ROOTS OF POLYNOMIAL EQUATIONS

In this unit we discuss polynomial equations. A **polynomial** in x of degree n, where $n \geq 0$ is an integer, is an expression of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$

where $a_n \neq 0$, $a_{n-1}, \ldots, a_0$ are constants. When $P_n(x)$ is set equal to zero, the resulting equation

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{2}$$

is called a **polynomial equation** of degree n. In this unit we are concerned with the number of solutions of polynomial equations, the nature of these solutions (be they real or complex, rational or irrational), and techniques for finding the solutions. We call values of x that satisfy equation (2) **roots** or **solutions** of the equation. They are also called **zeros** of the polynomial $P_n(x)$.

When $n = 1$, equation (2) is called a **linear** equation (or equation of degree 1),

$$a_1 x + a_0 = 0. \tag{3}$$

Its only solution is $x = -a_0/a_1$.

Quadratic equations (equations of degree 2) are obtained when $n = 2$. It is customary in this case to denote coefficients as follows

$$ax^2 + bx + c = 0. \tag{4}$$

They were solved in the material on Complex Numbers. In this unit we concentrate on polynomials of degree three and higher.

The next simplest polynomial equation after linear and quadratic is the cubic,

$$ax^3 + bx^2 + cx + d = 0, \tag{5}$$

and after that the quartic,

$$ax^4 + bx^3 + cx^2 + dx + e = 0. \tag{6}$$

There are procedures that give roots for both of these equations, but they are of so little practical use in this day of the electronic calculator and personal computer, we relegate them to the exercises at the end of this section (see Exercises 33 and 40). A more modern approach is to use the analytic methods of this unit, if possible, or numerical methods. It is interesting to note that no algebraic formulas can be given for roots of polynomial equations that have degree greater than or equal to five. For such equations, it is usually necessary to use numerical methods to find roots.

When an exact solution of a polynomial equation can be found, it can be removed from the equation, yielding a simpler equation to solve for the remaining roots. The process by which this is done is a result of the remainder and factor theorems.

Theorem 1 **Remainder Theorem** When a polynomial $P_n(x)$ is divided by $bx - a$, the remainder is $P_n(a/b)$; that is, $P_n(x)$ can be expressed in the form

$$P_n(x) = (bx - a)Q_{n-1}(x) + P_n(a/b), \tag{7}$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$.

Proof There is no question that when a polynomial of degree n is divided by $bx - a$, the quotient is a polynomial of degree $n - 1$ and the remainder is a constant,

$$P_n(x) = (bx - a)Q_{n-1}(x) + R,$$

(long division tells us this). Substitution of $x = a/b$ immediately yields $R = P_n(a/b)$.

Example 1 Find the remainders when $P(x) = x^4 - 2x^2 + x + 5$ is divided by (a) $x + 3$ and (b) $2x + 3$.

Solution (a) According to the remainder theorem, the remainder is

$$P(-3) = (-3)^4 - 2(-3)^2 + (-3) + 5 = 65.$$

This is much easier than using the long division to the right.

$$\begin{array}{rl} & \underline{x^3 - 3x^2 + 7x - 20} \\ x+3 \,) & x^4 - 2x^2 + x + 5 \\ & \underline{x^4 + 3x^3} \\ & -3x^3 - 2x^2 \\ & \underline{-3x^3 - 9x^2} \\ & 7x^2 + x \\ & \underline{7x^2 + 21x} \\ & -20x + 5 \\ & \underline{-20x - 60} \\ & 65 \end{array}$$

(b) Theorem 1 indicates that the remainder is

$$P(-3/2) = (-3/2)^4 - 2(-3/2)^2 + (-3/2) + 5 = 65/16. \bullet$$

Polynomial $P_n(x)$ and divisor $bx - a$ in the remainder theorem need not be real; they can be complex. We illustrate in the following example.

Example 2 Determine the remainder when $P(x) = ix^3 - 3x + 4 + i$ is divided by $ix + 2$.

Solution The remainder is

$$P(-2/i) = P(2i) = i(2i)^3 - 3(2i) + 4 + i = 8 - 6i + 4 + i = 12 - 5i. \bullet$$

An immediate consequence of the remainder theorem is the factor theorem.

Theorem 2 **Factor Theorem** $bx - a$ is a factor of $P_n(x)$ if and only if $P_n(a/b) = 0$.

