of areas
  - above the x–axis counted $+$, below counted $-$

Example 3.2.1 Find $\int_1^2 4x\,dx$ from the definition.



- area (using elementary geometry)
  - area of triangle

$$\frac{1}{2}bh = \frac{1}{2}(2-1)(8-4) = 2$$

  - area of rectangle

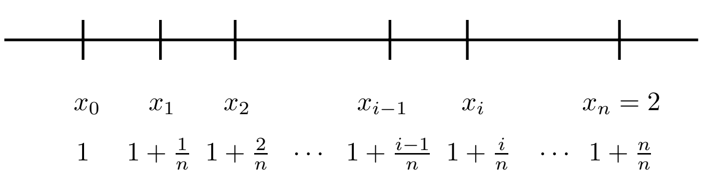
$$bh = (2-1)4 = 4$$

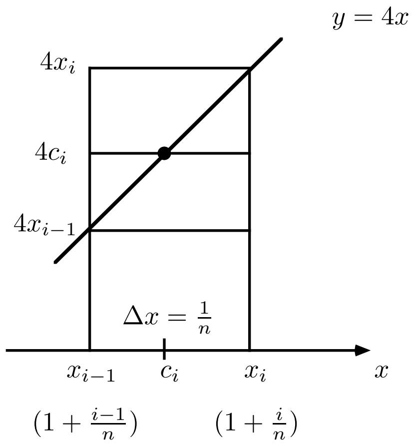
- total area

$$4 + 2 = 6$$

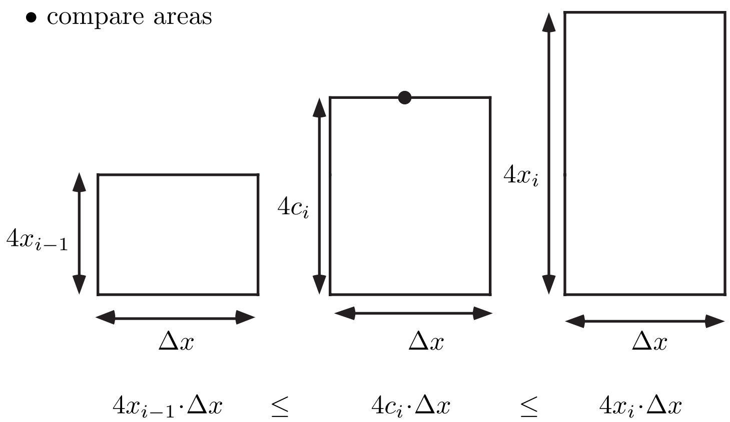
$$\int_1^2 4x\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 4c_i\,\Delta x$$

- show the limit exists and find its value
- divide the interval into n equal pieces.
- length of each piece is $\frac{1}{n}$ $(\Delta x = \frac{1}{n})$





- compare areas



- $x_{i-1} = 1 + \dfrac{i-1}{n}$
- $x_i = 1 + \dfrac{i}{n}$
- $\Delta x = \dfrac{1}{n}$

$$4\left(1 + \frac{i-1}{n}\right)\frac{1}{n} \;\leq\; 4c_i\frac{1}{n} \;\leq 4\left(1 + \frac{i}{n}\right)\frac{1}{n}$$

$$\sum_{i=1}^{n} \frac{4}{n} + \frac{4}{n^2}(i-1) \;\leq\; \int_1^2 4x\,dx \leq \sum_{i=1}^{n} \frac{4}{n} + \frac{4i}{n^2}$$

- to do the addition we need a formula

$$\boxed{\sum_{j=1}^{n} j = \frac{n(n+1)}{2}}$$

- the formula can be established by rewriting

$$1 + 2 + 3 + \ldots + 10$$

$$= (1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6)$$

$$= \frac{10}{2} \cdot 11 = 55$$

- a slightly different method is used for n odd

$$1 + 2 + 3 + \ldots + 9$$

$$= (1 + 8) + (2 + 7) + (3 + 6) + (4 + 5) + 9$$

$$= 9 \cdot \frac{10}{2} = 45$$

- a proof can also be given by mathematical induction
- set $n = 1$

$$\sum_{j=1}^{1} j = 1 = \frac{1(1+1)}{2}$$

- therefore the formula is true for $n = 1$

- assume true for some n, prove for $n + 1$

$$\begin{aligned}\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1), \quad \text{by inductive hypothesis} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

- this proves the formula with n replaced by $n + 1$
- the formula holds for all n by induction

$$\sum_{i=1}^{n} \frac{4}{n} + \frac{4}{n^2}(i-1) = \frac{4}{n}\sum_{i=1}^{n} 1 + \frac{4}{n^2}\sum_{i=1}^{n}(i-1)$$

$$= \frac{4}{n}n + \frac{4}{n^2}\sum_{j=0}^{n-1} j \qquad (\text{set } j = i-1)$$

$$= 4 + \frac{4}{n^2}\frac{(n-1)n}{2}$$

$$= 4 + \frac{2n^2 - 2n}{n^2}$$

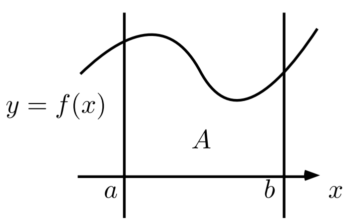
$$\sum_{i=1}^{n} \frac{4}{n} + \frac{4}{n^2} i = \frac{4}{n}n + \frac{4}{n^2}\sum_{i=1}^{n} i$$

$$= 4 + \frac{4}{n^2}\frac{n(n+1)}{2} = 4 + \frac{2n^2 + 2n}{n^2}$$

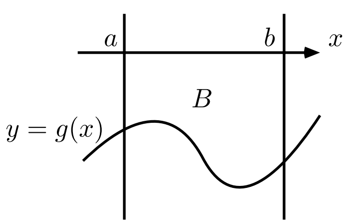
$$4 + 2\left(\frac{n^2 - n}{n^2}\right) \leq \int_1^2 4x\,dx \leq 4 + 2\left(\frac{n^2 + n}{n^2}\right)$$

- by the squeeze theorem $\int_1^2 4x\,dx = 6$

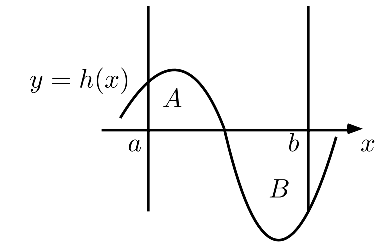
3.3 The definite integral and area



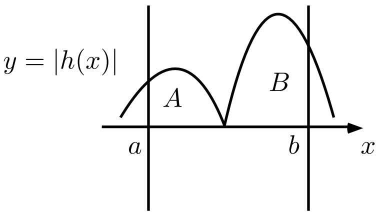
$$\int_a^b f = A$$



$$\int_a^b g = -B$$



$$\int_a^b h = A - B$$



$$\int_a^b |h| = A + B$$

$$\text{Area} = \int_a^b |f|$$

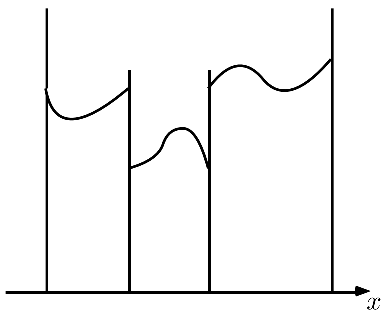
- gives the area bounded by
  - the curve $y = f(x)$
  - the x-axis
  - the lines $x = a$ and $x = b$, $a \leq b$

Summary

- start with the intuitive idea of area
  - this motivates the *definition* of definite integral
- give the definition of definite integral
  - definition does not depend on intuitive idea of area
- *define* area in terms of integral

3.4 Facts and miscellaneous information

- $\displaystyle\int_a^b f$ exists for (*piecewise*) continuous f on $[a, b]$
  - piecewise continuous



Example 3.4.1 (An integral which does not exist!)

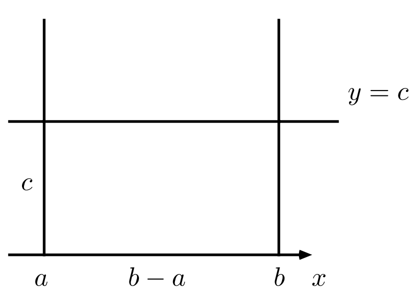
- let $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$
  - this is a "fringe"
- any interval contains rationals and irrationals
- for any Δx can choose a_i rational and b_i irrational

$$\sum f(a_i)\Delta x = 1 \qquad \sum f(b_i)\Delta x = 0$$

- $\displaystyle\lim_{\Delta x \to 0} \sum f(c_i)\Delta x$ does not exist

- for any constant c

$$\boxed{\int_a^b c\,dx = c(b-a)}$$



$$\boxed{\int_a^b f = \int_a^c f + \int_c^b f}$$

- clear if $a < c < b$
- we want this to be true for all values of a, b, and c
- therefore we need the following *definitions*

$$\boxed{\int_a^a f = 0} \qquad \boxed{\int_b^a f = -\int_a^b f}$$

- for k a constant

$$\boxed{\int_a^b kf = k\int_a^b f}$$

$$\boxed{\int_a^b (f+g) = \int_a^b f + \int_a^b g}$$

- both are to be read in the usual way
- they can be proved by looking at appropriate sums

- $f \geq 0,\ a \leq b \quad \Rightarrow \quad \int_a^b f \geq 0$
  - all terms in the sums are positive
- $f \leq g,\ a \leq b \quad \Rightarrow \quad \int_a^b f \leq \int_a^b g$
  - from the above since $g - f \geq 0$
- $m \leq f \leq M,\ a \leq b \ \Rightarrow \ m(b-a) \leq \int_a^b f \leq M(b-a)$
  - from the above since $\int_a^b m\,dx = m(b-a)$

- $a \le b \Rightarrow \left| \int_a^b f(x)\ dx \right| \le \int_a^b |f(x)|\ dx$

  $-|f(x)| \le f(x) \le |f(x)|$

  $\Rightarrow -\int_a^b |f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$

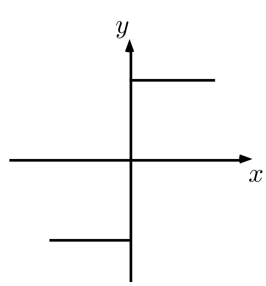
  $\Rightarrow \left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx$

3.5 Functions defined by integrals

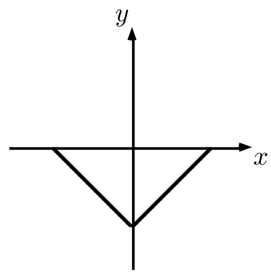
- $\int_a^x f(t)\, dt$ is a function of x
  - the algebraic sum of the area from a to x
- for continuous f, the function $\int_a^x f(t)\, dt$ is
  - continuous on $[a, b]$
  - differentiable on (a, b)

Example 3.5.1 Continuity of f is needed for differentiability of $\int_a^x f(t)\, dt$.

- let $f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$



- $\int_{-1}^{x} f(t)\,dt$



- not differentiable at $x = 0$

3.6 The Fundamental Theorems of Calculus

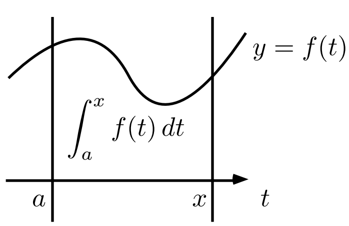
- two theorems
- differentiation and integration are "inverse processes"
  - "cancel" each another (in some sense)

  Fundamental Theorem: First Form

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$$

- hypothesis: f is continuous on $[a, b]$

- important to understand what this says
  - proof (later) will also help
- $\int_{a}^{x} f(t)\,dt$
  - gives "area" accumulated from a to x



- first form says

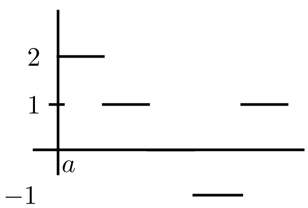
  – rate of change of area accumulation is equal to the value (height) of the function

- this makes sense

  – the bigger the function, the greater the area

- recall "area" means algebraic sum of area, area above axis counted positive, area below axis counted negative

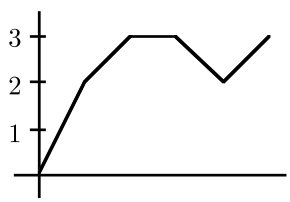
Example 3.6.1



$f(x)$

$\left(= \left(\int_a^x f(t)\,dt\right)',\right.$

when f is cont.)



$\int_a^x f(t)\,dt$

Example 3.6.2 Logarithm.

- $\ln x$ can be defined as an integral

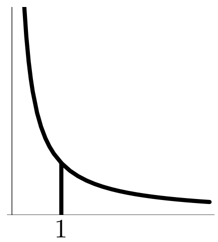
$$\ln x = \int_1^x \frac{dt}{t}$$

- fundamental theorem gives

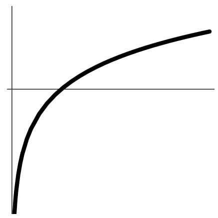
$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}$$

- this is an alternate way to develop logs and exponents

  – start with ln, define e^x as inverse of ln



$\frac{1}{x}$



$\ln x$

Fundamental Theorem: Second Form

$$\int_a^b \frac{dF}{dx}\,dx = F(b) - F(a)$$

- hypothesis: $\frac{dF}{dx}$ continuous on $[a, b]$
- notation $F(x)\Big|_a^b = F(b) - F(a)$
- so $\int_a^b \frac{dF}{dx}\,dx = F(x)\Big|_a^b = F(b) - F(a)$

! Example 3.6.3 Continuity is important.

- $\frac{1}{x^2}$ is not defined at $x = 0$
  - becomes infinite as approach 0
- blind application of fundamental theorem

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -1 - (- - 1) = -2$$

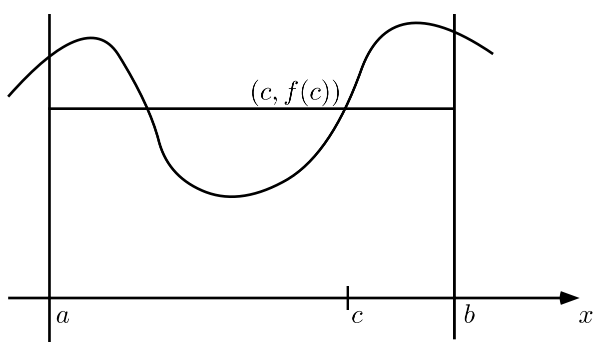
- this is clearly wrong (Why?)
- the integral does not exist

3.7 The Mean Value Theorem for Integrals

• assume f is continuous on $[a, b]$

• then there exists a c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$



Proof.

• the continuous function has a largest value $f(L) = M$ and a smallest value $f(l) = m$ on the closed interval $[a, b]$

• therefore, for all x in $[a, b]$,

$$f(l) = m \leq f(x) \leq M = f(L).$$

• $m \leq f \leq M$, $a \leq b$

$$\Rightarrow\ m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

- divide by $b-a$

$$f(l) = m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M = f(L).$$

- apply the Intermediate Value Theorem
  - says that for some c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

3.8 Proofs of Fundamental Theorems

First form:

- let $A(x) = \displaystyle\int_a^x f(t)\,dt$
- then $A'(x) = \displaystyle\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}$

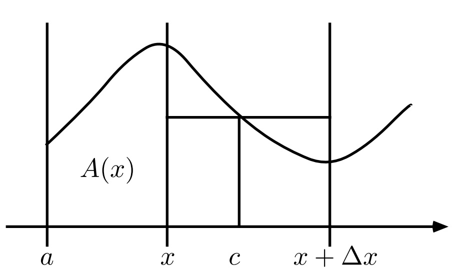
$$A(x+\Delta x) - A(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+\Delta x} f(t)\,dt$$

- by the mean value theorem for integrals

$$\int_x^{x+\Delta x} f(t)dt = f(c)\Delta x$$

for some c between x and $x+\Delta x$



$$\begin{aligned} A'(x) &= \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(c) \\ &= \lim_{c \to x} f(c) \\ &= f(x) \quad (f \text{ is cont.}) \end{aligned}$$

Second Form:

- recall $F' = G' \Rightarrow F = G + C$
- the first form says

$$F'(x) = \frac{d}{dx} \int_a^x F'(t)\,dt$$

- therefore

$$F(x) = \int_a^x F'(t)\,dt + C$$

- to find C, set $x = a$

$$F(a) = \int_a^a F'(t)\,dt + C = 0 + C$$

$$F(x) = \int_a^x F'(t)\,dt + F(a)$$

$$F(b) = \int_a^b F'(t)\,dt + F(a)$$

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

3.9 Examples

Example 3.9.1 Find $\int_2^5 3s^3\,ds$.

$$\int 3s^3\,ds = \frac{3}{4}s^4 \quad (= F(s);\ F'(s) = 3s^3)$$

$$\int_2^5 3s^3\,ds = \frac{3}{4}s^4\Big|_2^5$$

$$= \frac{3}{4}[5^4 - 2^4]$$

Example 3.9.2

$$\int_1^3 (x^2 + 3x + 5)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{3x^2}{2} + 5x\right)\Big|_1^3$$

$$= 9 + \frac{27}{2} + 15 - \frac{1}{3} - \frac{3}{2} - 5$$

$$= \dots$$

Example 3.9.3 Find $\int_{-5}^{15} |2x+6|\,dx$.

