Estimation of mean squared prediction error of empirically spatial predictor of small area means under a linear mixed model

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Abstract

Policy decisions regarding allocation of resources to subgroups in a population, called small areas, are based on reliable predictors of their underlying parameters. However, the information is collected at a different scale than these subgroups. Hence, we need to predict characteristics of the subgroups based on the coarser scale data. In view of this, there is a growing demand for reliable small area predictors by borrowing information from other related sources. For this purpose, mixed models have been commonly used in small area estimation assuming independent small areas. There are many situations, however, that the small area parameters are related to their locations. For instance, it is an interest of policy makers (and public) to know the spatial pattern of a chronic disease (e.g., asthma) to identify small areas with high risk of disease for possible preventions. In this paper, we propose small area models in the class of spatial linear mixed models to be able to predict small area parameters and also to obtain corresponding mean squared pre-

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diction error (MSPE). We also provide unbiased estimators of MSPE of small area predictors using Taylor series expansion and parametric bootstrap methods. In our simulations, we show that our MSPE estimators using Taylor expansion and parametric bootstrap perform very well in terms of precision of small area predictors. Performance of our proposed approach is also evaluated through a real application of physician visits for Total Respiratory Morbidity conditions in Manitoba, Canada.

Keywords: Conditional auto-regressive model; Random effects; Restricted maximum likelihood; Small area estimation; Spatial model

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1. Introduction

Sample surveys are conducted with the purpose of providing reliable predictors for the finite population characteristics such as totals or means. Methods used in deriving such predictors (direct survey predictors) are based on total sample size. However in the past few decades, there have been increasing demand in using same sample survey data to get predictions for sub-populations, such as health regions or gender-age groups. Such sub-populations for which reliable predictions are needed are called small areas in the literature. The term small area refers to small sample size compared to population size in that area.The traditional area-specific direct predictors tend to have inadequate precision due to small sample sizes corresponding to population sizes for each small area. Since policy decisions about implementing specific projects to these small areas are made using predictions on underlying characteristics, survey researchers are developing methods to provide more reliable predictions for small areas. To this end, model-based estimators (Jiang and Lahiri, 2006; Rao and Molina, 2015; Jiang, 2017, Chap. 4) have been proposed to borrow strength from other related sources such as past surveys and census. For this purpose, mixed models are commonly used in small area estimation.

In particular, in the context of linear mixed models (LMMs), such small area models may be classified into two broad types: (i) Area-level models that relate small area direct estimates to area-specific covariates; such models are used if unit-level data are not available. (ii) Unit-level models that relate the unit values of a study variable to associated unit-level covariates with known area means and area-specific covariates. A comprehensive account of model-based small area estimation under area-level and unit-level models is given by Rao and Molina (2015). Among other approaches, parameters of the LMM can be estimated using either the maximum likelihood (ML) or restricted ML (REML). Although it is somewhat straightforward to predict the small area parameters under the LMM, e.g., using the best linear unbiased predictor (BLUP), obtaining its prediction error and associated prediction interval is difficult.

Fay and Herriot (1979; hereafter by FH) used an area-level model assuming independent small areas to predict small area parameters. Following this seminal work, many developments have been then made in small area estimation to predict small area characteristics (e.g., mean) and to obtain the corresponding mean squared prediction error (MSPE) estimation. There are many situations, however, that the small area parameters are related to their locations. For instance, it is an interest of policy makers (and public) to know the spatial pattern of a chronic disease (e.g., asthma) to identify small areas with high risk of disease for possible preventions. There is limited literate in small area estimation assuming small areas are spatially correlated.

Spatial models on the area specific effects are used when "neighboring" areas can be defined for each small area. Such models induce correlations among the small areas on geographical proximity for example in the context of estimating area-level mortality disease rates. Cressie (1991) used conditional auto-regressive (CAR) to account for the area-level spatial random effects (Besag 1974) for small area estimation in the context of US census undercount. Petrucci and Salvati (2006) used simultaneous auto-regressive (SAR) to account for the area-level spatial random effects to estimate the amount of erosion delivered to streams in the Rathbun Lake Watershed in Iowa. Pratesi and Salvati (2008) used the same model to estimate the mean per capita income (PCI) in sub-regions of Tuscany using data from the 2001 Life Condition Survey from Tuscany.