The factor theorem is very useful in solving polynomial equations. It does not find solutions, however. What it does do is simplify the problem each time a solution is found. To illustrate, consider the quartic equation

$$P(x) = x^4 + 2x^3 + x^2 - 2x - 2 = 0. \tag{8a}$$

A moment's reflection indicates that $x = 1$ satisfies the equation. The factor theorem then implies that $x - 1$ is a factor of the quartic. The remaining cubic factor can be obtained by long division or synthetic division (if you have learned it). The division can also be done mentally. The result is

$$P(x) = x^4 + 2x^3 + x^2 - 2x - 2 = (x - 1)(x^3 + 3x^2 + 4x + 2).$$

What this means is that equation (8a) can be replaced by

$$P(x) = (x - 1)(x^3 + 3x^2 + 4x + 2) = 0. \tag{8b}$$

To find further solutions of quartic equation (8a), we need only examine the cubic $x^3 + 3x^2 + 4x + 2$ in (8b) for its zeros. Once we notice that a zero is $x = -1$, we may factor $x + 1$ from the cubic and replace (8b) with

$$P(x) = (x - 1)(x + 1)(x^2 + 2x + 2) = 0. \tag{8c}$$

The remaining two solutions are given by the quadratic formula,

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

Thus quartic equation (8a) has two real solutions $x = \pm 1$ and two complex solutions $x = -1 \pm i$.

A solution of the equation $2x^3 - x^2 + x - 6 = 0$ is $x = 3/2$ (we show how we found it shortly). We therefore factor $2x - 3$ from the cubic,

$$(2x-3)(x^2+x+2)=0.$$

The remaining two solutions are complex

$$x=\frac{-1\pm\sqrt{1^2-4(1)(2)}}{2}=\frac{-1\pm\sqrt{7}i}{2}.$$

The factor theorem enables us to remove a known solution from a polynomial equation, thereby replacing the original polynomial with a polynomial of lower degree. This presupposes two things. First that the equation has a solution, and secondly that we can find it. The following theorem addresses the first question.

Theorem 3 **Fundamental Theorem of Algebra 1** Every polynomial of degree $n \geq 1$ has exactly n linear factors (which may not all be different).

For example, the quartic polynomial in (8a) has four different linear factors

$$x^4+2x^3+x^2-2x-2=(x-1)(x+1)(x+1+i)(x+1-i); \tag{9}$$

the cubic polynomial x^3-3x^2+3x-1 has three linear factors all the same,

$$x^3-3x^2+3x-1=(x-1)^3; \tag{10}$$

and the following eighth degree polynomial has three distinct linear factors, but a total of eight factors,

$$x^8+7x^7-86x^5-95x^4+363x^3+486x^2-540x-648=(x+3)^4(x-2)^3(x+1). \tag{11}$$

Each linear factor in (9), (10), or (11) leads to a zero of the polynomial. For (9), the zeros are ± 1 and $-1\pm i$; for (10), they are 1, 1, 1; and for (11), they are -3, -3, -3, -3, 2, 2, 2, -1. In the case of (10) and (11), there are repetitions. We say that $x=1$ is a zero of multiplicity 3 for the polynomial in (10); the multiplicity corresponds to the number of times the factor $x-1$ appears in the factorization. Each of the zeros in (9) is of multiplicity 1. In (11), $x=-3$ has multiplicity 4, the zero $x=2$ has multiplicity 3, and $x=-1$ has multiplicity 1. These examples suggest that the sum of the multiplicities of the zeros of a polynomial is equal to the degree of the polynomial. This is confirmed in the following alternative version of the Fundamental Theorem of Algebra.

Theorem 4 **Fundamental Theorem of Algebra 2** Every polynomial of degree $n \geq 1$ has exactly n zeros (counting multiplicities).

Our discussions will be confined to polynomials with real coefficients, but Theorems 3 and 4 are valid even when coefficients are complex numbers. What is difficult to prove in Theorem 4 is the existence of one zero. (This is usually done with material from a course on complex function theory.) Once existence of one zero is established, the factor theorem immediately implies that there are precisely n zeros, and that the polynomial can be factored into n linear factors.

When coefficients of a polynomial are real, complex zeros must occur in complex conjugate pairs. This is proved in the next theorem.

Theorem 5 If z is a complex zero of a polynomial with real coefficients, then so also is $\overline{z}$.