- kind of thing needed for areas

$$|2x+6| = \begin{cases} 2x+6 & \text{if } 2x+6 \geq 0 \ \text{ i.e. } x \geq -3; \\ -(2x+6) & \text{if } 2x+6 \leq 0 \ \text{ i.e. } x \leq -3. \end{cases}$$

$$\int_{-5}^{15} |2x+6|\,dx = \int_{-5}^{-3} -(2x+6)\,dx + \int_{-3}^{15} (2x+6)\,dx$$

$$= -(x^2+6x)\Big|_{-5}^{-3} + (x^2+6x)\Big|_{-3}^{15}$$

$$= -[(9-18) - (25-30)] + 225 + 90 - (9-18)$$

$$= \dots$$

Example 3.9.4 Find the area bounded $y = x^2 - 4$ and the x–axis.

$$A = \int_a^b |f|$$

• need the limits of integration

---

sign $y = (x-2)(x+2)$

$$\begin{aligned}
A &= \int_{-2}^{2} |x^2 - 4|\, dx \\
&= \left| \int_{-2}^{2} (x^2 - 4)\, dx \right| \\
&= \left| \left[\frac{x^3}{3} - 4x \right]_{-2}^{2} \right| \\
&= \left| \frac{8}{3} - 8 - \left[\frac{-8}{3} + 8 \right] \right| \\
&= \left| \frac{16}{3} - 16 \right| = 16 - \frac{16}{3}
\end{aligned}$$

Example 3.9.5 Find the area between $y = x^3 + 2x^2 - x - 2$ and the x–axis.

---

sign $y = (x-1)(x+1)(x+2)$

$$\begin{aligned}
A &= \int_{-2}^{1} |x^3 + 2x^2 - x - 2|\, dx \\
&= \left| \int_{-2}^{-1} (x^3 + 2x^2 - x - 2)\, dx \right| + \left| \int_{-1}^{1} (x^3 + 2x^2 - x - 2)\, dx \right|
\end{aligned}$$

$$= \left| \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x \right]_{-2}^{-1} \right| + \left| \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^{1} \right|$$

$$= \left| \left[\frac{1}{4} + \frac{-2}{3} - \frac{1}{2} + 2 \right] - \left[4 - \frac{16}{3} - 2 + 4 \right] \right|$$

$$+ \left| \left[\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2 \right] - \left[\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2 \right] \right|$$

$$= \dots$$

Example 3.9.6 Find the area between $y = \sin x$, the x–axis, $x = -\pi/3$, and $x = \pi/3$.

• sign y ______________________________

$$\begin{aligned} A &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} |\sin x| \, dx \\ &= \left| \int_{-\frac{\pi}{3}}^{0} \sin x \, dx \right| + \left| \int_{0}^{\frac{\pi}{3}} \sin x \, dx \right| \\ &= 2 \int_{0}^{\frac{\pi}{3}} \sin x \, dx \\ &= 2(-\cos x) \Big|_{0}^{\frac{\pi}{3}} = 2[-\frac{1}{2} + 1] = 1 \end{aligned}$$

Example 3.9.7 Find $F'(3)$ if $F(x) = \int_1^x \frac{s^2}{s+5} \, ds$.

$$F'(x) = \frac{x^2}{x+5}$$

$$F'(3) = \frac{3^2}{3+5} = \frac{9}{8}$$

Example 3.9.8 Find $\frac{d}{dx}\int_{ax}^{bx}\frac{dt}{5+t^4}$.

- let $f(x)=\int_c^x \frac{dt}{5+t^4}$

- then $f'(x)=\frac{1}{5+x^4}$ (Fundamental Theorem)

$$\int_c^{bx}\frac{dt}{5+t^4}=f(bx)$$

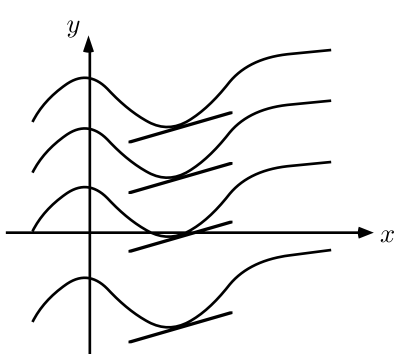
$$\int_{ax}^{bx}\frac{dt}{5+t^4}=\int_{ax}^{c}\frac{dt}{5+t^4}+\int_c^{bx}\frac{dt}{5+t^4}$$

$$=-f(ax)+f(bx)$$

$$\begin{aligned}\frac{d}{dx}\int_{ax}^{bx}\frac{dt}{5+t^4}&=(f(bx)-f(ax))'\\&=bf'(bx)-af'(ax)\\&=\frac{b}{5+(bx)^4}-\frac{a}{5+(ax)^4}\end{aligned}$$

3.10 Antiderivatives and indefinite integrals

Theorem. *If $F'(x)=G'(x)$ on some interval I, then $F(x)=G(x)+C$ on I, where C is constant.*



- if $F'(x) = f(x)$ on an interval I
  - then F is an *antiderivative* of f on I.

Example 3.10.1

$\frac{x^3}{3}$ is an antiderivative for x^2

$\frac{x^3}{3} + 17$ is also an antiderivative for x^2

- any constant is an antiderivative of 0
- the Theorem says that any two antiderivatives differ only by a constant (on some interval)
  - a constant C is the most general antiderivative of 0
  - $\frac{x^3}{3} + C$ is the most general antiderivative for x^2

- write

$$\int f(x)\,dx = F(x) + C \iff F'(x) = f(x)$$

- $\int f(x)\,dx$ is the *indefinite integral* of $f(x)$
  - it is the most general antiderivative of $f(x)$
- $\int$ is the *integral* sign

$$\int \frac{dF}{dx}\,dx = F(x) + C$$

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

- $\int$ and $\frac{d}{dx}$ "cancel"
- indefinite integration is the "inverse" of differentiation
  - the reason for the name "antidifferentiation"

- the fundamental theorem says

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Bigg|_a^b$$

- explains similar notation for two entirely different ideas
- we can ignore the constant of integration
  - let $F = G + C$

$$F(x)\Big|_a^b = F(b) - F(a) = [G(b) + C] - [G(a) + C]$$

$$= G(b) - G(a) = G(x)\Big|_a^b$$

Example 3.10.2

$$\int x^6 + x^3\,dx = \frac{x^7}{7} + \frac{x^4}{4} + C$$

checking

$$\left(\frac{x^7}{7} + \frac{x^4}{4}\right)' = x^6 + x^3$$

Example 3.10.3

$$\int 5x^2\,dx = 5\int x^2\,dx = 5\left(\frac{x^3}{3}\right) + C$$

General Rules:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 3.10.4

$$\int \frac{1}{\sqrt{x+1}}\,dx$$

- know $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$
- try $(2\sqrt{x+1})' = \dfrac{1}{\sqrt{x+1}}$
- so $\displaystyle\int \frac{1}{\sqrt{x+1}}\,dx = 2\sqrt{x+1} + C$

3.11 Applications

Example 3.11.1 Find $f(x)$ if $f'(x) = x^2 - x + 2$ and $f(1) = 3$. (An initial value problem.)

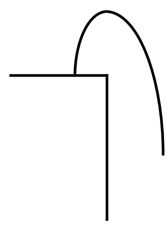
$$f(x) = \int (x^2 - x + 2)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + 2x + C$$

$$3 = f(1) = \frac{1}{3} - \frac{1}{2} + 2 + C$$

$$C = 3 - \frac{1}{3} + \frac{1}{2} - 2 = \frac{7}{6}$$

$$f(x) = \frac{x^3}{3} - \frac{x^2}{2} + 2x + \frac{7}{6}$$

Example 3.11.2 A ball is thrown upwards at 1 m/sec from the top of a 100 m high building. When does it hit the ground and how fast is it going then? When does it reach its highest point and how high is it then? (Acceleration due to gravity is 9.8 m/sec^2.)



$$a = \frac{d^2s}{dt^2} = -9.8$$

$$v = \int a\,dt = \int -9.8\,dt = -9.8t + C$$

- when $t = 0$, $v = v(0) = v_0 = C$, the *initial* velocity

$$v = -9.8t + v_0$$

- in this case $v_0 = 1$

$$\boxed{v = -9.8t + 1}$$

$$s = \int v\,dt = \int (-9.8t + 1)\,dt = \frac{-9.8t^2}{2} + t + s_0$$

- s_0 is the position when $t = 0$
- in this case, $s_0 = 100$

$$\boxed{s = -4.9t^2 + t + 100}$$

- $s = 0$ when $t = \dfrac{-1 \pm \sqrt{1 - 4(-4.9)100}}{2(-4.9)}$

The ball hits the ground when $t = \dfrac{-1 - \sqrt{1961}}{-9.8} \approx 4.6$ sec.

Its velocity then is $v \approx -9.8(4.6) + 1 \approx -44$ m/sec.

– note that $v < 0$; the ball is going down

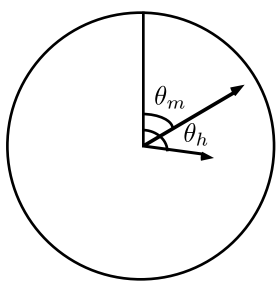
- the highest point occurs when $v = 0$

– when $-9.8t + 1 = 0$

The ball is highest when $t = \dfrac{1}{9.8}$ sec.

Highest point reached is $\dfrac{-9.8}{2}\left(\dfrac{1}{9.8}\right)^2 + \dfrac{1}{9.8} + 100$ m.

Example 3.11.3 Find the first time after 3 o'clock when the hands of a watch are together.



- θ_m – the angle the minute hand makes with noon
- θ_h – the angle the hour hand makes with noon

$$\frac{d\theta_m}{dt} = 1 \text{ rev/hr} = 2\pi \text{ radians/hr}$$

$$\frac{d\theta_h}{dt} = \frac{1}{12} \text{ rev/hr} = \frac{2\pi}{12} \text{ radians/hr}$$

$$\theta_m = \int 2\pi\, dt = 2\pi t + C_m$$

$$\theta_h = \int \frac{2\pi}{12}\, dt = \frac{2\pi}{12} t + C_h$$

$$C_m = 0;\ C_h = \frac{\pi}{2}\ (t = 0 \text{ at 3 o'clock})$$

$$\theta_m = 2\pi t;\ \theta_h = \frac{2\pi}{12}t + \frac{\pi}{2}$$

$$\text{Solve } 2\pi t = \frac{2\pi}{12}t + \frac{\pi}{2}$$

$$2\pi\left(\frac{11}{12}\right)t = \frac{\pi}{2}$$

$$t = \frac{3}{11} \text{ hours}$$

The hands are together at 3/11 hours = 180/11 min past 3 o'clock.

Chapter 4. Integration by Substitution

In which

We learn that integration is not the neat algorithm that differentiation is and that special methods abound.

MEMORIZE!

$dx^{r+1} = (r+1)x^r\,dx$	$d\ln x = \frac{1}{x}\,dx$
$de^x = e^x\,dx$	$da^x = a^x \ln a\,dx$
$d\sin x = \cos x\,dx$	$d\cos x = -\sin x\,dx$
$d\tan x = \sec^2 x\,dx$	$d\cot x = -\csc^2 x\,dx$
$d\sec x = \sec x \tan x\,dx$	$d\csc x = -\csc x \cot x\,dx$
$d\tan^{-1}\frac{x}{a} = \frac{a}{x^2+a^2}\,dx$	$d\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}\,dx$

• note: $\arctan x = \tan^{-1} x$; $\arcsin x = \sin^{-1} x$; etc.

4.1 Substitution

• chain rule

$$[F(g(x))]' = F'(g(x)) \cdot g'(x)$$

• integrate both sides

$$[F(g(x))] + C = \int [F(g(x))]'\,dx = \int F'(g(x)) \cdot g'(x)\,dx$$

• gives **substitution formula**

$$\boxed{\int F'(g(x)) \cdot g'(x)\,dx = F(g(x)) + C}$$

- let $f = F'$ or equivalently $\int f(u)\,du = F(u) + C$

- when $u = g(x)$, substitution becomes

$$\boxed{\int f(g(x)) \cdot g'(x)\,dx = \int f(g(x))\,dg(x)}$$

(says we can replace $g'(x)dx$ by $dg(x)$ inside an integral)

- in the other notation

$$\boxed{\int f(u)\,\frac{du}{dx}\,dx = \int f(u)\,du \quad (= F(u) + C)}$$

4.2 Examples

Example 4.2.1 $\quad I = \int (2x+4)(x^2+4x+9)^7\,dx$

- let $u = x^2 + 4x + 9; \qquad du = \dfrac{du}{dx}\,dx = (2x+4)\,dx$

$$\begin{aligned} I &= \int u^7\,du \\ &= \frac{u^8}{8} + C = \frac{(x^2+4x+9)^8}{8} + C \end{aligned}$$

- check $\left[\dfrac{(x^2+4x+9)^8}{8}\right]' = \dfrac{8(x^2+4x+9)^7}{8}(2x+4)$

Example 4.2.2 $\quad I = \int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

- let $u = \sqrt{x}$

- then $du = \dfrac{1}{2\sqrt{x}}\,dx; \quad \dfrac{dx}{\sqrt{x}} = 2\,du$

$$I = 2\int \sin u\,du \qquad \text{where } u = \sqrt{x}$$

$$= 2(-\cos u) + C = -2\cos\sqrt{x} + C$$

- check $(-2\cos\sqrt{x})' = -2(-\sin\sqrt{x})\dfrac{1}{2\sqrt{x}} = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$

Example 4.2.3 $I = \int \frac{\ln t}{t}\, dt$

- let $u = \ln t$
- then $du = \frac{1}{t}\, dt$

$$I = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln t)^2}{2} + C$$

- check $\left[\frac{(\ln t)^2}{2}\right]' = \frac{2\ln t}{2} \cdot \frac{1}{t} = \frac{\ln t}{t}$

4.3 Definite Integrals

$$\int_a^b F'(g(x))\, g'(x)\, dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

- let $B = g(b)$, and $A = g(a)$)

$$\begin{aligned} F(g(b)) - F(g(a)) = F(B) - F(A) &= F(u)\Big|_A^B \\ &= \int_A^B F'(u)\, du \end{aligned}$$

- write $F' = f, \quad u = g(x)$

$$\boxed{\int_a^b f(u(x))\, \frac{du}{dx}\, dx = \int_A^B f(u)\, du}$$

Example 4.3.1 $I = \int_0^1 x\sqrt{2-x}\, dx$

- let $u = \sqrt{2-x} \quad (\Rightarrow x = 2 - u^2)$
- then $dx = -2u\, du$

x	$u = \sqrt{2-x}$
1	1
0	$\sqrt{2}$

$$\int_0^1 \quad x \quad \sqrt{2-x} \quad dx$$

$$= \int_{\sqrt{2}}^1 \quad (2-u^2) \quad u \quad (-2u\, du)$$

$$= \int_{\sqrt{2}}^{1} (2u^4 - 4u^2)\, du$$

$$= \left[\frac{2}{5}u^5 - \frac{4}{3}u^3\right]_{\sqrt{2}}^{1}$$

$$= \left(\frac{2}{5} - \frac{4}{3}\right) - \left(\frac{2}{5}(\sqrt{2})^5 - \frac{4}{3}(\sqrt{2})^3\right) = \frac{16}{15}\sqrt{2} - \frac{14}{15}$$

Alternate Method:

$$\int x\sqrt{2-x}\, dx = \frac{2}{5}u^5 - \frac{4}{3}u^3$$

$$= \frac{2}{5}(2-x)^{5/2} - \frac{4}{3}(2-x)^{3/2}$$

$$\int_0^1 x\sqrt{2-x}\, dx = \left[\frac{2}{5}(2-x)^{5/2} - \frac{4}{3}(2-x)^{3/2}\right]_0^1$$

$$= \left(\frac{2}{5} - \frac{4}{3}\right) - \left(\frac{2}{5}2^{5/2} - \frac{4}{3}2^{3/2}\right)$$

[!]

- $I \neq \int_0^1 (2u^4 - 4u^2)\, du$
- $\int_0^1 (2u^4 - 4u^2)\, du = \left[\frac{2}{5}u^5 - \frac{4}{3}u^3\right]_0^1 = \frac{2}{5} - \frac{4}{3}$
  - don't write something which is wrong!
- $I = \int_{x=0}^{x=1} (2u^4 - 4u^2)\, du$ is occasionally used
  - reminder to replace u by x before evaluation
- first method is usually easier and is preferable

Example 4.3.2 $I = \int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\, dx.$

• let $u = \pi \ln x$; $du = \pi \frac{dx}{x}$;

x	$u = \pi \ln x$
$\sqrt{e}$	$\pi \ln \sqrt{e} = \pi/2$
1	$\pi \ln 1 = 0$

$$I = \frac{1}{\pi} \int_0^{\pi/2} \sin(u)\, du$$

$$= -\frac{\cos(u)}{\pi}\Big|_0^{\pi/2}$$

$$= -\frac{\cos(\pi/2)}{\pi} + \frac{\cos(0)}{\pi} = \frac{1}{\pi}$$

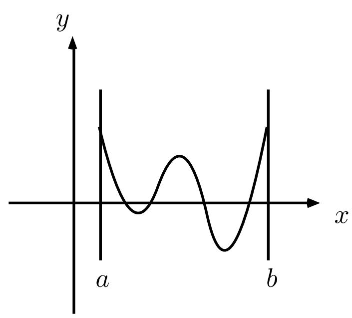
Chapter 5. Applications of Integration

In which

We use our knowledge of integration to find areas and volumes. We also learn how to average functions.

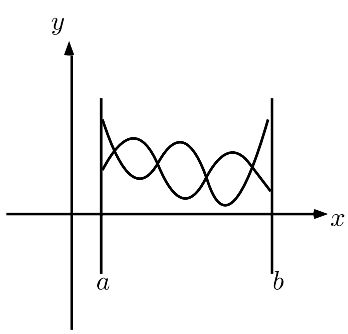
5.1 Area

• area between a curve $y = f(x)$ and the x-axis.