In terms of spatial model parameters estimation, Cressie and Chan (1989) studied ML estimation of spatial model parameters for the spatial FH model. Since spatial FH models are special cases of general linear mixed model, the MSPE of the BLUP has been provided in the literature (Kackar and Harville, 1984). However, a rigorous second-order MSPE approximation for the spatial empirical BLUP (EBLUP) can not be obtained from the general framework of Das, Jiang, and Rao (2004) because of the presence of correaltion among observations from different small areas.

Singh, Shukla, and Kundu (2005) also studied the spatial FH models in small area estimation. In particular, they considered the SAR to account for the spatial random effects and ML approach to estimate the model parameters. They heuristically derived the second-order MSPE of EBLUP of small area mean and obtained the corresponding second-order unbiased estimator of MSPE using Taylor expansion assuming that the number of small areas is finite. Petrucci and Salvati (2006) also used the MSPE estimators using Taylor expansion with model parameters estimated by REML or ML, respectively, in the case of SAR to account for the spatial random effects. Molina, Salvati, and Pratesi (2009) considered a spatial FH model with SAR random effects and obtained the EBLUP of the small area mean and used bootstrap MSPE estimator for spatial EBLUP of small area means.

Singh et al. (2005) also used saptio-temporal FH models to develop EBLUP estimators to study the relative performance of spatial and spatio-temporal models on monthly data on per captia concumer expenditure from India. Marhuenda, Molina, and Morales (2013) also considered the same set-up as Molina, Salvati, and Pratesi (2009) but for spatio-temporal FH model by assuming that area effects in the Rao-Yu model (Rao and Yu, 1994; Torabi and Shokoohi, 2012) follow a SAR to account for the spatial random effects. Schmid and M*ü*nnich (2014) extended the theory of robust EBLUP (Sinha and Rao, 2009) to spatial linear mixed models. Using Bayesian inference, Porter, Holan, Wikle, and Cressie (2014) used spatial FH model to analyze relative change of percent household Spanish-speaking in the U.S using direct estimators for the states (small areas) from the American Community Survey (ACS) and big data covariates from Google Trends searches over time as functional covariates. Porter, Wikle, and Holan (2015) extended the FH model to the multivariate FH models with latent spatial dependence in the Bayesian framework. Chandra, Salvati, and Chambers (2015) extended the

FH model that accounts for the SAR spatial non-stationarity where the parameters of regression model vary spatially. Baldermann, Salvati, and Schmid (2018) extended the work of Schmid and Munnich (2014) to the SAR spatial non-stationary model.

It is well-known that the conditional spatial dependence parameter defined through the SAR model can be inconsistent unlike the CAR model (Schabenberger and Gotway, 2004, page 340; Banerjee, Carlin, and Gelfand, 2014). In this paper, we introduce the spatial FH model by considering CAR to account for the spatial random effects (Banerjee, Carlin, and Gelfand, 2014) and use the generalized weighted least squared approach to estimate the regression coefficients and the REML to estimate the variance components. We also rigorously obtain the MSPE of EBLUP of small area means as well as the estimators of MSPEs of the EBLUP of small area means.

The rest of the paper is organized as follows. In section 2, we introduce the general set-up for spatial linear mixed model and the BLUP of small area means and corresponding MSPE of the BLUP of small area means. In section 3, we use the REML method to estimate the variance components to get the EBLUP of small area means. In section 4, we provide asymptotic expression of the MSPE of EBLUP of small area means. The estimation of MSPE of EBLUP of small area means is considered in section 5 using the Taylor expansion and parametric bootstrap approaches. We spell-out the spatial linear mixed model theory for the special case of FH model (section 6). In section 7, performance of the proposed approach is evaluated using a real application of physician visits for Total Respiratory Morbidity conditions in Manitoba, Canada, during 2000–2010. We also evaluate our proposed approach using a simulation study in section 8. Finally, some concluding remarks are given in section 9. Technical details and computer codes are provided as supplementary materials.