Proof Suppose z is a zero of $P(x)=a_nx^n+\cdots+a_1x+a_0$, where coefficients $a_n,\ldots,a_0$ are real. Then,

$$a_nz^n+\cdots+a_1z+a_0=0.$$

We show that $P(\overline{z})=0$. If we take complex conjugates of both sides of this equation, and use the results of Exercise 37 in the Complex Numbers unit,

$$
\begin{aligned}
0 &= \overline{a_n z^n + \cdots + a_1 z + a_0} \\
&= \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} && \text{(part (a) of the exercise)} \\
&= a_n \overline{z^n} + \cdots + a_1 \overline{z} + a_0 && \text{(part (c) of the exercise)} \\
&= a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 && \text{(part (e) of the exercise)} \\
&= P(\overline{z}).
\end{aligned}
$$

We have seen a number of examples of this result. The quartic equation (8) has two real solutions $x = \pm 1$ and two complex conjugate solutions $x = -1 \pm i$. The quartic equation $x^4 + 5x^2 + 4 = (x^2 + 1)(x^2 + 4) = 0$ has two pairs of complex conjugate roots, $x = \pm i$ and $x = \pm 2i$.

When a real polynomial $P(x)$ has a pair of complex zeros, say $x = a \pm bi$, then two linear factors of $P(x)$ are $x - a - bi$ and $x - a + bi$; that is, $P(x)$ can be expressed in the form

$$P(x) = (x - a - bi)(x - a + bi)Q(x), \tag{12a}$$

where $Q(x)$ is a polynomial of degree two less that $P(x)$. If the complex factors are multiplied together the result is

$$P(x) = [x^2 - 2ax + (a^2 + b^2)]Q(x), \tag{12b}$$

where $-2a$ and $a^2 + b^2$ are real. In other words, the pair of complex linear factors in (12a) is equivalent to the real quadratic factor in (12b). Since this can be done for each and every pair of complex conjugate roots, we have the following result.

Theorem 6 Every real polynomial can be factored into the product of real linear and irreducible real quadratic factors.

What we mean by irreducible is that the quadratic factor cannot be factored into real linear factors. For instance, $x^2 + 7$ is an irreducible quadratic, $x^2 - 7$ is not.

EXERCISES

In Exercises 1–4, what is the remainder when the first polynomial is divided by the second?

1. $x^4 + 3x^3 - 2x + 1, \quad x - 2$

2. $x^3 - 2x^2 + 4x + 5, \quad x + 1$

3. $x^4 - 3x^3 + 2x^2 + x + 10, \quad 3x + 4$

4. $x^3 + 3x^2 - 2x + 2, \quad 3x - 1$

In Exercises 5–8, find a polynomial with as low a degree as possible with the given zeros. Assume each zero has multiplicity 1 unless otherwise specified.

5. 3, −2, 4

6. 2 (multiplicity 2), −3, 4 (multiplicity 3)

7. −1 (multiplicity 3), 4

8. 1, −1, 3, −2 (multiplicity 2)

9. Can $x^4 + 5x^2 + 2$ have a real zero?

10. If $-2 + 3i$ is a zero of the polynomial $x^3 + 7x^2 + 25x + 39$, find its other zeros.

11. If $1 + i$ is a zero of the polynomial $x^4 + x^3 + 3x^2 - 8x + 14$, find its other zeros.

∗12. Prove that a polynomial of odd degreee with real coefficients must have at least one real zero. Is this also true for complex polynomials?

∗13. Prove that if $P(x)$ is a polynomial having only even powers of x, and $P(a) = 0$, then $P(x)$ is divisible by $x^2 - a^2$.

In Exercises 14–18, find k in order that the first polynomial be a factor of the second.

$*$14. $x-2,\quad x^3+kx^2+5x-10$

$*$15. $x+1,\quad x^3+4x^2+kx+9$

$*$16. $x+3,\quad x^4+7x^3+kx^2-21x-36$

$*$17. $2x-3,\quad 2x^4+kx^3-6x^2-8x-15$

$*$18. $2x+1,\quad 6x^4+x^3+53x^2+kx-9$

$*$19. Find the remainder when $x^{999}+2x^{998}+x^2-1$ is divided by $x+1$.

$*$20. For what value(s) of k will the polynomial $5x^5+4x^4+kx^3+2x^2+x+1$ have remainder 15 when divided by $x-1$.

$*$21. Find the value(s) of k so that k is the remainder when $x^3-kx^2-14x+15k$ is divided by $x-5$.