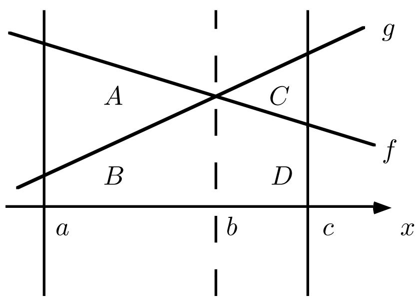
$$\text{Area} = \int_a^b |f(x)|\, dx$$

• area bounded by $f(x)$ and $g(x)$



$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx$$

- formula for area between curves
  - follows from area between curve and x-axis
  - just subtract
  - draw a series of pictures which illustrates this



- area between f and g (from a to c) $= A + C$

$$A + B = \int_a^b f\ ; \qquad B = \int_a^b g$$

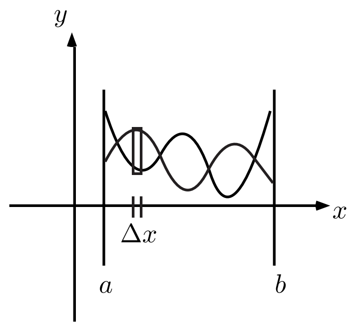
$$A = (A + B) - B = \int_a^b f - \int_a^b g = \int_a^b f - g$$

$$C + D = \int_b^c g\ ; \qquad D = \int_b^c f$$

$$C = (C + D) - D = \int_b^c g - \int_b^c f = \int_b^c g - f$$

$$A + C = \int_a^b f - g + \int_b^c g - f = \int_a^c |f - g|$$

- formula can also be obtained
  - directly from the definition of the integral



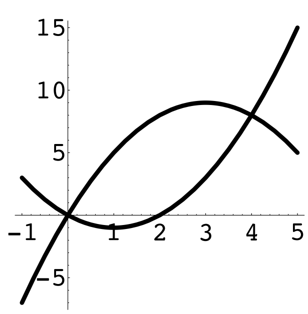
$$A \approx \sum_{i=1}^{n} |f(c_i) - g(c_i)| \Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} |f(c_i) - g(c_i)| \Delta x$$

$$= \int_a^b |f(x) - g(x)|\, dx$$

Example 5.1.1 Find the area bounded between

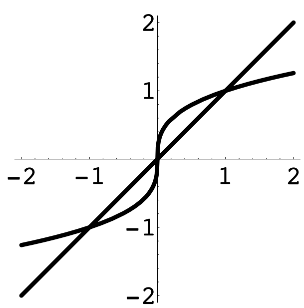
$$y = x^2 - 2x \text{ and } y = 6x - x^2.$$



$$
\begin{aligned}
A &= \int_0^4 |x^2 - 2x - (6x - x^2)|\,dx \\
&= \left|\int_0^4 x^2 - 2x - (6x - x^2)\,dx\right| \\
&= \left|\int_0^4 (2x^2 - 8x)\,dx\right| \\
&= \left|\left[\frac{2x^3}{3} - \frac{8x^2}{2}\right]_0^4\right| = \left|\frac{2\cdot 4^3}{3} - \frac{8\cdot 4^2}{2}\right|
\end{aligned}
$$

Example 5.1.2 Find the area between

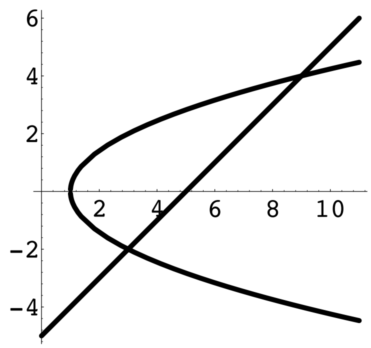
$$y = x \text{ and } y = x^{1/3}.$$



$$
\begin{aligned}
A &= \int_{-1}^1 |x - x^{1/3}|\,dx \\
&= \int_{-1}^0 (x - x^{1/3})\,dx + \int_0^1 (x^{1/3} - x)\,dx \\
&= \left|\int_{-1}^0 (x - x^{1/3})\,dx\right| + \left|\int_0^1 (x - x^{1/3})\,dx\right| \\
&= \ldots
\end{aligned}
$$

Example 5.1.3 Find the area between

$$y^2 = 2x - 2 \text{ and } y = x - 5.$$



- to find where the curves intersect solve

$$(x-5)^2 = 2x - 2$$

$$x^2 - 10x + 25 = 2x - 2$$

$$x^2 - 12x + 27 = (x-3)(x-9) = 0$$

$$x = 3; \quad y = 3 - 5 = -2$$

$$x = 9; \quad y = 9 - 5 = 4$$

- note that $2x - 2\ (= y^2) \geq 0$, so $x \geq 1$.

$$y^2 = 2x - 2; \quad y = \pm\sqrt{2x-2}$$

$$A = \int_1^3 2\sqrt{2x-2}\,dx + \int_3^9 \sqrt{2x-2} - (x-5)\,dx$$

- interchange the roles of x and y
- area between $x = \dfrac{y^2+2}{2}$ and $x = y + 5$

$$A = \int_{-2}^4 y + 5 - \left(\frac{y^2+2}{2}\right)\,dy$$

- this is clearly a better strategy

5.2 Area for Parametric Curves

- for y above the x-axis

$$A = \int_a^b y\, dx$$

- if y is continuous and x is differentiable

$$A = \int_\alpha^\beta y\, \frac{dx}{dt}\, dt$$

- essentially a change of variable
- gives area if $y\, \dfrac{dx}{dt} \geq 0$ and $\alpha \leq \beta$

Example 5.2.1 Find the area bounded by
$x = t^2 - t;\ y = t^2 + t$ and the x-axis.

- we sketched this earlier
- $y = 0$ when $t = -1$ and $t = 0$

$$\int_{-1}^{0} (t^2 + t)(2t - 1) dt = \int_{-1}^{0} (2t^3 + t^2 - t)\, dt$$
$$= \frac{t^4}{2} + \frac{t^3}{3} - \frac{t^2}{2}\Big|_{-1}^{0} = \frac{1}{3}$$

! This "area" is below the x-axis. What's going on?

- y and $\dfrac{dx}{dt}$ are ≤ 0, curve is below axis, moving left

Example 5.2.2 Find the area bounded by
$x = t^2 - t;\ y = t^2 + t$ and the line $x = 3/4$.

- solve $t^2 - t = 3/4$

$$t^2 - t - 3/4 = (t + 1/2)(t - 3/2)$$
$$t = -1/2,\ t = 3/2$$

$$A = \int_{-1/2}^{3/2} (t^2 + t)(2t - 1)\, dt$$

- why is this the area wanted?
  - look at the sketch and analyze the signs

sign $y = t^2 + t$ ________________________

sign $\frac{dx}{dt} = 2t - 1$ ________________________

sign product ________________________

Example 5.2.3 Find the area bounded by
$x = 4\cos^3 t;\ y = 4\sin^3 t.$

- range of t to get the curve; $0 \le t \le 2\pi$

$$\int_0^{2\pi} 4(\sin^3 t)[12\cos^2 t(-\sin t)]\ dt$$

! • this integral does *not* give the area

- sign of integrand is negative
  - counterclockwise around curve
  - correct value, wrong sign

Example 5.2.4 Find the area of a circle.

- parametric equations: $x = a\cos\theta,\ y = a\sin\theta$
- circle is covered counterclockwise

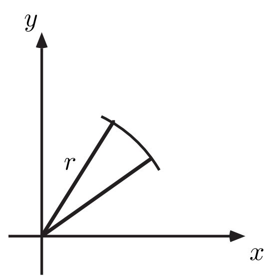
$$A = -\int_0^{2\pi} a\sin\theta(-a\sin\theta)\ d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi}(1 - \cos 2\theta)\ d\theta$$

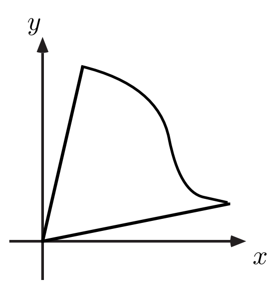
$$= \frac{1}{2}a^2\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \pi a^2$$

5.3 Area in Polar Coordinates

- area of a sector of a circle



- area $= \frac{r^2}{2}\Delta\theta \qquad \left(\frac{A}{\pi r^2} = \frac{\Delta\theta}{2\pi}\right)$



$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta}$$

Example 5.3.1 Find the area of a circle.

- equation of circle, centre $(0,0)$, radius a, is $r = a$
- area

$$A = \frac{1}{2}\int_0^{2\pi} a^2\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} d\theta$$

$$= \frac{1}{2}a^2\theta\Big|_0^{2\pi} = \frac{1}{2}a^2\cdot 2\pi = \pi a^2$$

Example 5.3.2 Find the area of one leaf of $r = \sin 2\theta$.

- review sketch (example 2.4.2)
  - one leaf is swept out when $0 \le \theta \le \dfrac{\pi}{2}$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2 \, d\theta$$

  - other choices are possible

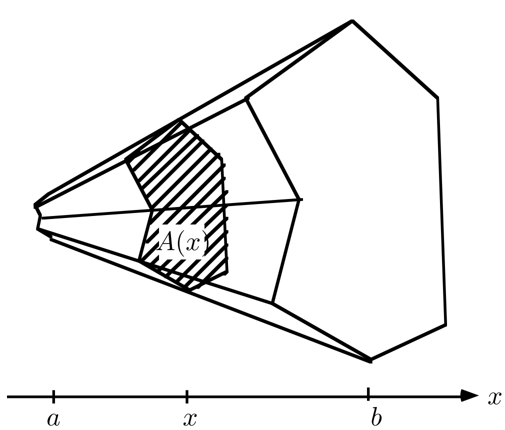
$$A = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (\sin 2\theta)^2 \, d\theta$$

Example 5.3.3 Find the area of $r = 1 - \sin\theta$.

- review sketch (example 2.4.6)
  - completely cover area when $0 \le \theta \le 2\pi$

$$A = \frac{1}{2}\int_0^{2\pi} (1 - \sin\theta)^2 \, d\theta$$

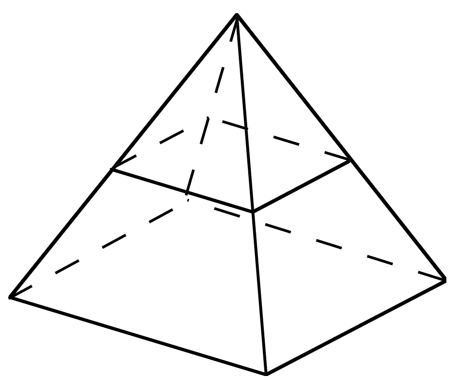
5.4 Volume

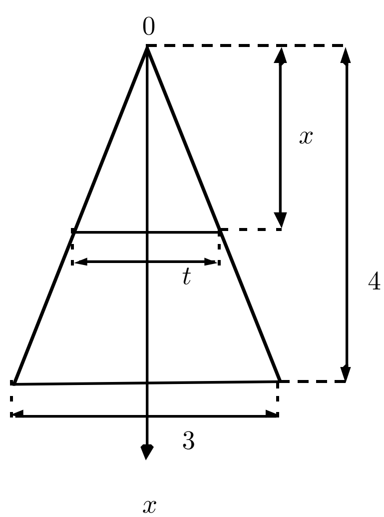


- slice volume at x
  - with a plane $\perp$ to x-axis
  - $A(x)$ is cross-sectional area at x
  - volume is "sum" of these areas

$$V = \int_a^b A(x)\,dx$$

Example 5.4.1 Find the volume of a pyramid which is 4m high and has a square base 3m on each side.





• by similar triangles

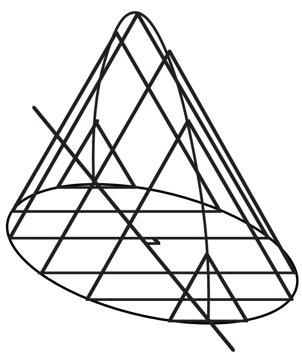
$$\frac{x}{t} = \frac{4}{3} \Rightarrow t = \frac{3}{4}x$$

$$A(x) = \left(\frac{3}{4}x\right)^2$$

$$V = \int_0^4 A(x)\,dx = \frac{9}{16}\int_0^4 x^2\,dx$$

$$= \frac{9}{16} \cdot \frac{x^3}{3}\bigg|_0^4 = 12 \text{ m}^3$$

Example 5.4.2 Find the volume of a solid which has a circular base of radius r and sections perpendicular to a diameter which are equilateral triangles.



$$A(x) = \frac{1}{2}2\sqrt{r^2 - x^2}\sqrt{3}\sqrt{r^2 - x^2} = \sqrt{3}(r^2 - x^2)$$

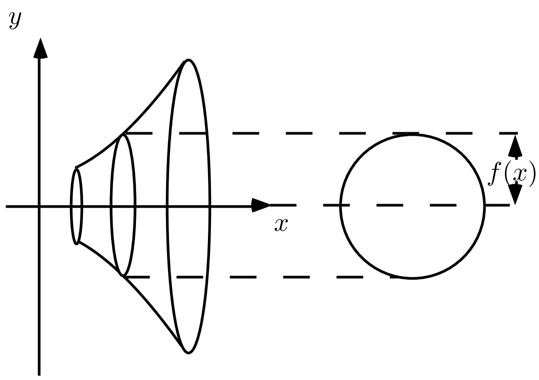
$$V = \int_{-r}^{r} A(x)\,dx = 2\int_0^r \sqrt{3}(r^2 - x^2)\,dx$$

$$= 2\sqrt{3}\left[r^2x - \frac{x^3}{3}\right]_0^r$$

$$= 2\sqrt{3}\left[r^3 - \frac{r^3}{3}\right] = \frac{4}{\sqrt{3}}r^3$$

5.5 Solids of revolution by discs (and washers)

- revolution of $f(x)$ about the x-axis



- cross section is a circle

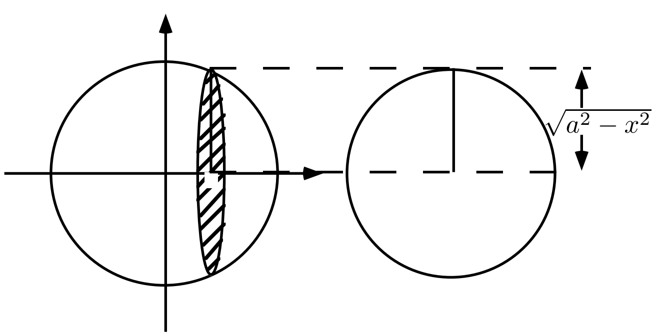
$$A(x) = \pi\,[f(x)]^2$$

- sum these “discs”

$$\boxed{V = \pi \int_a^b (f(x))^2\; dx}$$

- disc method

Example 5.5.1 Find the volume of a sphere of radius a.



- $A(x) = \pi(\sqrt{a^2 - x^2})^2 = \pi(a^2 - x^2)$

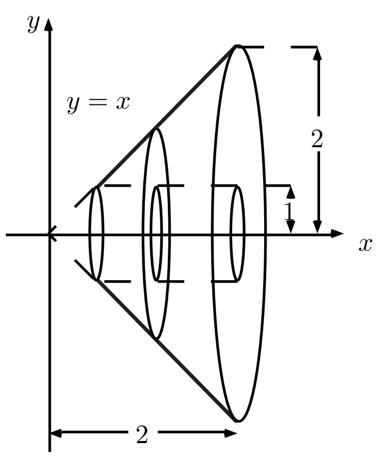
$$V = \pi \int_{-a}^{a} (a^2 - x^2)\, dx$$

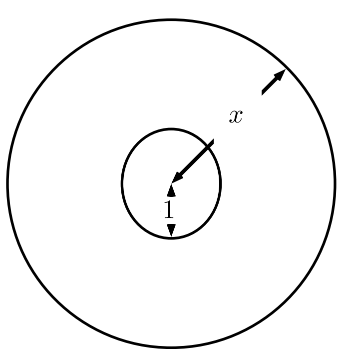
$$= 2\pi \int_{0}^{a} (a^2 - x^2)\, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\pi \cdot 2\frac{a^3}{3} = \frac{4}{3}\pi a^3$$

Example 5.5.2 A hole of radius 1 is bored along the axis of a cone of height 2 and with radus of base 2. Find a formula for the volume left in the cone.





- cross section at x

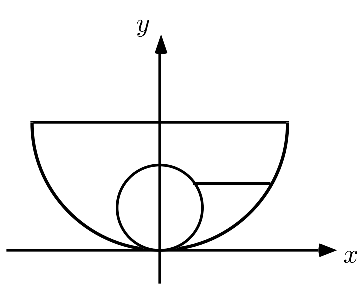
- the cross section is a "washer"
- area of cross section is difference of area of two circles
  - smaller of radius 1; $A = \pi 1^2 = \pi$
  - larger of radius x; $A = \pi x^2$
- total area at x

$$A(x) = \pi x^2 - \pi$$

- limits $x = 1$ to $x = 2$

$$V = \int_1^2 \pi x^2 - \pi \, dx$$

Example 5.5.3 A ball of radius 10 is placed in a hemisphere shaped bowl of radius 30. How much water is in the bowl when the depth is h?