2. Spatial linear mixed model

The model is given by

$$y_i = x_i^\top \beta + z_i^\top v + \epsilon_i; \qquad i = 1, ..., n$$

In matrix notation, one can also write

$$\mathrm{y} = X\beta + Zv + \epsilon, \tag{2.1}$$

where X_{np} and Z_{nn} are known $n \times n$ and $n \times p$ matrices, respectively, β is a vector of regression coefficients with dimension p, $v \sim (0, G)$ and $\epsilon \sim (0, R)$ with G and R depend on some variance parameters $\sigma = (\sigma^{(0)}, \sigma^{(1)}) = (\sigma_{01}, ..., \sigma_{0q_0}, \sigma_{11}, ..., \sigma_{1q_1})$ assuming that σ belongs to a specified subset of the q-dimensional Euclidean space such that

$$cov(\mathrm{y}) = \Sigma = R + ZGZ^\top$$

is non-singular for all σ belonging to the subset where $q = q_0 + q_1$. In the case of spatial model, one can have $Z = I_n$, with I_n being the identity matrix with dimension n,$v = (v_1, ..., v_n)^\top$ where $G = \sigma_{11} S(\sigma_{12}, ..., \sigma_{1q_1}), R = R(\sigma^{(0)})$ with known $\sigma^{(0)}$, and $S(\cdot)$ is spatial covariance; noting that σ_{11} captures spatial dispersion and $(\sigma_{12}, ..., \sigma_{1q_1})$ account for spatial correlation. In particular, one popular approach is to use the proper CAR model (Cressie and Chan, 1989; Stern and Cressie, 1999) with

$$G = \sigma_{11}(I_n - \sigma_{12} D)^{-1} C^{-1}, \tag{2.2}$$

where σ_{11} and σ_{12} account for spatial dispersion and spatial correlation, respectively; D is a $n \times n$ matrix with elements $D_{ii} = 0, D_{ij} = w_{ij}/w_{i+}$ if area i and j are adjacent (shown by $i \sim j$) and $D_{ij} = 0$ otherwise, where w_{ij} is the corresponding weight and $w_{i+} = \sum_{i \sim j} w_{ij}$; and $C = diag(w_{i+})$. See section 4 for regularity conditions of X, Z, R, and G.

Our interest is to find a predictor of the following small area mean

$$\mu = l^\top \beta + m^\top v, \tag{2.3}$$

for specified matrices, l and m, of constants. BLUP of μ under model (2.1) [see Henderson, 1975] is:

$$t(\sigma) = t(\sigma, \mathrm{y}) = l^\top \tilde{\beta} + s^\top(\sigma)(\mathrm{y} - X\tilde{\beta}), \tag{2.4}$$

where $\tilde{\beta} = \tilde{\beta}(\sigma) = (X^\top \Sigma^{-1} X)^{-1} X^\top \Sigma^{-1} \mathrm{y}$, $s(\sigma) = \Sigma^{-1} Z G m$, and corresponding $Var(\tilde{\beta}) = (X^\top \Sigma^{-1} X)^{-1}$. Note that $t(\sigma, \mathrm{y})$ is an unbiased predictor of μ in the sense that $E[t(\sigma, \mathrm{y}) - \mu] = 0$. Also, the measure of variability of $t(\sigma)$ can be accounted through:

$$MSE[t(\sigma)] = E[t(\sigma) - \mu][t(\sigma) - \mu]^\top = g_1(\sigma) + g_2(\sigma), \tag{2.5}$$

where

$$g_1(\sigma) = m^\top (G - GZ^\top \Sigma^{-1} Z G) m,$$

and

$$g_2(\sigma) = [l - X^\top s(\sigma)]^\top (X^\top \Sigma^{-1} X)^{-1} [l - X^\top s(\sigma)].$$