$*$22. Find the value(s) of k so that k^2 is the remainder when $2x^3-x^2+(k+1)x+10$ is divided by $x+1$.

In Exercises 23–25, find h and k in order that the first two polynomials be factors of the third.

$*$23. $x+4,\quad x-6,\quad x^4+hx^3-44x^2+kx+576$

$*$24. $x-2,\quad x-5,\quad x^5-15x^4+hx^3+kx^2+274x-120$

$*$25. $x+3,\quad x-3,\quad x^6+16x^4+hx^3+kx^2-1296$

$*$26. Show that when α_1, α_2 and α_3 are the zeros of a cubic polynomial ax^3+bx^2+cx+d, then:

(a) $\alpha_1+\alpha_2+\alpha_3=-\dfrac{b}{a}$

(b) $\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3=\dfrac{c}{a}$

(c) $\alpha_1\alpha_2\alpha_3=-\dfrac{d}{a}$

$*$27. (a) What are the equations in Exercise 26 for the cubic polynomial $P(x)=x^3+3x^2+2x+5$.

(b) The equations in (a) represent three equations in the three zeros of $P(x)$. If we solve them, we find the three zeros. Eliminate two of the zeros to find a single equation in the remaining zero. How do you like the equation?

$*$28. This exercise generalizes the results of Exercise 26. Suppose that the zeros of a polynomial $a_nx^n+\cdots+a_1x+a_0$ are α_1, α_2,…, α_n (where some of the α_j may be repeated).
Show that:

(a) $\alpha_1+\alpha_2+\cdots+\alpha_n=-\dfrac{a_{n-1}}{a_n}$

(b) $\alpha_1\alpha_2\cdots\alpha_n=(-1)^n\dfrac{a_0}{a_n}$

(c) the sum of the products of the roots in pairs is a_{n-2}/a_n; that is,

$$\begin{aligned}&\big(\alpha_1\alpha_2+\cdots+\alpha_1\alpha_n\big)+\big(a_2\alpha_3+\cdots+\alpha_2\alpha_n\big)\\&\qquad+\cdots+\big(\alpha_{n-2}\alpha_{n-1}+\alpha_{n-2}\alpha_n\big)+\alpha_{n-1}\alpha_n=\frac{a_{n-2}}{a_n}.\end{aligned}$$

$*$29. Show that the only way $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$ can have $n+1$ distinct zeros is for all coefficients to be zero.

$*$30. Suppose that $P_1(x)$ and $P_2(x)$ are polynomials of degree n. Use the result of Exercise 29 to show that if $P_1(x)=P_2(x)$ at $n+1$ distinct points, then $P_1(x)=P_2(x)$ for all x.

$*$31. Give a formal definition of what it means for $x=a$ to be a zero of multiplicity k for a polynomial $P(x)$.

$*$32. Suppose $n\geq 2$ points are chosen on the circumference of a circle. Each point is joined to every other point. This divides the interior of the circle into various parts. Let $N(n)$ be the maximum number of parts so formed. Drawings indicate that $N(1)=1$, $N(2)=2$, $N(3)=4$, $N(4)=8$, and $N(5)=16$. One might be led to believe that $N(n)=2^{n-1}$. This is however incorrect since $N(6)=31$. It is known that the formula for $N(n)$ is a quartic polynomial. Find it. What is $N(7)$?

∗33. In this exercise we develop a procedure for solving cubic equations. Such equations can always be expressed in the form

$$x^3 + bx^2 + cx + d = 0.$$

(a) Show that the change of variable $y = x + b/3$ replaces the cubic in x with the following equation in y

$$y^3 + py + q = 0, \qquad \text{where} \quad p = c - \frac{b^2}{3}, \quad q = d - \frac{bc}{3} + \frac{2b^3}{27}.$$

This equation is called the **reduced cubic** (reduced in the sense that there is no y^2 term).

(b) Show that the change of variable $y = z - p/(3z)$ replaces the reduced cubic in y with

$$z^6 + qz^3 - \frac{p^3}{27} = 0,$$

a quadratic equation in z^3. Once this equation is solved for the two values of z^3, cube roots yield values for z, and the transformation $x = z - p/(3z) - b/3$ gives solutions of the original cubic. At most three distinct values result. Should values of z^3 be complex, it would be necessary to take cube roots of complex numbers. Although we learned how to do this in Unit '`Complex Numbers`', the cubics in Exercises 34-39 are specially chosen so that z^3 is real.