- revolution about the y-axis
  - the sections are washers
  - could put the bowl on its side
- big sphere equation

  $x^2 + (y - 30)^2 = 30^2$

- small sphere

  $x^2 + (y - 10)^2 = 10^2$

- if $h \leq 20$

$$V = \pi \int_0^h (30^2 - (y-30)^2) - (10^2 - (y-10)^2)\, dy$$

- if $h > 20$

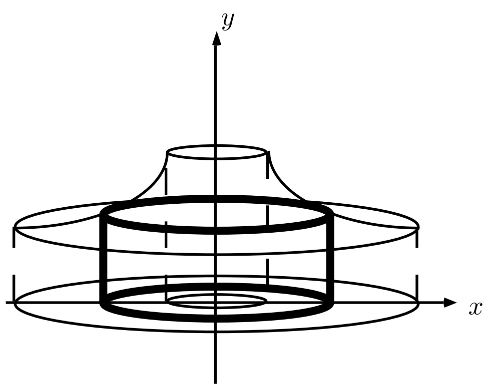
$$V = \pi \int_0^{20} (30^2 - (y-30)^2) - (10^2 - (y-10)^2)\, dy$$

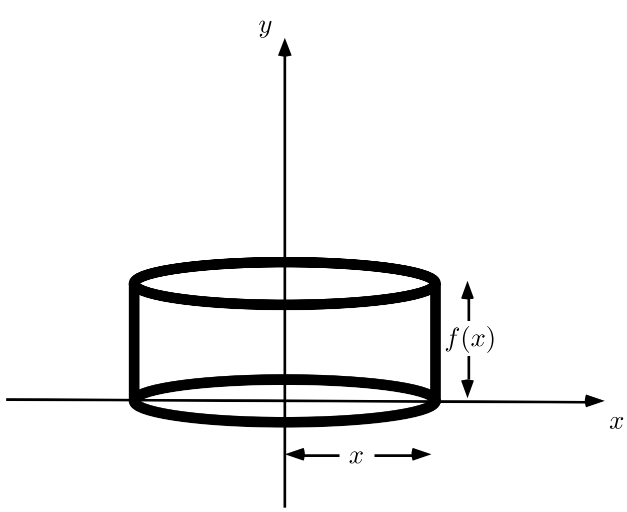
$$+\pi \int_{20}^h (30^2 - (y-30)^2)\, dy$$

- this is an example of revolution of $g(y)$ about y-axis

5.6 Volumes of revolutions by shells

- revolution of $f(x)$ about y-axis



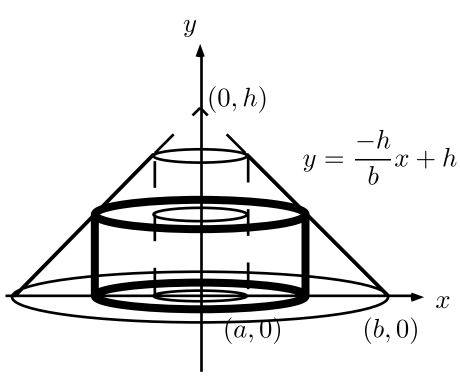


- surface area of the cylinder: $2\pi x f(x)$
- thickness of the wall of the cylinder: dx
- volume of shell: $2\pi x f(x)\, dx$
- volume by adding up the shells

$$\boxed{V = 2\pi \int_a^b x f(x)\, dx}$$

- the shell method

Example 5.6.1 The hole through the cone with this method.



- the shell method

$$V = 2\pi \int_a^b x \left(\frac{-h}{b} x + h \right) dx$$

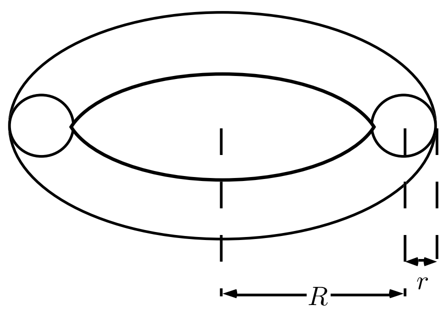
- of course this problem can be done without calculus
- volume of a cone is 1/3 volume of enclosing cylinder

$$V(\text{cone}) = \frac{1}{3}\pi r^2 h$$

- take full cone, subtract little top cone, cylindrical hole

$$V = \frac{1}{3}\pi b^2 h - \frac{1}{3}\pi a^2 \left(\frac{ah}{b}\right) - \pi a^2 \left(h - \frac{ah}{b}\right)$$

Example 5.6.2 Find the volume of a torus.



- put y-axis through centre of hole
- formed by revolving $(x-R)^2 + y^2 = r^2$ about y-axis

$$y = \pm\sqrt{r^2 - (x-R)^2}$$

$$V = 2\pi \int_{R-r}^{R+r} x \times 2\sqrt{r^2 - (x-R)^2}\, dx$$

- let $u = x - R$, then $du = dx$; $x = u + R$

x	$u = x - R$
$R + r$	r
$R - r$	$-r$

$$V = 4\pi \int_{-r}^{r} (u + R)\sqrt{r^2 - u^2}\, du$$

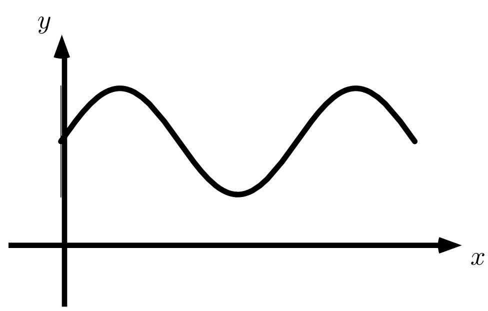
$$= 4\pi \int_{-r}^{r} u\sqrt{r^2 - u^2}\, du + 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du$$

- the first integrand is an odd function
  - the integral is 0
- the second integral represents the area of a semicircle

$$V = 4\pi R \frac{\pi r^2}{2} = 2\pi^2 r^2 R$$

5.7 When to use discs and when to use shells

Example 5.7.1 Revolve $y = 2 + \sin x$, $(0 \leq x \leq 3\pi)$ about the x-axis, the y-axis, ,the line $y = -1$, and the line $x = -1$. Describe the objects and find their volumes.



- about the x-axis
  - the object looks like a table leg
  - turned on a lathe between centres
- discs
  - perpendicular to rotation axis

$$V = \pi \int_{0}^{3\pi} (2 + \sin x)^2\, dx$$

- about the y-axis
  - the object looks like a checker
  - turned on a lathe at the end
- shells
  - around the rotation axis

  $$V = 2\pi \int_0^{3\pi} x\,(2 + \sin x)\; dx$$

- about the line $y = -1$
  - the object looks like a table leg
  - turned on a lathe between centres
- discs
  - perpendicular to rotation axis
  - centred at -1, radius $1 + (\sin x + 2)$

  $$V = \pi \int_0^{3\pi} (1 + (2 + \sin x))^2\; dx$$

- about the line $x = -1$
  - the object looks like a checker with a hole
  - turned on a lathe at the end
- shells
  - parallel to rotation axis
  - centred at 1, radius is $x + 1$

  $$V = 2\pi \int_0^{3\pi} (x + 1)\,(2 + \sin x)\; dx$$

Example 5.7.2 Revolve $x = 2 + \sin y$, $(0 \leq y \leq 3\pi)$ about the x-axis and the y-axis. Describe the objects and find their volumes.

- similar to the above example

  –the roles of x and y are interchanged.

- Which uses discs and which shells?

  – draw the pictures

  – write down the integrals

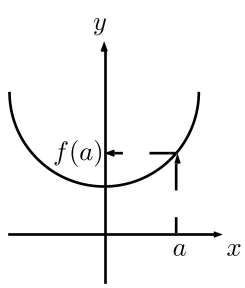
Chapter 6. Inverse Trig Functions

In which

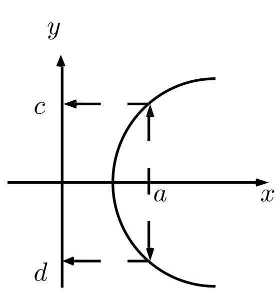
We review the definition of the inverse of a function and the formula for its derivative. We then study the inverses of the trigonometric functions in detail.

6.1 Functions

• a function



• not a function



Function:

- each x determines a *unique* y
- two different y's cannot come from the same x
- let $y_1 = f(x_1)$ and $y_2 = f(x_2)$

$$\boxed{y_1 \neq y_2 \Rightarrow x_1 \neq x_2}$$

- domain ($Dom(f)$): set of inputs (x's)
- range ($Ran(f)$): set of outputs (y's)
- notation

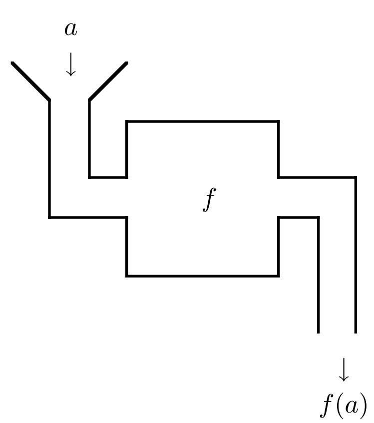
  $f : A \to B$ where $Dom(f) = A$, $Ran(f) \subseteq B$

  ($\subseteq$ – subset, $Ran(f)$ is part of B)

  – f "acts" on all of A

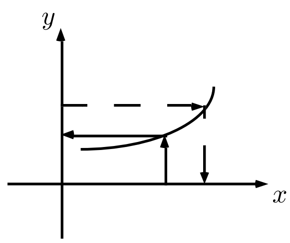
  – f takes on values in B

  – does not necessarily "hit" all of B



6.2 Inverse of a Function

- function is a rule which associates a y with each x
- want to use the rule "backwards"
- want each y to determine an x

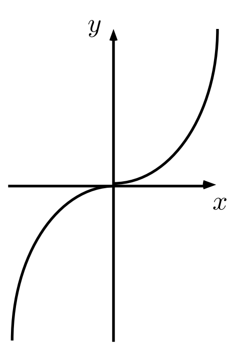


[!] this will not always work

- to get a function this way
  - **need** each y to determine a unique x
- let $y_1 = f(x_1)$ and $y_2 = f(x_2)$

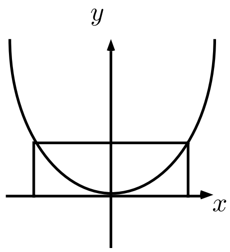
$$\boxed{x_1 \neq x_2 \Rightarrow y_1 \neq y_2}$$

- a function with this property is *one to one* (1–1)



1–1

- has an inverse



not 1–1

- does not have an inverse

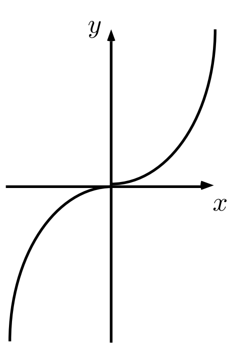
- let f be 1-1
- the *inverse* is denoted by f^{-1}

$$\boxed{a = f^{-1}(b) \iff f(a) = b}$$

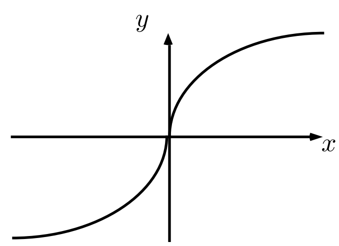
$$\boxed{b = f(f^{-1}(b))}$$

$$\boxed{a = f^{-1}(f(a))}$$

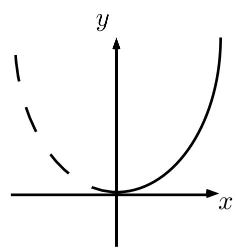
Example 6.2.1 $f(x) = x^3$



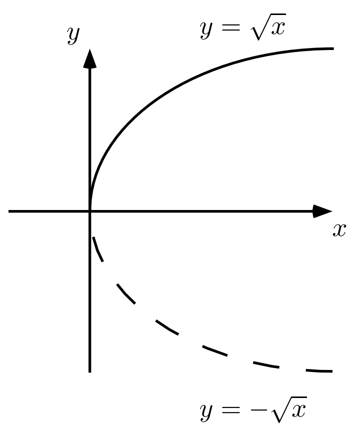
$f^{-1}(x) = x^{\frac{1}{3}}$



Example 6.2.2 $y = x^2,\ x \geq 0$

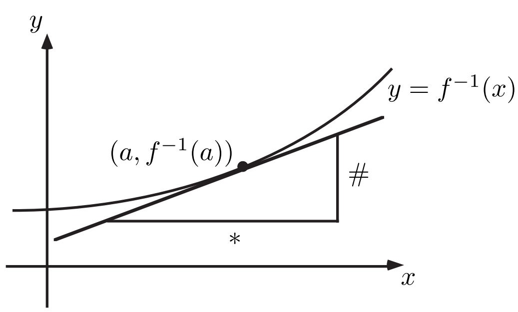


- $y = x^2$ is *not* 1–1
- $y = x^2,\ x \geq 0$ *is* 1–1
- $y = \sqrt{x}$ is the inverse of *that* function

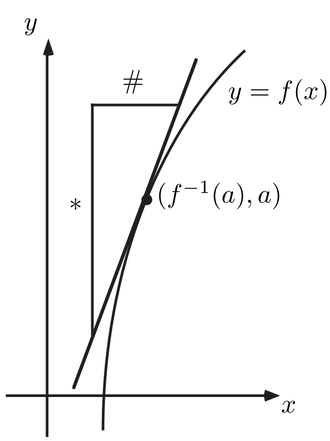


- $(\sqrt{b})^2 = b$?
  - true for all b for which both sides make sense
  - all b in the domain of $\sqrt{x}$
- $\sqrt{a^2} = a$?
  - [!] true only for $a \geq 0$. Why?
  - but $\sqrt{a^2}$ is defined for all a!
  - What is another notation for $\sqrt{a^2}$?

6.3 Derivatives of Inverse Functions



$$(f^{-1})'(a) = \frac{\#}{*}$$



$$f'(f^{-1}(a)) = \frac{*}{\#}$$

$$y = f^{-1}(x)$$

$$f(y) = f(f^{-1}(x)) = x$$

• differentiate both sides of the equation

$$f'(y)\, y' = 1$$

$$\boxed{y' = \frac{1}{f'(y)} \quad \text{or} \quad \big(f^{-1}\big)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}}$$

• the method is more important than the formula

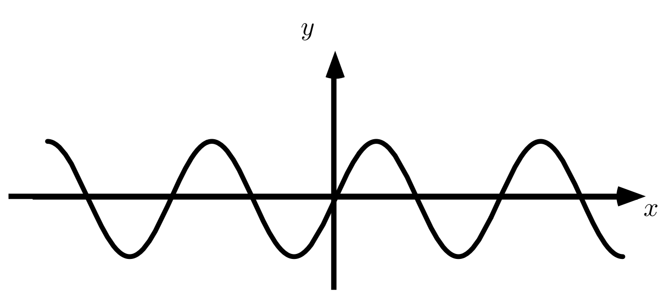
Example 6.3.1 Find $(\sqrt{x})'$.

$$\begin{aligned} y &= \sqrt{x} \\ y^2 &= x \\ 2yy' &= 1 \\ y' &= \frac{1}{2y} = \frac{1}{2\sqrt{x}} \end{aligned}$$

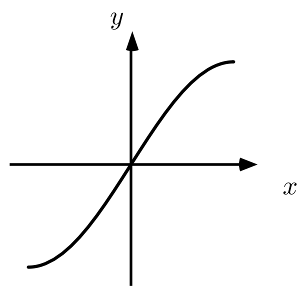
$$\boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$$

6.4 Inverse sine

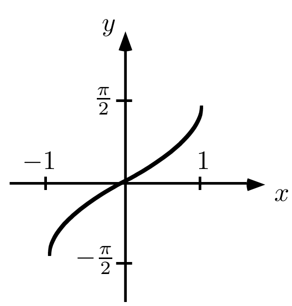
• $y = \sin x$ is not 1–1



$$\boxed{y = \text{Sin } x \iff y = \sin x \text{ and } -\frac{\pi}{2} \le x \le \frac{\pi}{2}}$$



- $y = \sin^{-1} x = \text{Arcsin } x$



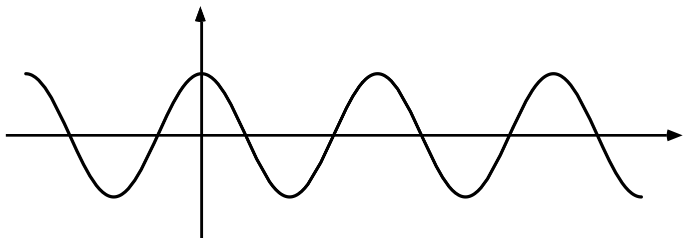
$$y = \sin^{-1} x \iff x = \text{Sin}\, y$$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{d\,\text{Sin}\, y}{dy} = \cos y \\
\frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \text{(why is } \cos y > 0 \text{ ?)} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

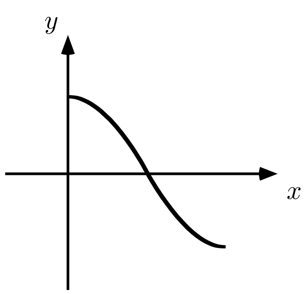
$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad (-1 < x < 1)}$$