3. EBLUP estimator

The BLUP estimator $t(\sigma, \mathrm{y})$ given by (2.4) depends on the variance parameters σ which $\sigma^{(1)}$ are unknown in practical applications. We estimate $\sigma^{(1)}$ using the REML approach. To that end, the log-likelihood function of model (2.1) is

$$\ell(\sigma) = c - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(\mathrm{y} - X\beta)^\top \Sigma^{-1}(\mathrm{y} - X\beta), \tag{3.1}$$

where c is a constant. Then there exist some T such that $T^\top X = 0$ and $rank(T) = n - p$. For any T satisfying the above condition, the log-likelihood is

$$\ell_R(\sigma) = c - \frac{1}{2}[\log(|T^\top \Sigma T|) + \mathrm{y}^\top P\mathrm{y}], \tag{3.2}$$

where $P = \Sigma^{-1} - \Sigma^{-1}X(X^\top\Sigma^{-1}X)^{-1}X^\top\Sigma^{-1}$. Define the REML estimator $\hat{\sigma}^{(1)}$ as the solution to the following score function:

$$\frac{\partial \ell_R(\sigma)}{\partial \sigma^{(1)}}|_{\sigma^{(1)}=\hat{\sigma}^{(1)}} = 0,$$

where

$$\frac{\partial \ell_R(\sigma)}{\partial \sigma_j^{(1)}} = \frac{1}{2}[\mathrm{y}^\top PV_jP\mathrm{y} - tr(PV_j)]$$

$$= \frac{1}{2}[u^\top PV_jPu - tr(PV_j)], \qquad j = 1, ..., q_1,$$

with $u = \mathrm{y} - X\beta; u \sim N(0, \Sigma); V_j = \frac{\partial \Sigma}{\partial \sigma_j^{(1)}}(j = 1, ..., q_1)$. We now provide a class of convergence of $\hat{\sigma}^{(1)}$ in the following Lemma 1.

Lemma 1: Let $d_i^2 = \max_{j,k}\{tr(PV_iPV_i), tr(P\frac{\partial V_i}{\partial \sigma_j^{(1)}}P\frac{\partial V_i}{\partial \sigma_j^{(1)}}), tr(P\frac{\partial^2 V_i}{\partial \sigma_j^{(1)}\partial \sigma_k^{(1)}}P\frac{\partial^2 V_i}{\partial \sigma_j^{(1)}\partial \sigma_k^{(1)}})\}$ and $d_* = \min_{1\le i\le q_1} d_i$. Then there exists $\hat{\sigma}^{(1)}$ such that for any $0 < q_0 < 1$, there is a set $\Im$ satisfying for large n and on $\Im$,

$$\hat{\sigma}^{(1)} - \sigma^{(1)} = -A^{-1}a + o_p(d_*^{-2q_0}u),$$

where $a = \frac{\partial \ell(\sigma)}{\partial \sigma^{(1)}}, A = E(\frac{\partial^2 \ell(\sigma)}{\partial \sigma^{(1)} \partial \sigma^{(1)\top}})$, and $E(u^g)$ is bounded; and $Pr(\Im^c) \leq cd_*^{-\tau g}$ where $\tau = \frac{1}{4} \wedge (1 - q_0)$ and c is a constant.

Proof: See the Supplementary Materials A.

Hence, the EBLUP estimator of μ is given by $t(\hat{\sigma}, \mathrm{y})$. We can also obtain the variance of model parameters estimate $\hat{\sigma}^{(1)}$ using the inverse of corresponding Fisher information matrix. In particular, the matrix of expected second derivatives of $\ell_R(\sigma)$ with respect to $\sigma^{(1)}$ is given by $I(\sigma^{(1)})$ with (j,k)th element $I_{j,k}(\sigma^{(1)}) = \frac{1}{2} tr[PV_j PV_k], (j,k = 1, ..., q_1)$. Then, we have $Var(\hat{\sigma}^{(1)}) = I^{-1}(\sigma^{(1)})$.