In Exercises 34–39, use the procedure in Exercise 33 (and the suggestions) to find solutions for the cubic equation.

∗34. $x^3 - 6x^2 + 11x - 6 = 0$ (In this exercise, it is not necessary to proceed past the substitution $y = x + b/3$.)

∗35. $x^3 + 12x^2 + 48x + 64 = 0$

∗36. $x^3 - 6x^2 + 24x + 31 = 0$

∗37. $x^3 + 4x^2 + 12x + 9 = 0$

∗∗38. $x^3 - 2x^2 + 5x - 10 = 0$ Show that the equation in z is $729z^6 - 5292z^3 - 1331 = 0$, and that both real solutions for z^3 lead to $x = 2$. Now factor $x - 2$ from the cubic.

∗39. $x^3 - 23x^2 - 21x - 72 = 0$ Show that the equation in z is $729z^6 - 826\,875z^3 + 207\,474\,688 = 0$, and that both real solutions for z^3 lead to $x = 24$. Now factor $x - 24$ from the cubic.

∗40. In this exercise we develop a procedure for solving quartic equations. Such equations can always be expressed in the form

$$x^4 + bx^3 + cx^2 + dx + e = 0.$$

(a) Show that this equation can be rewritten

$$\left(x^2 + \frac{bx}{2}\right)^2 = \left(\frac{b^2}{4} - c\right)x^2 - dx - e.$$

(b) Verify that when $(x^2 + bx/2)y + y^2/4$ is added to both sides of the equation in (a), it can be expressed as

$$\left(x^2 + \frac{bx}{2} + \frac{y}{2}\right)^2 = \left(\frac{b^2}{4} - c + y\right)x^2 + \left(\frac{by}{2} - d\right)x + \left(\frac{y^2}{4} - e\right).$$

(c) The right side of the equation in (b) is quadratic in x, and can always be factored. But because the left side is a perfect square, these factors must be identical. It follows that the discriminant must vanish. Show that this requires y to satisfy the cubic equation

$$y^3 - cy^2 + (bd - 4e)y + (4ce - b^2e - d^2) = 0,$$

called the **resolvent** equation. When y is a solution of the resolvent equation, corresponding values for x can be obtained by substituting into the equation in (b) and taking square roots. It can be shown that all three solutions for y lead to the same four solutions of the quartic. Usually we would choose a real solution for y.

In Exercises 41–45, use the procedure of Exercise 40 to solve the quartic equation.

$*$**41.** $x^4 - 16 = 0$ (This is a very simple example.)

$*$**42.** $x^4 - 5x^2 + 4 = 0$

$*$**43.** $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$

$*$**44.** $x^4 + 7x^3 + 9x^2 - 21x - 36 = 0$ ($y = 9$ is a solution of the resolvent equation.)

$*$**45.** $x^4 - 3x^3 - 3x^2 + 11x - 6 = 0$

Answers

1. 37 **2.** -2 **3.** 486 **4.** 46/27 **5.** $(x-3)(x+2)(x-4)$
6. $(x-2)^2(x+3)(x-4)^2$ **7.** $(x+1)^3(x-4)$ **8.** $(x-1)(x+1)(x-3)(x+2)^2$
9. No **10.** $-2-3i$, -3 **11.** $1-i$, $(-3/2) \pm (\sqrt{19}/2)i$ **12.** No
14. -2 **15.** 12 **16.** 9 **17.** 9 **18.** 9 **19.** 1 **20.** 2 **21.** 5
22. 2, -3 **23.** -4, 96 **24.** 85, -225 **25.** 0, -81 **27.** (b) $\beta^3 + 3\beta^2 + 2\beta + 5 = 0$
31. $P(x) = (x-a)^k Q(x)$ where $Q(a) \neq 0$
32. $(n^4 - 6n^3 + 23n^2 - 18n + 24)/24$ **34.** 1, 2, 3 **35.** -4 of multiplicity 3
36. -1, $(7/2) \pm (5\sqrt{3}/2)i$ **37.** -1, $(-3/2) \pm (3\sqrt{3}/2)i$ **38.** 2, $\pm\sqrt{5}i$
39. 24, $(-1/2) \pm (\sqrt{11}/2)i$ **41.** ± 2, $\pm 2i$ **42.** ± 1, ± 2
43. $-1 \pm \sqrt{2}$ both of muliplicity 2 **44.** -3, -4, $\pm\sqrt{3}$
45. 1 of multiplicity 2, -2, 3