6.5 Inverse cosine

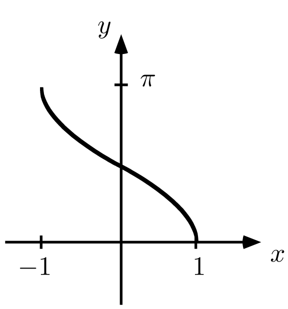
- $y = \cos x$ is not 1–1



$$\boxed{y = \text{Cos } x \iff y = \cos x \text{ and } 0 \leq x \leq \pi}$$



- $y = \cos^{-1} x = \text{Arccos } x$



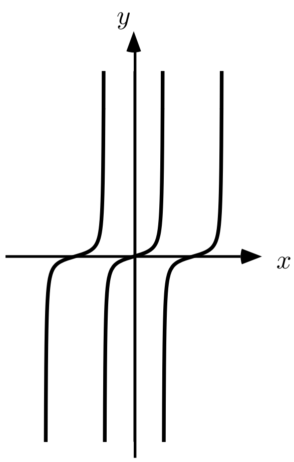
$$y = \cos^{-1} x \iff x = \operatorname{Cos} y$$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{d \operatorname{Cos} y}{dy} = -\sin y \\
\frac{dy}{dx} &= \frac{1}{dx/dy} = -\frac{1}{\sin y} \\
&= -\frac{1}{\sqrt{1-\cos^2 y}} \qquad \text{(why is } \sin y > 0 \text{ ?)} \\
&= -\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

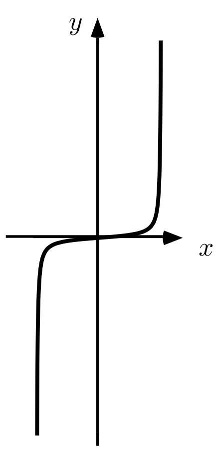
$$\boxed{\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad (-1 < x < 1)}$$

6.6 Inverse tangent

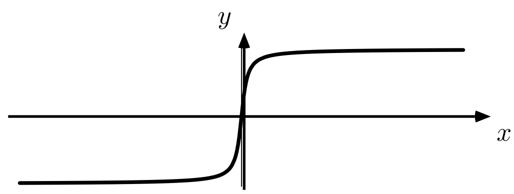
- $y = \tan x$ is not 1–1



$$\boxed{y = \operatorname{Tan}\, x \iff y = \tan x \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}}$$



- $y = \tan^{-1} x = \text{Arctan } x$



$$y = \tan^{-1} x \iff x = \operatorname{Tan} y$$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{d \operatorname{Tan} y}{dy} = \sec^2 y \\
\frac{dy}{dx} &= \frac{1}{dx/dy} = \frac{1}{\sec^2 y} \\
&= \frac{1}{1 + \tan^2 y} \\
&= \frac{1}{1 + x^2}
\end{aligned}$$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}}$$

6.7 Examples

Example 6.7.1

$$\sin[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)] = \frac{\sqrt{3}}{2}$$

$\sin[\sin^{-1}(x)] = x$ for all x in the domain of $\sin^{-1}$

Example 6.7.2 $\quad \sin^{-1}[\sin\left(\frac{7\pi}{4}\right)] \neq \frac{7\pi}{4} \qquad$ (Why?)

$$\sin^{-1}[\sin\left(\frac{7\pi}{4}\right)] = -\frac{\pi}{4}$$

$$\left(\sin\left(-\frac{\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) \text{ and } -\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}\right)$$

Example 6.7.3

$$\sin(\cos^{-1}(\frac{3}{5})) = \frac{4}{5}; \quad \cos(\cos^{-1}(\frac{3}{5})) = \frac{3}{5}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

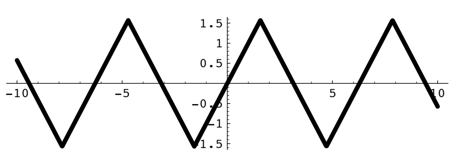
$$\sin(2\cos^{-1}(\frac{3}{5})) = 2\sin(\cos^{-1}(\frac{3}{5}))\cos(\cos^{-1}(\frac{3}{5}))$$

$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$$

Example 6.7.4 Plot $f(x) = \sin^{-1}(\sin x)$.

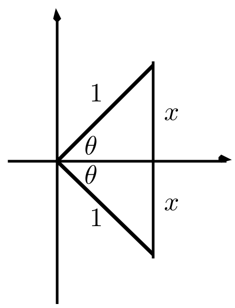
$$f'(x) = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|}$$

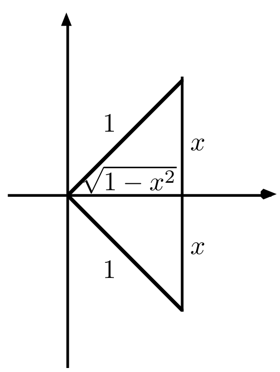
$$= \begin{cases} 1, & \text{if } \cos x > 0 \\ -1, & \text{if } \cos x < 0. \end{cases}$$



Example 6.7.5 Simplify $\tan(\sin^{-1} x)$.

- let $\theta = \sin^{-1} x$ (recall $-\pi/2 \leq \theta \leq \pi/2$)
  - then $\sin\theta = x$ (θ is an angle whose sin is x)





$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}, \qquad |x| < 1$$

Example 6.7.6

$$\frac{d}{dx}\left[\sin^{-1}(x^3)\right]^2$$

$$= 2\left[\sin^{-1}(x^3)\right]\frac{1}{\sqrt{1-x^6}}\,3x^2$$

Example 6.7.7 Assume that $b, ax-b \geq 0$. Show that

$$\int \frac{1}{x\sqrt{ax-b}}\,dx = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C.$$

$$\left(\frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}}\right)'$$

$$= \frac{2}{\sqrt{b}} \cdot \frac{1}{1+\dfrac{ax-b}{b}} \cdot \frac{1}{2\sqrt{\dfrac{ax-b}{b}}} \cdot \frac{a}{b}$$

$$= \frac{2}{\sqrt{b}} \cdot \frac{b}{ax} \cdot \frac{\sqrt{b}}{2\sqrt{ax-b}} \cdot \frac{a}{b} = \frac{1}{x\sqrt{ax-b}}$$

6.8 Integration with inverse trig functions

Example 6.8.1 $I = \displaystyle\int \frac{x^2}{2+x^6}\,dx$

- let $u = x^3$ why?
- then $du = 3x^2\,dx$

$$I = \frac{1}{3}\int \frac{du}{2+u^2}$$

$$= \frac{1}{3}\frac{1}{\sqrt{2}}\text{Arctan}\frac{u}{\sqrt{2}} + C = \frac{1}{3\sqrt{2}}\text{Arctan}\frac{x^3}{\sqrt{2}} + C$$

- check $\left[\dfrac{1}{3\sqrt{2}}\text{Arctan}\dfrac{x^3}{\sqrt{2}}\right]' = \dfrac{1}{3\sqrt{2}} \cdot \dfrac{1}{1+(x^6/2)} \cdot \dfrac{3}{\sqrt{2}}x^2$

Example 6.8.2 $I = \displaystyle\int \frac{dx}{e^x+e^{-x}}$

- let $u = e^x$
- then $du = e^x\,dx$

$$I = \int \frac{e^x\,dx}{e^x(e^x+e^{-x})} = \int \frac{e^x\,dx}{(e^x)^2+1}$$

$$= \int \frac{du}{u^2+1} = \text{Arctan} u + C = \text{Arctan } e^x + C$$

- check $[\text{Arctan } e^x]' = \dfrac{1}{1+(e^x)^2}e^x = \dfrac{1}{e^{-x}+e^x}$

Chapter 7. Techniques of Integration

In which

We study several more special methods of finding antiderivatives.

7.1 Integration by Parts

- product rule

$$d(uv) = u\,dv + v\,du$$

- integrate both sides

$$uv = \int d(uv) = \int u\,dv + \int v\,du$$

- rearrange

$$\boxed{\int u\,dv = uv - \int v\,du}$$

- to use
  - split the integrand into "parts"
  - one part is u, the other is dv
- to find v
  - must be able to integrate dv
  - integrate a "part"
- must also be able to find $\int v\,du$

Example 7.1.1 $I = \int x \sin x \, dx.$

- possible choices for parts

(i) $u = x \sin x \qquad dv = dx$

(ii) $u = \sin x \qquad dv = x \, dx$

(iii) $u = x \qquad dv = \sin x \, dx$

- which one to choose?

- can find v in each case
- must also be able to find $\int v \, du$
- the correct choice is (iii)

$$u = x \qquad dv = \sin x \, dx$$

$$du = dx \qquad v = -\cos x$$

$$I = \overset{u}{x}\overset{v}{(-\cos x)} - \int \overset{v}{(-\cos x)} \overset{du}{dx}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

- checking the result

$$(-x \cos x + \sin x)' = -\cos x - x(-\sin x) + \cos x$$

$$= x \sin x$$

- differentiate polynomials, integrate sin and cos
  - polynomials become simpler
  - sin and cos don't change in complexity

- what about the other choices?

  – choice (ii)

  $u = \sin x \qquad dv = x\,dx$

  $du = \cos x\,dx \qquad v = \dfrac{x^2}{2}$

$$I = \overset{u}{(\sin x)}\overset{v}{\left(\frac{x^2}{2}\right)} - \int \overset{v}{\frac{x^2}{2}}\overset{du}{\cos x\,dx}$$

- the resulting polynomial is harder

- try case (i); it is even worse

Example 7.1.2 $\quad I = \displaystyle\int x \ln x\,dx.$

- choices

(i) $\quad u = \ln x \qquad dv = x\,dx$

(ii) $\quad u = x \qquad dv = \ln x\,dx$

- using (ii)

  – simplifies, but requires $\displaystyle\int \ln x\,dx$

  – can do this by parts (next example)

- using (i)

  $u = \ln x \qquad dv = x\,dx$

  $du = \dfrac{1}{x}\,dx \qquad v = \dfrac{x^2}{2}$

$$I = \overset{v}{\frac{x^2}{2}}\overset{u}{\ln x} - \int \overset{v}{\frac{x^2}{2}}\overset{du}{\frac{1}{x}\,dx}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C$$

Example 7.1.3 $I = \int \ln x \, dx$.

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\, dx \qquad v = x$$

$$I = \overbrace{x}^{v}\overbrace{\ln x}^{u} - \int \overbrace{x}^{v}\, \overbrace{\frac{1}{x}\, dx}^{du}$$

$$= x \ln x - \int dx$$

$$= x \ln x \; - x + C$$

• can do $I = \int x \ln x \, dx$ (previous example) with this

$$u = x \qquad dv = \ln x dx$$

$$du = dx \qquad v = x \ln x - x$$

$$I = \overbrace{x}^{u}\overbrace{(x \ln x - x)}^{v} \quad - \int \overbrace{(x \ln x - x)}^{v} \quad \overbrace{dx}^{du}$$

$$= x^2 \ln x - x^2 \qquad - I \qquad + \int x \, dx$$

$$2I = x^2 \ln x - x^2 \; + \frac{x^2}{2} + C \qquad I = \frac{x^2}{2} \ln x - \frac{x^2}{4} + K$$

Example 7.1.4 $I = \int x e^{\sqrt{x}} \, dx$

• could do parts directly $dv = e^{\sqrt{x}} \, d\sqrt{x} = e^{\sqrt{x}} \frac{1}{2\sqrt{x}} \, dx$

• substitution simplifies

$$t = \sqrt{x}$$

$$dt = \frac{1}{2\sqrt{x}}\, dx; \quad dx = 2t \, dt$$

$$I = \int t^2 e^t 2t \, dt = 2 \int t^3 e^t \, dt = 2I_1$$

$u = t^3; \quad du = 3t^2\,dt$

$dv = e^t\,dt; \quad v = e^t$

$$I_1 = t^3 e^t - 3\int t^2 e^t\,dt$$

- effect? t^3 reduced to t^2
- repeat this process twice more
- finally only need to integrate e^t

Example 7.1.5 $I = \displaystyle\int \sin 3x \sin 2x\,dx.$

- parts (can also be evaluated using trig. identities)

$u = \sin 3x \qquad dv = \sin 2x\,dx$

$du = 3\cos 3x\,dx \qquad v = -\dfrac{\cos 2x}{2}$

$$I = \underbrace{\sin 3x}_{u}\ \underbrace{\left(-\frac{\cos 2x}{2}\right)}_{v} - \int \underbrace{-\frac{\cos 2x}{2}}_{v}\ \underbrace{3\cos 3x\,dx}_{du}$$

$$= -\frac{1}{2}\sin 3x\,\cos 2x + \frac{3}{2}\int \cos 2x \cos 3x\,dx$$

- $I_1 = \displaystyle\int \cos 2x \cos 3x\,dx$ is as difficult as the original
- there are two possible ways to proceed

  – the correct one is done first

$u = \cos 3x \qquad dv = \cos 2x\,dx$

$du = -3\sin 3x\,dx \qquad v = \dfrac{\sin 2x}{2}$

$$I_1 = \cos 3x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}(-3\sin 3x)\,dx$$

$$= \frac{1}{2}\cos 3x \sin 2x + \frac{3}{2} I$$

$$\begin{aligned} I &= -\frac{1}{2}\sin 3x\cos 2x + \frac{3}{2}I_1 \\ &= -\frac{1}{2}\sin 3x\cos 2x + \frac{3}{2}\left(\frac{1}{2}\cos 3x\sin 2x + \frac{3}{2}I\right) \\ &= -\frac{1}{2}\sin 3x\cos 2x + \frac{3}{4}\cos 3x\sin 2x + \frac{9}{4}I \end{aligned}$$

- solve for I

$$I - \frac{9}{4}I = -\frac{5}{4}I = -\frac{1}{2}\sin 3x\cos 2x + \frac{3}{4}\cos 3x\sin 2x$$

$$I = -\frac{4}{5}\left(-\frac{1}{2}\sin 3x\cos 2x + \frac{3}{4}\cos 3x\sin 2x\right) + C$$

- what happens if the wrong choice is made?

$$u = \cos 2x \qquad dv = \cos 3x\,dx$$

$$du = -2\sin 2x\,dx \qquad v = \frac{\sin 3x}{3}$$

$$I_2 = \cos 2x\left(\frac{\sin 3x}{3}\right) + \frac{2}{3}I$$

$$I = -\frac{1}{2}\sin 3x\cos 2x + \frac{3}{2}\left(\frac{1}{3}\cos 2x\sin 3x + \frac{2}{3}I\right)$$

$$I = -\frac{1}{2}\sin 3x\cos 2x + \frac{1}{2}\cos 2x\sin 3x + I$$

$$I = I!$$

Example 7.1.6 $I = \int \sin^2 x\,dx.$

$$u = \sin x \qquad dv = \sin x\,dx$$

$$du = \cos x\,dx \qquad v = -\cos x$$

$$\begin{aligned} I &= -\cos x\sin x - \int(-\cos x)\cos x\,dx \\ &= -\cos x\sin x + \int \cos^2 x\,dx \end{aligned}$$

- now what?
  - effect is to change $\sin^2 x$ to $\cos^2 x$

- don't change $\cos^2 x$ to $\sin^2 x$ (Why not?)
- use $\cos^2 x = 1 - \sin^2 x$

$$I = -\cos x \sin x + \int (1 - \sin^2 x)\, dx$$

$$= -\cos x \sin x + \int dx - I$$

$$2I = -\cos x \sin x + x$$

$$I = \frac{x}{2} - \frac{1}{2}\cos x \sin x + C$$

Example 7.1.7 $I = \displaystyle\int \sec^3 x\, dx.$

$u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \qquad v = \tan x$

$$I = \sec x \tan x - \int \tan x(\sec x \tan x\, dx)$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

- use $\tan^2 x = \sec^2 x - 1$

$$I = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$= \sec x \tan x - I + \int \sec x\, dx$$

$$2I = \sec x \tan x + \int \sec x\, dx$$

$$I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

7.2 Trigonometric Integrals

• for powers of sin and cos remember

$$\sin^2 x + \cos^2 x = 1$$

$$d\sin x = \cos x\, dx$$

$$d\cos x = -\sin x\, dx$$

Example 7.2.1 $\quad I = \int \sin^3 x\, \cos^4 x\, dx$

• use one of the sin's to make $d\cos$

$$\begin{aligned} I &= \int \sin^2 x\, \cos^4 x\, \sin x\, dx \\ &= -\int \sin^2 x\, \cos^4 x\, d\cos x \end{aligned}$$

• let $c = \cos x$; then $\sin^2 x = 1 - \cos^2 x = 1 - c^2$

$$\begin{aligned} I &= -\int (1 - c^2) c^4\, dc = \int (c^6 - c^4)\, dc \\ &= \frac{c^7}{7} - \frac{c^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C \end{aligned}$$

$$\int \sin^m x\, \cos^n x\, dx$$

• for above technique to work for $m, n \geq 0$

– at least one of m and n must be odd

• does *not* work for

$$\int \sin^2 x\, \cos^2 x\, dx$$

$$\int \sin^2 x\, dx$$

Example 7.2.2 $\int \cos^3 x\, dx$

$$
\begin{aligned}
&= \int \cos^2 x\, \cos x\, dx \\
&= \int (1 - \sin^2 x)\, d \sin x \\
&= \int (1 - s^2)\, ds \\
&= s - \frac{s^3}{3} + C \\
&= \sin x - \frac{\sin^3 x}{3} + C
\end{aligned}
$$

- if m and n both even;
  - need more trig. identities (or use parts see later)

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin 2x &= 2 \sin x \cos x \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos 2x &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 = 1 - 2\sin^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x)
\end{aligned}
$$

Example 7.2.3 $I = \int \sin^2 x\, dx$

- use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$I = \frac{1}{2} \int (1 - \cos 2x)\, dx = \frac{1}{2}(x - \frac{1}{2} \sin 2x) + C$$

- this example can also be done with “parts”

Example 7.2.4 $\quad I = \displaystyle\int \sin^4 x \, dx$

- use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$
\begin{aligned}
I &= \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1 + \cos 4x}{2}\, dx \\
&= \ldots
\end{aligned}
$$

Example 7.2.5 $\quad I = \displaystyle\int \sin^2 x \cos^2 x \, dx$

- various ways to proceed
  - reduce $\sin^2$ and $\cos^2$ by formulas
  - use $\sin^2 x + \cos^2 x = 1$, reduce to previous examples

$$I = \int \frac{(1 - \cos 2x)}{2}\frac{(1 + \cos 2x)}{2}\, dx = \int \frac{(1 - \cos^2 2x)}{4}\, dx$$

$$I = \int \sin^2 x(1 - \sin^2 x)\, dx = \int \sin^2 x \; dx - \int \sin^4 x \, dx$$

- for powers of sec and tan remember

$$
\boxed{
\begin{aligned}
&\tan^2 x + 1 = \sec^2 x \\
&d\tan x = \sec^2 x \, dx \\
&d\sec x = \sec x \tan x \, dx
\end{aligned}
}
$$

- need $\sec^2$ to get $d\tan$
  - then even power of sec to reduce all to tan
- need $\sec\tan$ to get $d\sec$
  - then even power of tan to reduce all to sec

Example 7.2.6 Easy powers of sec and tan, $(m, n \geq 0)$.

$$\int \sec^2 x \tan^n x \, dx = \int \tan^n x \, d \tan x$$

$$\int \sec^m x \tan x \, dx = \int \sec^{m-1} x \, d \sec x \quad (m \geq 1)$$

$$\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \, d \sec x$$

$$= \int \sec^2 x (\sec^2 x - 1) \, d \sec x$$

- when the "easy" method does *not* work
  - odd powers of sec
  - even powers of tan with odd powers of sec
- odd powers of sec can be done by parts

Example 7.2.7

$$\begin{aligned} \int \sec x \tan^2 x \, dx &= \int \sec x (\sec^2 x - 1) \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \end{aligned}$$

- for powers of csc and cot
- similar to sec and tan
- relevant formulas

$$\boxed{\begin{aligned} &1 + \cot^2 x = \csc^2 x \\ &d \cot x = -\csc^2 x \, dx \\ &d \csc x = -\csc x \cot x \, dx \end{aligned}}$$

- hard to do odd powers of csc
  - includes even cot with odd csc

7.3 Trig. substitutions

- handles $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$
- some of these are easy

Example 7.3.1 $\quad I = \int x\sqrt{1 - x^2}\, dx$

- set $u = \sqrt{1 - x^2}$ then $u^2 = 1 - x^2$ and $2u\, du = -2x\, dx$

$$\begin{aligned} I &= -\int \sqrt{1 - x^2}(-x\, dx) = -\int u(u\, du) \\ &= -\frac{u^3}{3} + C = -\frac{(1 - x^2)^{3/2}}{3} + C \end{aligned}$$

- this substitution will *not* work for

$$\int \sqrt{1 - x^2}\, dx \text{ or } \int x^2\sqrt{1 - x^2}\, dx$$

- When will it work?