4. Asymptotic expression for the MSPE of $t(\hat{\sigma}, \mathrm{y})$

We now obtain a second-order approximation to the MSPE of EBLUP $t(\hat{\sigma}, \mathrm{y})$, note that $\hat{\sigma} = (\sigma^{(0)}, \hat{\sigma}^{(1)})$. Under normality (Kackar and Harville, 1984), we have

$$MSPE[t(\hat{\sigma})] = E[t(\hat{\sigma}) - \mu][t(\hat{\sigma}) - \mu]^\top$$

$$= MSPE[t(\sigma)] + E[t(\hat{\sigma}) - t(\sigma)][t(\hat{\sigma}) - t(\sigma)]^\top, \tag{4.1}$$

where MSPE$[t(\sigma)] = g_1(\sigma) + g_2(\sigma)$. One of the key steps in obtaining the asymptotic expression for MSPE in (4.1) is to establish an approximation of $E[t(\hat{\sigma}) - t(\sigma)][t(\hat{\sigma}) - t(\sigma)]^\top$ using the Taylor expansion of $t(\hat{\sigma})$ around $\sigma^{(1)}$. Under some regularity conditions, our interest is to establish

$$E[t(\hat{\sigma}) - t(\sigma)][t(\hat{\sigma}) - t(\sigma)]^\top = E[\frac{\partial t(\sigma)}{\partial \sigma^{(1)}}(\hat{\sigma}^{(1)} - \sigma^{(1)})][\frac{\partial t(\sigma)}{\partial \sigma^{(1)}}(\hat{\sigma}^{(1)} - \sigma^{(1)})]^\top + [o(d_*^{-2})]_{n \times n}. \tag{4.2}$$

The covariance matrix of our spatial random effects has a form of $\sigma_{11}S(\sigma_{12},...,\sigma_{1q_1})$ which is not linear in σ. Note that the asymptotic is in the sense of increasing number of small areas n. The following regularity conditions (referred to as RC later on) will be assumed throughout the paper:

1) The elements of X, Z, and R are bounded. Also, G is bounded ($[\lambda_{\max}(G^\top G)]^{1/2} < C$) for some constant C in such a way that the off-diagonal elements of G decay excluding (first) neighbours of each small area which are bounded.

2) The smallest and largest eigenvalues of $\Sigma, \Sigma^{-1}, \frac{\partial\Sigma}{\partial\sigma_j^{(1)}}, \frac{\partial^2\Sigma}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}}$, $\Sigma^{-1}\frac{\partial\Sigma}{\partial\sigma_j^{(1)}}$, $\Sigma^{-1/2}\frac{\partial\Sigma}{\partial\sigma_j^{(1)}}\Sigma^{-1/2}$, $\Sigma^{-1}\frac{\partial\Sigma}{\partial\sigma_j^{(1)}}\Sigma^{-1}$, $\frac{\partial\Sigma^{-1}}{\partial\sigma_j^{(1)}}\Sigma\frac{\partial\Sigma^{-1}}{\partial\sigma_j^{(1)}}$, $\Sigma\frac{\partial\Sigma^{-1}}{\partial\sigma_j^{(1)}}\Sigma\frac{\partial\Sigma^{-1}}{\partial\sigma_j^{(1)}}\Sigma$, $(1 \le j, k \le q_1)$, and $(X^\top\Sigma^{-1}X)^{-1}$ are bounded away from zero and infinity, respectively.

3) $s(\sigma)$ has bounded second derivatives w.r.t. $\sigma^{(1)}(2 \le i, j \le q_1)$.

4) $l - X^\top s(\sigma) = [O(1)]_{p\times n}$, $[\frac{\partial X^\top s(\sigma)}{\partial\sigma_j^{(1)}}] = [O(1)]_{p\times n}$, and $[\frac{\partial^2 X^\top s(\sigma)}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}}] = [O(1)]_{p\times n}$, $(1 \le j, k \le q_1)$, noting that $[O(1)]_{p\times n}$ means that each element of the matrix is $O(1)$.