Example 7.3.2 $\quad I = \int x^3\sqrt{1 - x^2}\, dx$

$$I = -\int x^2\sqrt{1 - x^2}\,(-x\, dx)$$

$$= -\int (1 - u^2)u\, u\, du \quad (u^2 = 1 - x^2 \Rightarrow x^2 = 1 - u^2)$$

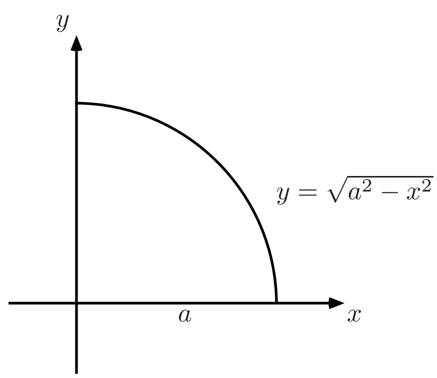
- to handle $\sqrt{\square}$
- rewrite $\square$ as a square
- use the trig identities

$$1 - \sin^2\theta = \cos^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

Example 7.3.3 Find the area of a circle of radius a.



- $A = 4\int_0^a \sqrt{a^2 - x^2}\, dx$
- to write $a^2 - x^2$ as a square

$$1 - \sin^2\theta = \cos^2\theta$$

$$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$$

- set $x = a\sin\theta$

$$a^2 - x^2 = a^2 - (a\sin\theta)^2 = a^2(1 - \sin^2\theta)$$

$$= a^2\cos^2\theta$$

$$dx = a\cos\theta\, d\theta$$

x	$\sin^{-1}(x/a) = \theta$
a	$\sin^{-1} 1 = \pi/2$
0	$\sin^{-1} 0 = 0$

$$A = 4\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= 4\int_0^{\pi/2} a\cos\theta \quad a\cos\theta\, d\theta \qquad \text{(Why is cos +ve?)}$$

$$= 4a^2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4a^2 \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 2a^2 \left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2}$$

$$= 2a^2 \left(\frac{\pi}{2} + 0 - (0 + 0)\right)$$

$$= \pi a^2$$

Example 7.3.4 $\quad I = \displaystyle\int \frac{dx}{\sqrt{3 + 2x^2}}.$

- to create the "square" use $\tan^2\theta + 1 = \sec^2\theta$
- the 3 is factored out $2x^2 + 3 = 3(\tan^2\theta + 1) = 3\sec^2\theta$
- need $2x^2 = 3\tan^2\theta$
- get $\sqrt{2}x = \sqrt{3}\tan\theta$

$$\sqrt{2}\, dx = \sqrt{3}\sec^2\theta\, d\theta$$

$$I = \int \frac{1}{\sqrt{3\sec^2\theta}} \frac{\sqrt{3}}{\sqrt{2}} \sec^2\theta\, d\theta$$

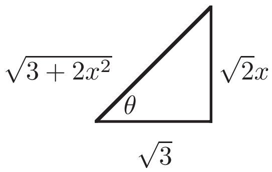
$$= \frac{1}{\sqrt{2}} \int |\sec\theta|\, d\theta$$

- notice that the substitution is actually $\theta = \text{Arctan}\dfrac{\sqrt{2}x}{\sqrt{3}}$
- therefore $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$
- so $\sec\theta \geq 0$

$$I = \frac{1}{\sqrt{2}} \int \sec\theta\, d\theta = \frac{1}{\sqrt{2}} \ln|\sec\theta + \tan\theta| + C$$

• to change back to x

$$\tan\theta = \frac{\sqrt{2}x}{\sqrt{3}}$$



$$\sec\theta = \frac{\sqrt{3+2x^2}}{\sqrt{3}}$$

$$I = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{3+2x^2}}{\sqrt{3}} + \frac{\sqrt{2}x}{\sqrt{3}}\right| + C$$

• checking

$$I' = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{3+2x^2}+\sqrt{2}x} \cdot \frac{1}{\sqrt{3}} \cdot \left(\frac{4x}{2\sqrt{3+2x^2}} + \sqrt{2}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3+2x^2}+\sqrt{2}x} \cdot \frac{2x+\sqrt{2}\left(\sqrt{3+2x^2}\right)}{\sqrt{3+2x^2}}$$

$$= \frac{1}{\sqrt{3+2x^2}}$$

Example 7.3.5 $\quad I = \displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}.$

• set $x = a\sec\theta \quad (\theta = \text{Arcsec}\frac{x}{a})$

• then $dx = a\sec\theta\tan\theta\, d\theta$

• and $x^2 - a^2 = a^2\tan^2\theta$

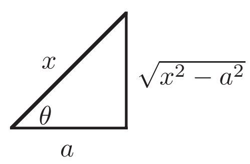
$$I = \int \frac{a\sec\theta\tan\theta}{a^2\sec^2\theta \cdot a|\tan\theta|}\, d\theta$$

• assume $\tan\theta \geq 0$ and $0 \leq \theta < \pi/2$

 – depends on definition of arcsec (may require $x \geq a$)

$$I = \frac{1}{a^2} \int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{a^2} \int \cos\theta \, d\theta = \frac{\sin\theta}{a^2} + C$$



$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{1}{a^2} \frac{\sqrt{x^2 - a^2}}{x} + C$$

Example 7.3.6 $\quad I = \int \frac{dx}{x^2 + x + 1}$

• complete the square $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

• set $u = x + \frac{1}{2}$ then $du = dx$

$$I = \int \frac{du}{u^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \text{ Arctan} \frac{2u}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}} \text{ Arctan} \frac{2(x + \frac{1}{2})}{\sqrt{3}} + C$$

Example 7.3.7 $\quad I = \int \frac{x \, dx}{x^2 - 2x + 3}$

• compute $d(x^2 - 2x + 3) = (2x - 2) \, dx$

• write $\frac{x}{x^2 - 2x + 3} = \frac{1}{2}\left(\frac{2x - 2}{x^2 - 2x + 3}\right) + \frac{1}{x^2 - 2x + 3}$

• complete the square $x^2 - 2x + 3 = (x - 1)^2 + 2$

$$I = \frac{1}{2} \int \frac{(2x - 2) \, dx}{x^2 - 2x + 3} + \int \frac{dx}{(x - 1)^2 + 2}$$

$$= \frac{1}{2} \ln |x^2 - 2x + 3| + \frac{1}{\sqrt{2}} \text{ Arctan} \frac{x - 1}{\sqrt{2}} + C$$

Example 7.3.8 $I = \int_1^3 \frac{dx}{\sqrt{4x - x^2}}$

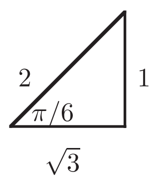
- set $u = 2 - x, \quad du = -dx$

x	$2 - x = u$
3	$2 - 3 = -1$
1	$2 - 1 = 1$

$$I = \int_1^{-1} \frac{-du}{\sqrt{4 - u^2}}$$

$$= \left[\text{Arcsin}\frac{u}{2}\right]_{-1}^{1}$$

$$= \left[\frac{\pi}{6} - (\frac{-\pi}{6})\right] = \frac{\pi}{3}$$



7.4 Partial Fractions

- integrals of form $\int \frac{P(x)}{Q(x)}\, dx$
  - where $P(x)$, $Q(x)$ are polynomials
- must be able to factor $Q(x)$
  - linear terms *e.g.* $x - a$
  - quadratics with no linear factors

    $ax^2 + bx + c$ where $b^2 - 4ac < 0$
- always possible in theory, may be difficult in practice

- first step – divide if possible

Example 7.4.1 $I = \int \frac{x^2}{x - 4}\, dx.$

$$\begin{array}{r|ll} & x & +4 \\ \hline x - 4 & x^2 & \\ & x^2 - 4x & \\ \hline & 4x & \\ & 4x & -16 \\ \hline & & 16 \end{array}$$

$$I = \int \left(x + 4 + \frac{16}{x - 4}\right) dx = \frac{x^2}{2} + 4x + 16 \ln|x - 4| + C$$

- second step
  - factor denominator
  - split up the fraction

Example 7.4.2 $\quad I = \displaystyle\int \frac{dx}{5-x^2}$.

- cannot divide
- factor

$$5 - x^2 = (\sqrt{5} - x)(\sqrt{5} + x)$$

- partial fractions

$$\boxed{\frac{1}{5-x^2} = \frac{A}{\sqrt{5}-x} + \frac{B}{\sqrt{5}+x}}$$

- bring back to a common denominator

$$\frac{A}{\sqrt{5}-x} + \frac{B}{\sqrt{5}+x} = \frac{A(\sqrt{5}+x) + B(\sqrt{5}-x)}{5-x^2}$$

- equate numerators

$$A(\sqrt{5}+x) + B(\sqrt{5}-x) = 1$$

- polynomials are equal for all values of x
- coefficients of corresponding powers of x must be equal
- substitute $x = \sqrt{5}$

$$A(\sqrt{5}+\sqrt{5}) + B(\sqrt{5}-\sqrt{5}) = 1$$

$$2\sqrt{5}A = 1; \qquad A = \frac{1}{2\sqrt{5}}$$

- equating coefficients of x

$$A - B = 0; \qquad A = B$$

$$\frac{1}{5-x^2} = \frac{1}{2\sqrt{5}}\left(\frac{1}{\sqrt{5}+x} + \frac{1}{\sqrt{5}-x}\right)$$

$$I = \frac{1}{2\sqrt{5}} \left(\int \frac{dx}{\sqrt{5}+x} + \int \frac{dx}{\sqrt{5}-x} \right)$$

$$= \frac{1}{2\sqrt{5}} \left(\ln|\sqrt{5}+x| - \ln|\sqrt{5}-x| \right) + C$$

$$= \frac{1}{2\sqrt{5}} \ln\left| \frac{\sqrt{5}+x}{\sqrt{5}-x} \right| + C$$

Example 7.4.3 $I = \int \frac{dx}{x^3+9x}$.

$$x^3 + 9x = x(x^2+9)$$

$$\boxed{\frac{1}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}}$$

$$\frac{1}{x(x^2+9)} = \frac{A(x^2+9) + x(Bx+C)}{x(x^2+9)}$$

$$A(x^2+9) + x(Bx+C) = 1$$

• set $x = 0$ (or equate the constant coefficients)

$$9A = 1; \qquad A = \frac{1}{9}$$

• equate the coefficients of x

$$C = 0$$

• equate the coefficients of x^2

$$A + B = 0; \qquad B = -A = -\frac{1}{9}$$

$$
\begin{aligned}
I &= \frac{1}{9}\left(\int \frac{dx}{x} - \int \frac{x\,dx}{x^2+9}\right) \\
&= \frac{1}{9}\left(\int \frac{dx}{x} - \frac{1}{2}\int \frac{d(x^2+9)}{x^2+9}\right) \\
&= \frac{1}{9}\left(\ln|x| - \frac{1}{2}\ln(x^2+9)\right) + K
\end{aligned}
$$

Example 7.4.4 $\quad I = \displaystyle\int \frac{x^3-1}{x^2(x-2)^3}\,dx.$

$$
\boxed{\frac{x^3-1}{x^2(x-2)^3} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^2} + \frac{E}{x-2}}
$$

$$
= \frac{[A+Bx](x-2)^3 + [C + D(x-2) + E(x-2)^2]x^2}{x^2(x-2)^3}
$$

$$
[A+Bx](x-2)^3 + [C + D(x-2) + E(x-2)^2]x^2 = x^3 - 1
$$

- set $x = 2$

$$
C(2)^2 = 2^3 - 1; \qquad C = \frac{7}{4}
$$

- set $x = 0$

$$
A(-2)^3 = -1; \qquad A = \frac{1}{8}
$$

- coefficients of x

$$
12A - 8B = 0; \qquad B = \frac{12A}{8} = \frac{12}{64} = \frac{3}{16}
$$

- coefficients of x^4

$$B + E = 0; \qquad E = -B = -\frac{3}{16}$$

- coefficients of x^3

$$A - 6B + D - 4E = 1$$

$$D = 1 + 4E + 6B - A$$

$$= 1 - \frac{12}{16} + \frac{18}{16} - \frac{1}{8} = \frac{5}{4}$$

$$I = \frac{1}{8}\int \frac{dx}{x^2} + \frac{3}{16}\int \frac{dx}{x} + \frac{7}{4}\int \frac{dx}{(x-2)^3}$$

$$+ \frac{5}{4}\int \frac{dx}{(x-2)^2} - \frac{3}{16}\int \frac{dx}{(x-2)}$$

$$= -\frac{1}{8x} + \frac{3}{16}\ln|x| - \frac{7}{8(x-2)^2}$$

$$- \frac{5}{4(x-2)} - \frac{3}{16}\ln|x-2| + K$$

Example 7.4.5 $\quad I = \int \frac{x\,dx}{(x^2 - x + 1)^2}$.

- $x^2 - x + 1$ cannot be factored
  - the partial fractions are done!
- notice $d(x^2 - x + 1) = (2x - 1)\,dx$

$$I = \frac{1}{2}\int \frac{(2x-1)\,dx}{(x^2 - x + 1)^2} + \frac{1}{2}\int \frac{dx}{(x^2 - x + 1)^2}$$

$$= \frac{1}{2} \qquad I_1 \qquad + \frac{1}{2} \qquad I_2$$

- $I_1 = \int \frac{du}{u^2}$; where $u = x^2 - x + 1$
- for I_2
  - complete the square

$$x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

- set $v = x - \frac{1}{2}$; $dv = dx$

$$I_2 = \int \frac{dv}{\left(v^2 + \frac{3}{4}\right)^2}$$

- set $v = \frac{\sqrt{3}}{2}\tan\theta$; $dv = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$v^2 + \frac{3}{4} = \frac{3}{4}(\tan^2\theta + 1) = \frac{3}{4}\sec^2\theta$$

$$I_2 = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\left(\frac{3}{4}\sec^2\theta\right)^2}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{16}{9}\int\frac{d\theta}{\sec^2\theta} = \frac{8}{3\sqrt{3}}\int\cos^2\theta\,d\theta = \ldots$$

7.5 Reduction Formulas

Example 7.5.1 $I = \int \sin^n x\,dx, \quad n \geq 2.$

- $n = 2$ done above
- use a similar technique

$$u = \sin^{n-1} x \qquad dv = \sin x\,dx$$

$$du = (n-1)(\sin^{n-2} x)\cos x\,dx \qquad v = -\cos x$$

$$I = (\sin^{n-1} x)(-\cos x)$$

$$-\int(-\cos x)(n-1)\sin^{n-2} x\cos x\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

- replace $\cos^2 x$ by $1 - \sin^2 x$

$$I = -\sin^{n-1} x \cos x$$

$$+(n-1) \left[\int \sin^{n-2} x \, dx - \int \sin^n x \, dx \right]$$

$$(1 + (n-1))I = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx$$

$$\boxed{\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx}$$

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx$$

$$= -\frac{1}{2} \sin x \cos x + \frac{x}{2} + C$$

$$\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx$$

$$= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

$$= -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left[-\frac{1}{2} \sin x \cos x + \frac{x}{2} \right] + C$$

- for higher powers, reduce by powers of 2
  - end up at 0 or 1 as above

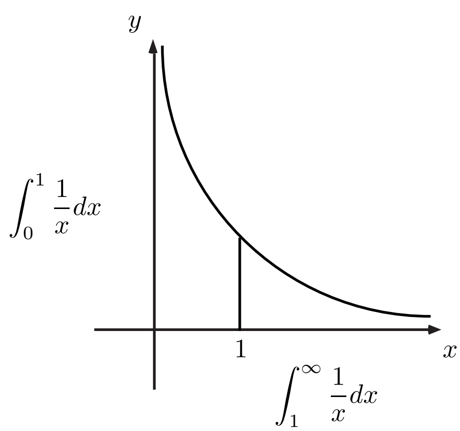
Chapter 8. Improper Integrals

In which

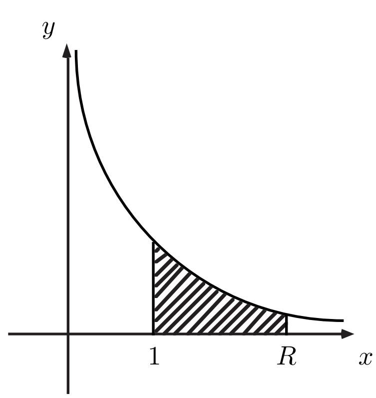
We extend the definition of integration to allow for areas with an infinite boundary.

8.1 Definitions

- two types



- meaning for $\int_1^\infty \frac{1}{x}\, dx$

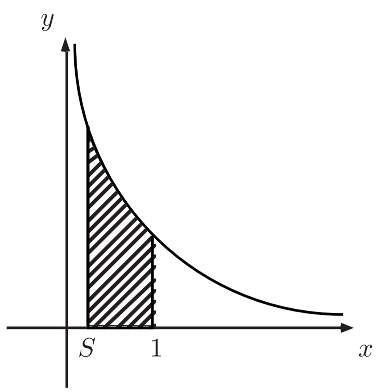


- for any $R > 1$, $\int_1^R \frac{1}{x}\,dx$ makes sense
- the bigger R, the closer to what we want
- *define*

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \ln x\Big|_1^R = \lim_{R\to\infty} \ln R = \infty$$

- meaning for $\int_0^1 \frac{1}{x}\,dx$

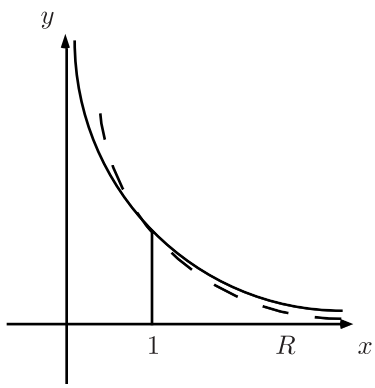


- for any $0 < S < 1$, $\int_S^1 \frac{1}{x}\,dx$ makes sense
- the smaller S, the closer to what we want
- *define*

$$\int_0^1 \frac{1}{x}\,dx = \lim_{S\to 0^+} \int_S^1 \frac{1}{x}\,dx$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{S\to 0^+} \ln x\Big|_S^1 = \lim_{S\to 0^+} 0 - \ln S = -(-\infty) = \infty$$

- Is "area" with an infinite edge is always infinite?
  - look at area under $\frac{1}{x^2}$ from 1 to ∞



- this area is smaller than the area under $\frac{1}{x}$
- $\frac{1}{x^2}$ get small fast enough so the area is finite!

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^2}\,dx$$

$$= \lim_{R\to\infty} -\frac{1}{x}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{1}{R} + 1 = 1$$

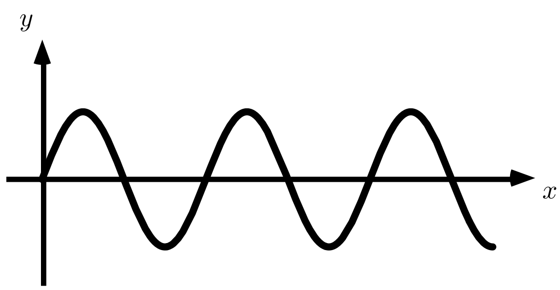
- notice $\frac{1}{x^2} \geq \frac{1}{x}$ on $(0,1]$
- here the area under $\frac{1}{x^2}$ is larger
  - therefore this area is infinite
- by calculation

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{S\to 0^+}\int_S^1 \frac{1}{x^2}\,dx$$

$$= \lim_{S\to 0^+} -\frac{1}{x}\Big|_S^1 = \lim_{S\to 0^+} -1 + \frac{1}{S} = \infty$$

- some integrals simply do not exist

Example 8.1.1 $\int_0^\infty \sin x\, dx$



$$\int_0^\infty \sin x\, dx = \lim_{R\to\infty} \int_0^R \sin x\, dx$$

$$= \lim_{R\to\infty} -\cos x\Big|_0^R$$

$$= \lim_{R\to\infty} -\cos R + 1$$

$$= ?$$

- $\cos x$ varies from -1 to 1
  - the limit doesn't settle on any value

Example 8.1.2

$$\int_{-\infty}^{-1} \frac{dx}{x^2+1} = \lim_{R\to-\infty} \int_R^{-1} \frac{dx}{x^2+1}$$

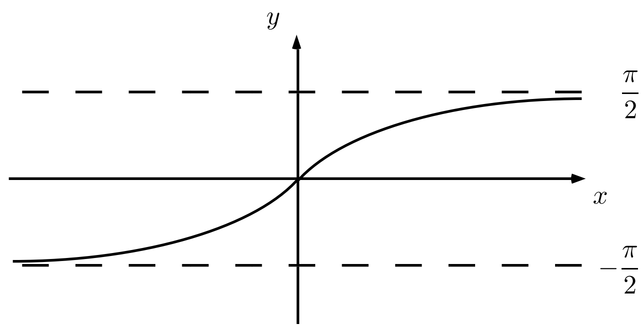
$$= \lim_{R\to-\infty} \arctan x\Big|_R^{-1}$$

$$= \lim_{R\to-\infty} \arctan(-1) - \arctan(R)$$

$$= -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$$

- see picture

$y = \arctan x$



Example 8.1.3 $\int_{-1}^{1} \frac{dx}{x^2}$

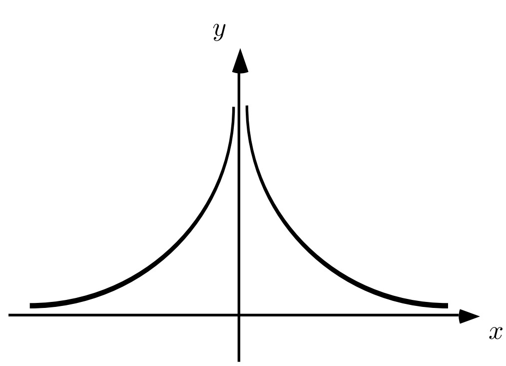
- this integral is improper at 0

! • what happens if we ignore this?

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1 - - - 1 = -2$$

- this is clearly wrong

  $\frac{1}{x^2} > 0$, so integral would have to be +ve



$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$= \lim_{R\to 0^-} \int_{-1}^{R} \frac{dx}{x^2} + \lim_{S\to 0^+} \int_{S}^{1} \frac{dx}{x^2}$$

$$= \lim_{R\to 0^-} -\frac{1}{x}\Big|_{-1}^{R} + \lim_{S\to 0^+} -\frac{1}{x}\Big|_{S}^{1}$$

$$= \infty - - - 1 - 1 - -\infty = \infty$$

! Example 8.1.4 $\displaystyle\int_{-1}^{1} \frac{dx}{x}$

- improper at 0

$$\int_{-1}^{1} \frac{dx}{x} = \lim_{R\to 0^-} \int_{-1}^{R} \frac{dx}{x} + \lim_{S\to 0^+} \int_{S}^{1} \frac{dx}{x}$$

$$= \lim_{R\to 0^-} \ln|x| \Big|_{-1}^{R} + \lim_{S\to 0^+} \ln|x| \Big|_{S}^{1}$$

$$= -\infty + 0 + 0 - (-\infty) = \infty - \infty, \text{ undefined}$$

! • the function is odd, but the integral is not 0

Example 8.1.5 $\displaystyle\int_{1}^{\infty} \frac{dx}{(x-1)^2}$

- improper at both ends

$$\int_{1}^{\infty} \frac{dx}{(x-1)^2} = \int_{1}^{2} \frac{dx}{(x-1)^2} + \int_{2}^{\infty} \frac{dx}{(x-1)^2}$$

- 2 could be any number > 1

$$= \lim_{R\to 1^+} \int_{R}^{2} \frac{dx}{(x-1)^2} + \lim_{S\to\infty} \int_{2}^{S} \frac{dx}{(x-1)^2}$$

$$= \lim_{R\to 1+} \frac{-1}{x-1}\Big|_{R}^{2} + \lim_{S\to\infty} \frac{-1}{x-1}\Big|_{2}^{S}$$

$$= -1 + \infty - 0 + 1 = \infty$$

8.2 Convergence, divergence and comparisons

- convergent *vs* divergent
  - the above integral is "divergent" to ∞
  - $\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent (showed = 1)
- the above example splits to show
  - $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$ is divergent
  - $\displaystyle\int_2^\infty \frac{dx}{(x-1)^2}$ is convergent

- comparison

$$0 \le f \le g \Rightarrow 0 \le \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

$$\int_a^b g(x)\,dx \text{ convergent } \Rightarrow \int_a^b f(x)\,dx \text{ convergent}$$

$$\int_a^b f(x)\,dx \text{ divergent } \Rightarrow \int_a^b g(x)\,dx \text{ divergent}$$

Example 8.2.1 Does $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx$ converge?

- compare with

$$\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \lim_{R\to\infty} \left.\frac{-2}{\sqrt{x}}\right|_1^R = 2$$

- converges

$$0 < \frac{1}{\sqrt{x^3+5}} < \frac{1}{\sqrt{x^3}} \text{ for } x \ge 1$$

- comparison means that

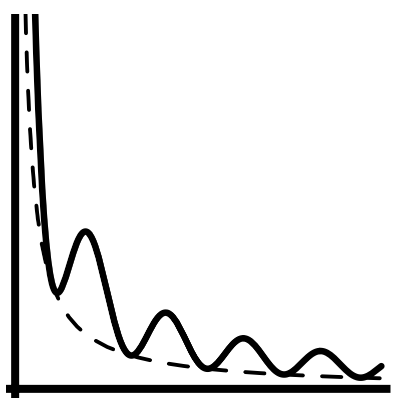
$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \text{ converges and } \le 2$$

Example 8.2.2 Does $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$ converge?

- behaves like $\frac{1}{x}$, so probably diverges
- $2+\cos x \geq 1$ so $0 \leq \frac{1}{x} \leq \frac{2+\cos x}{x}$

$$\infty = \int_{\pi}^{\infty} \frac{1}{x}\, dx \leq \int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$$

- $\int_{\pi}^{\infty} \frac{2+\cos x}{x}\, dx$ diverges



Example 8.2.3 For comparisons.

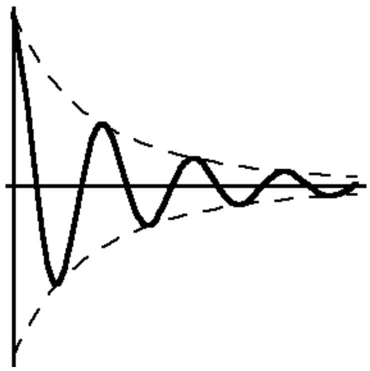
$$\int_{a}^{\infty} \frac{dx}{x^p} \quad \begin{cases} \infty & \text{for } p \leq 1 \\ \text{convergent} & \text{for } p > 1. \end{cases}$$

$$\int_{0}^{a} \frac{dx}{x^p} \quad \begin{cases} \infty & \text{for } p \geq 1 \\ \text{convergent} & \text{for } p < 1. \end{cases}$$

- (a can be replaced by any positive number)

$$\int_{1}^{\infty} e^{-x}\, dx = \lim_{R\to\infty} \left[\frac{e^{-x}}{-1}\right]_1^R = \lim_{R\to\infty} \left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e}$$

Example 8.2.4 Find $\int_0^\infty e^{-x} \cos x \, dx$



- integral is clearly less than

$$2 \int_0^\infty e^{-x} \, dx$$

- this integral converges

so $\int_0^\infty e^{-x} \cos x \, dx$ converges

- can actually evaluate by parts

$$\int e^{-x} \cos x \, dx = \frac{1}{2} e^{-x} (\sin x - \cos x)$$

$$\int_0^\infty e^{-x} \cos x \, dx = \lim_{R \to \infty} \frac{1}{2} e^{-x} (\sin x - \cos x) \Big|_0^R$$

$$= \lim_{R \to \infty} \frac{1}{2} e^{-R} (\sin R - \cos R) + \frac{1}{2}$$

$$= \frac{1}{2}$$

Example 8.2.5 Show $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x+x^3}}$ converges.

$$\int_0^\infty \frac{dx}{\sqrt{x+x^3}} = \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}}$$

• on $(0,1]$, $\sqrt{x+x^3} > \sqrt{x}$

$$\int_0^1 \frac{dx}{\sqrt{x+x^3}} < \int_0^1 \frac{dx}{\sqrt{x}} < \infty$$

• on $[1,\infty)$, $\sqrt{x+x^3} > \sqrt{x^3}$

$$\int_1^\infty \frac{dx}{\sqrt{x+x^3}} < \int_1^\infty \frac{dx}{\sqrt{x^3}} < \infty$$

Example 8.2.6 Show that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges and find an upper bound for its value.

On $[0,1]$, $0 < e^{-x^2} \le 1$, so

$$0 \le \int_0^1 e^{-x^2}\,dx \le \int_0^1 dx = 1$$

On $[1,\infty)$, $x^2 \ge x$, $-x^2 \le -x$, $0 \le e^{-x^2} \le e^{-x}$, so

$$0 < \int_0^\infty e^{-x^2}\,dx \le \int_0^\infty e^{-x}\,dx = \frac{1}{e}$$

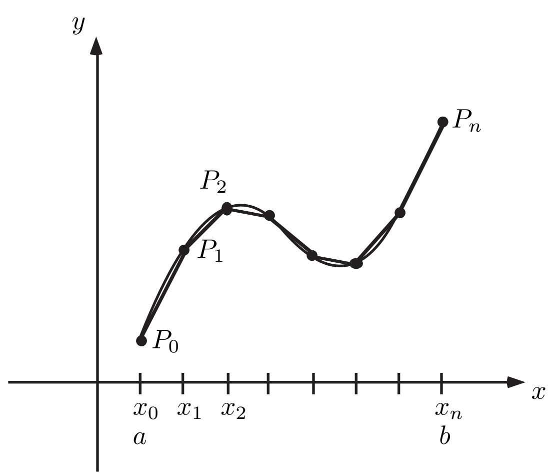
$$\int_0^\infty e^{-x^2}\,dx \le 1 + \frac{1}{e}$$

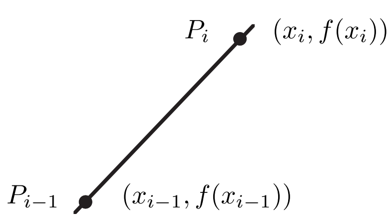
Chapter 9. Arc Length and Surface Area

In which

We apply integration to study the lengths of curves and the area of surfaces.

9.1 Arc Length





- distance from P_{i-1} to P_i

$$\boxed{\sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}}$$

• let $\Delta x = x_i - x_{i-1}$ be the same for all i

$$s \sim \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$= \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{\Delta x}\right)^2} \cdot \Delta x$$

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2} \cdot \Delta x$$

• for some $x_{i-1} < c_i < x_i$ by the M. V. Th.

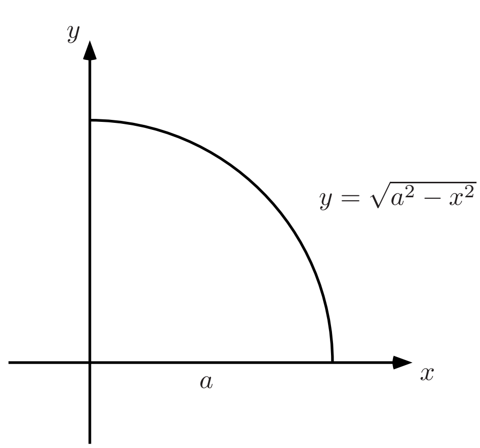
• take the limit as $\Delta x \to 0$

$$\boxed{s = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx}$$

• notice that for the limit (integral) to exist we need some hypothesis on the integrand

– continuity of f' will do

Example 9.1.1 Find the circumference of a circle.



$$\frac{dy}{dx} = \frac{-\not{2}x}{\not{2}\sqrt{a^2 - x^2}}$$

$$\begin{aligned}
s &= 4\int_0^a \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\, dx \\
&= 4\int_0^a \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\, dx = 4a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \\
&= 4a \lim_{R\to a^-} \left[\text{Arcsin}\frac{x}{a}\right]_0^R \\
&= 4a[\text{Arcsin}(1) - \text{Arcsin}(0)] = 4a\frac{\pi}{2} \\
&= 2\pi a
\end{aligned}$$

Example 9.1.2 Find the length along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 9$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{1}{4x}}$$

$$L = \int_0^9 \sqrt{1 + \frac{1}{4x}}\, dx = \int_0^9 \frac{\sqrt{4x + 1}}{2\sqrt{x}}\, dx$$

- this is not an easy integral
- substitute $t^2 = 4x$; $\qquad 2t\,dt = 4\,dx$

x	$t = 2\sqrt{x}$
9	6
0	0

$$\begin{aligned}
L &= \int_0^6 \frac{\sqrt{t^2 + 1}}{t}\,\frac{t\,dt}{2} \\
&= \frac{1}{2}\int_0^6 \sqrt{t^2 + 1}\, dt
\end{aligned}$$

- now the substitution is $t = \tan\theta$
- the integral that results is $\int \sec^3\theta\, d\theta$

- alternate method – interchange the roles of x and y
- Find the length of $x = y^2$ from $y = 0$ to $y = 3$

$$\frac{dx}{dy} = 2y$$

$$L = \int_0^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^3 \sqrt{1 + 4y^2}\, dy$$

- substitute $2y = \tan\theta$ to give $\displaystyle\int \sec^3\theta \, d\theta$

Example 9.1.3 Find the length along the curve $y = \ln x$ from $x = 1$ to $x = e$.

$$\frac{dy}{dx} = \frac{1}{x}$$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx$$

- substitute $t^2 = x^2 + 1$ to give $\displaystyle\int \frac{t^2}{t^2 - 1}\, dt$

9.2 Arc Length of Parametric Curves

- approximate length by straight lines

$$L \approx \sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$x_i = f(t_i) \Rightarrow \Delta x_i = f'(c_i)\Delta t \quad \text{by MVTh.}$$

$$y_i = g(t_i) \Rightarrow \Delta y_i = g'(d_i)\Delta t \quad \text{by MVTh.}$$

$$L \approx \sum_i \sqrt{(f'(c_i)^2 + (g'(d_i)^2}\Delta t$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- requires
  - $f'(t)$, $g'(t)$ continuous for $\alpha \le t \le \beta$
  - the curve is traversed once
- gives old formula if parameter is x

Example 9.2.1 The circumference of a circle.

- parametric equations of a circle

$$x = a\cos t,\ y = a\sin t; \quad 0 \le t \le 2\pi$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt \\ &= \int_0^{2\pi} a\, dt = at \Big|_0^{2\pi} = 2\pi a \end{aligned}$$

Example 9.2.2 Find the length of the curve
$x = t^2 - t$; $y = t^2 + t$ from $t = -1/2$ to $t = 3/2$.

$$L = \int_{-1/2}^{3/2} \sqrt{(2t-1)^2 + (2t+1)^2}\, dt$$

$$= \int_{-1/2}^{3/2} \sqrt{8t^2 + 2}\, dt = \sqrt{2} \int_{-1/2}^{3/2} \sqrt{(2t)^2 + 1}\, dt$$

- to do this integral substitute $\tan\theta = 2t$
- recall $\int \sec\theta = \ln|\sec\theta + \tan\theta|$

Example 9.2.3 Find the length of the curve
$x = 4\cos^3 t;\ y = 4\sin^3 t.$

$$L = \int_0^{2\pi} \sqrt{[12\cos^2 t(-\sin t)]^2 + [12\sin^2 t(\cos t)]^2}\, dt$$

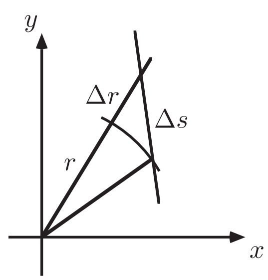
$$= 12\int_0^{2\pi} |\cos t \sin t|\, dt$$

$$= 12 \times 4 \left(\frac{\sin^2 t}{2}\right)\Bigg|_0^{\pi/2} = 24$$

[!] • What happens if we ignore the absolute value signs?

9.3 Arc Length of Polar Curves

• length of a sector of a circle



• compare length along the curve with length along the arc of a circle

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\theta)^2$$

$$= \left(\left(\frac{\Delta r}{\Delta\theta}\right)^2 + r^2\right)(\Delta\theta)^2$$

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2}\, d\theta$$

- formula can also be obtained from

$$\int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

- using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Example 9.3.1 Find the circumference of a circle.

- circle, centre $(0, 0)$, radius a is $r = a$
- circumference

$$C = \int_0^{2\pi} \sqrt{0^2 + a^2}\, d\theta$$

$$= a\int_0^{2\pi} d\theta$$

$$= a\theta\Big|_0^{2\pi} = 2\pi a$$

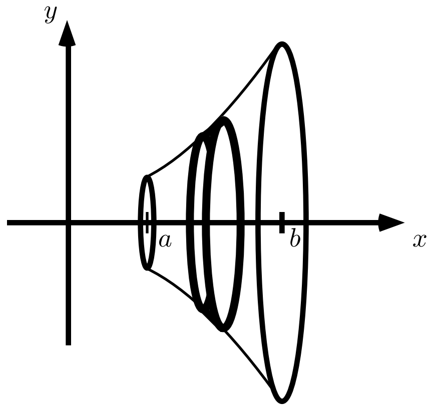
Example 9.3.2 Find the arclength of one leaf of $r = \sin 2\theta$.

- review sketch (example 8.7.4)
  - one leaf is swept out when $0 \le \theta \le \frac{\pi}{2}$

$$s = \int_0^{\frac{\pi}{2}} \sqrt{(2\cos 2\theta)^2 + (\sin 2\theta)^2}\, d\theta$$

- this integral is not easy to do

9.4 Surface Area

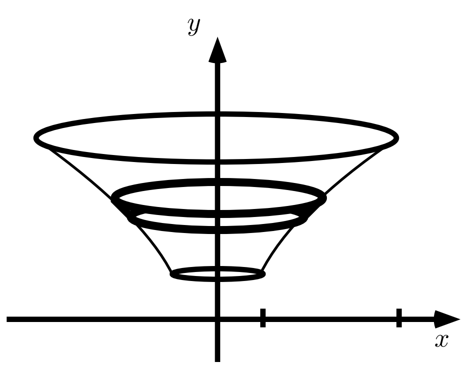


- approximate the surface using cylinders
  - radius of the cylinder is $f(c_i)$
  - height of cylinder is length along the curve
- for Δx, length is

$$\sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x \quad = \Delta s$$

$$S \approx \sum 2\pi f(c_i)\sqrt{1+\left(f'(c_i)\right)^2}\,\Delta x \quad = \sum 2\pi f(c_i)\Delta s$$

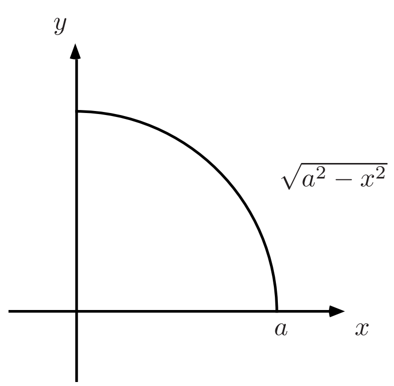
$$\boxed{S = 2\pi \int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx = 2\pi \int_a^b f(x)\,ds}$$



- rotation about the y-axis
  - radius of the cylinder is x
  - length along the curve as above

$$\boxed{S = 2\pi \int_a^b x\sqrt{1 + \left(f'(x)\right)^2}\, dx = 2\pi \int_a^b x\, ds}$$

Example 9.4.1 Find the surface area of a sphere.



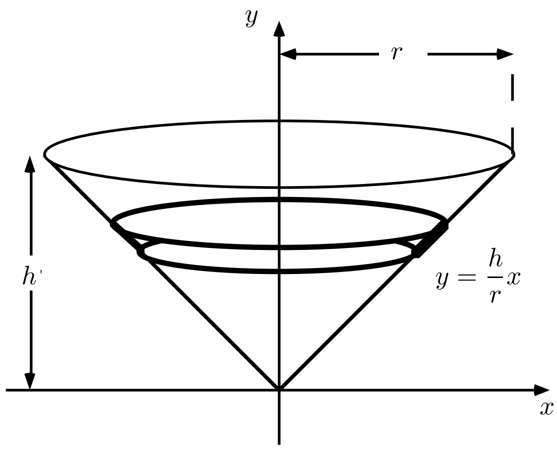
$$S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\, dx$$

$$= 2 \cdot 2\pi \int_0^a \sqrt{a^2 - x^2}\sqrt{1 + \left(\frac{-2x}{2\sqrt{a^2 - x^2}}\right)^2}\, dx$$

$$= 4\pi \int_0^a \sqrt{a^2 - x^2}\left(\frac{a}{\sqrt{a^2 - x^2}}\right) dx$$

$$= 4\pi a \int_0^a dx = 4\pi a^2$$

Example 9.4.2 Find the surface area of a cone.



$$S = 2\pi \int_0^r x \sqrt{1 + \left(\frac{h}{r}\right)^2}\, dx$$

$$= 2\pi \sqrt{1 + \left(\frac{h}{r}\right)^2}\, \left[\frac{x^2}{2}\right]_0^r$$

$$= 2\pi \sqrt{1 + \left(\frac{h}{r}\right)^2}\, \left[\frac{r^2}{2}\right]$$

$$= \pi r \sqrt{r^2 + h^2}$$

Example 9.4.3 Find the surface area of the object formed when $y = x^{3/2}$ from $x = 1$ to $x = 9$ is revolved about the y-axis.

$$S = 2\pi \int_1^9 x\sqrt{1 + \left((x^{3/2})'\right)^2}\, dx \qquad \left(2\pi \int x\, ds\right)$$

$$= 2\pi \int_1^9 x\sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= 2\pi \int_1^9 x\sqrt{1 + \frac{9}{4}x}\, dx$$

- by first formula with $x \leftrightarrow y$

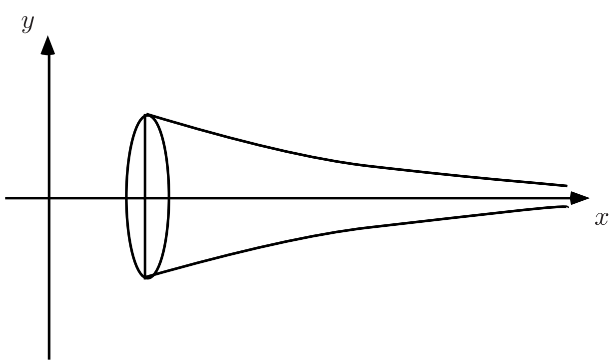
$$x = y^{2/3}$$

x	$y = x^{3/2}$
1	1
9	27

$$S = 2\pi \int_1^{27} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \left(2\pi \int x\, ds\right)$$

$$= 2\pi \int_1^{27} y^{2/3}\sqrt{1 + \left(\frac{2}{3}y^{-1/3}\right)^2}\, dy$$

Example 9.4.4 Find the volume and surface area of the infinitely long horn formed when $\frac{1}{x}$ is rotated about the x-axis from $x = 1$ to $x = \infty$.



$$V = \int_1^{\infty} \pi y^2\, dx$$

$$= \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \int_1^R \frac{dx}{x^2}$$

$$= \pi \lim_{R\to\infty} \left[\frac{-1}{x}\right]_1^R = \pi \lim_{R\to\infty} \left[\frac{-1}{R} + 1\right]$$

$$= \pi(0 + 1) = \pi$$

- Conclusion: The horn can be filled with π units of paint.

$$S = 2\pi \int_1^\infty y\sqrt{1+(y')^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$$

- this integral need not be evaluated

- notice

$$\frac{1}{x} \le \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}} \qquad \text{for } x \ge 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty}\int_1^R \frac{dx}{x}$$

$$= \lim_{R\to\infty} \ln x \Big|_1^R = \lim_{R\to\infty} \ln R - 0 = \infty$$

- by comparison $\displaystyle\int_1^\infty \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx = \infty$

- Conclusion: The surface cannot be painted!

9.5 Surface Area and Parametric Curves

$$S = \int 2\pi y\; ds$$

- for rotation about the x-axis

- for parametric $x = f(t)$, $y = g(t)$, just showed

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- needs f', g' continuous and $g \ge 0$

- summary

  – for rotation about the x-axis

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

  –for rotation about the y-axis

$$S = \int_{\gamma}^{\delta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 9.5.1 The surface area of a sphere.

- parametric equations of a semicircle

$$x = a\cos t,\ y = a\sin t; \quad 0 \le t \le \pi$$

- rotate about the x-axis to get a sphere
- $ds = a\ dt$ for the circle

$$S = \int_0^{\pi} 2\pi y\ ds = 2\pi \int_0^{\pi} (a\sin t)\ a\ dt$$

$$= 2\pi a^2(-\cos t)\Big|_0^{\pi} = 4\pi a^2$$

Example 9.5.2 The surface area of $x = e^t - t,\ y = 4e^{t/2}$, $0 \le t \le 1$, revolved about the x-axis.

$$S = 2\pi \int_0^1 4e^{t/2}\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\ dt$$

$$= 8\pi \int_0^1 e^{t/2}\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\ dt$$

$$= 8\pi \int_0^1 e^{t/2}\sqrt{(e^t+1)^2}\ dt$$

$$= 8\pi \int_0^1 e^{3t/2} + e^{t/2}\ dt = \ldots$$

Example 9.5.3 The surface area of $x = e^t - t,\ y = 4e^{t/2}$, $0 \le t \le 1$, revolved about the y-axis.

$$S = 2\pi \int_0^1 (e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\ dt$$

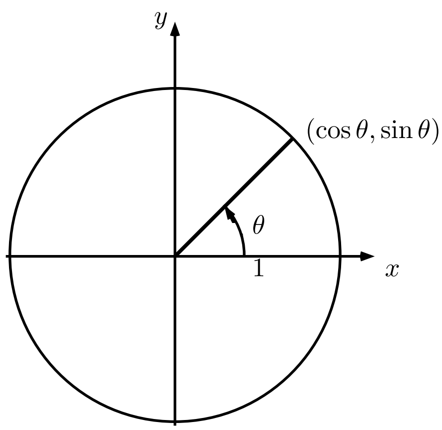
$$= 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\ dt$$

$$= 2\pi \int_0^1 (e^t - t)(e^t + 1)\ dt$$

$$= 2\pi \int_0^1 (e^{2t} - te^t + e^t - t)\ dt = \dots$$

Chapter 10. Review of trigonometry

10.1 Definitions



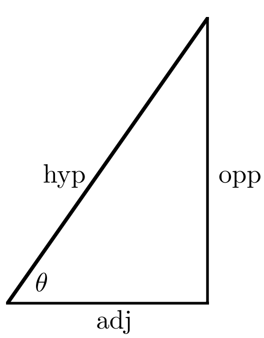
- circle of radius 1
- radian measure of angle θ = length of the arc
- arc length of circle = circumference = 2π

$$\boxed{2\pi \text{ radians } = 360^\circ}$$

- area of circle = π

$$\frac{\text{area sector}}{\pi} = \frac{\text{area sector}}{\text{area circle}} = \frac{\theta}{2\pi}$$

$$\boxed{\text{area sector } = \frac{\theta}{2}}$$

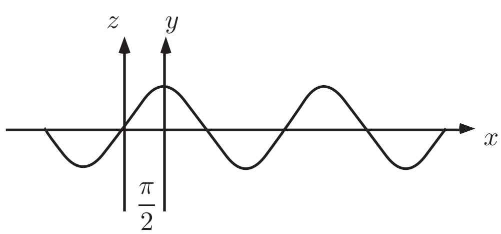


$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{\text{adj}}$$

10.2 Properties of the functions



$z = \sin x \quad y = \cos x$

- cos is even

  $\cos(-\theta) = \cos\theta$

- sin is odd

  $\sin(-\theta) = -\sin\theta$

- both are periodic, period 2π

  $\sin(\theta + 2\pi) = \sin\theta$

  $\cos(\theta + 2\pi) = \cos\theta$

- shifting by $\frac{\pi}{2}$

$$\cos x = \sin(x + \frac{\pi}{2}) = -\sin(x - \frac{\pi}{2})$$

$$\sin x = -\cos(x + \frac{\pi}{2}) = \cos(x - \frac{\pi}{2})$$

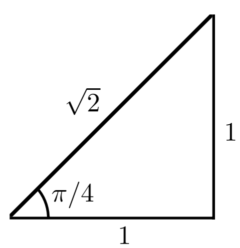
- shifting by π

$$\cos x = -\cos(x + \pi) = -\cos(\pi - x)$$

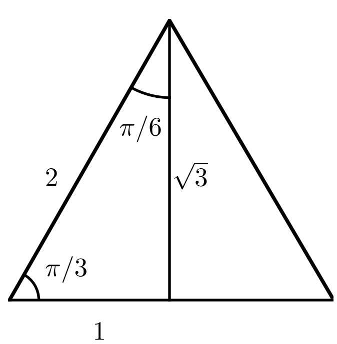
$$\sin x = -\sin(x + \pi) = \sin(\pi - x)$$

- values

$$\sin\frac{\pi}{2} = 1 \quad \cos\frac{\pi}{2} = 0 \quad \sin 0 = 0 \quad \text{etc.}$$



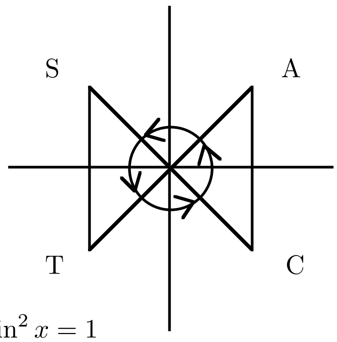
$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \tan\frac{\pi}{4} = 1$$



$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos\frac{\pi}{3} = \frac{1}{2} \quad \tan\frac{\pi}{3} = \frac{\sqrt{3}}{1}$$

$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

- other quadrants
  - cast iron rule



$\cos^2 x + \sin^2 x = 1$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$

$\cos x = \pm\sqrt{\frac{\cos 2x + 1}{2}} \quad \sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$