Theorem 1. *Suppose that the above conditions RC (1-4) are satisfied. Let* $t(\hat\sigma) = l^\top\hat\beta + m^\top\hat v$ *be the EBLUP of* μ *and* $\hat\sigma^{(1)}$ *be the REML estimator of* $\sigma^{(1)}$*. We then have:*

$$E[t(\hat\sigma)-t(\sigma)][t(\hat\sigma)-t(\sigma)]^\top = E[\frac{\partial t(\sigma)}{\partial\sigma^{(1)}}(\hat\sigma^{(1)}-\sigma^{(1)})][\frac{\partial t(\sigma)}{\partial\sigma^{(1)}}(\hat\sigma^{(1)}-\sigma^{(1)})]^\top + [o(d_*^{-2})]_{n\times n}$$

$$= -\nabla^\top s(\sigma)[A^{-1}\otimes\Sigma]\nabla s(\sigma) + [o(d_*^{-2})]_{n\times n}$$

$$:= g_3(\sigma) + [o(d_*^{-2})]_{n\times n},$$

where $\nabla s(\sigma) = col_{1\le j\le q_1}[\nabla_j s(\sigma)]$ *with* $\nabla_j s(\sigma) = \frac{\partial s(\sigma)}{\partial\sigma_j^{(1)}}$*, and* $\otimes$ *is Kronecker product. Note that the* $g_3(\sigma)$ *can be also written as* $g_3(\sigma) = -\sum_{j=1}^{q_1}\sum_{k=1}^{q_1} A_{jk}^{-1}(\sigma)$

$.[\nabla'_j s(\sigma)\Sigma\nabla_k s(\sigma)]$, *where* $A^{-1}_{jk}(\sigma)$ *is the* (j,k)*th element of* $A^{-1}(\sigma)$. *The MSPE of the EBLUP* $t(\hat{\sigma})$ *is then*

$$MSPE[t(\hat{\sigma})] = g_1(\sigma) + g_2(\sigma) + g_3(\sigma) + [o(d_*^{-2})]_{n\times n}. \tag{4.3}$$

Proof: See the Supplementary Materials A.

Note that the MSPE expression (4.3) is different from the Das, Jiang, and Rao, 2004; hereafter by DJR) derived for the longitudinal model. The variance components in G are not linear in parameters σ unlike the DJR model.

5. Estimation of the MSPE of $t(\hat{\sigma}, \mathrm{y})$

5.1 Taylor expansion

Since the approximated MSPE (4.3) is a function of unknown parameters $\sigma^{(1)}$, it is not computable. We now obtain the estimation of MSPE$[t(\hat{\sigma})]$ which is second-order unbiased in the sense that

$$E\{mspe[t(\hat{\sigma})]\} = MSPE[t(\hat{\sigma})] + [o(d_*^{-2})]_{n\times n}. \tag{5.1}$$

The following Theorem summarizes the asymptotic properties of $mspe[t(\hat{\sigma})]$.

Theorem 2. *Under the RC (1-4), the MSPE estimation of* $t(\hat{\sigma})$ *is given by*

$$mspe[t(\hat{\sigma})] = g_1(\hat{\sigma}) + g_2(\hat{\sigma}) + 2g_3(\hat{\sigma}) - \triangle(\hat{\sigma}), \tag{5.2}$$

which is second-order unbiased in the sense of (5.1), noting that $\hat{\sigma}$ *is the REML estimator of* σ, *and* $\triangle(\sigma) = \frac{-1}{2}\sum_{j=1}^{q_1}\sum_{k=1}^{q_1} A^{-1}_{jk}(\sigma)[m^\top \frac{\partial^2 G}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}} m +$

$s^\top \frac{\partial^2 G}{\partial \sigma_j^{(1)} \partial \sigma_k^{(1)}} s - 2m^\top \frac{\partial^2 G}{\partial \sigma_j^{(1)} \partial \sigma_k^{(1)}} Z^\top s].$

Proof: See the Supplementary Materials A.

5.2 Parametric Bootstrap

We now obtain an estimator of MSPE$[t(\hat{\sigma})]$ using the parametric bootstrap approach. As we have explicit forms of the MSPE approximation of EBLUP of small area means in terms of $g_1(\sigma) + g_2(\sigma) + g_3(\sigma)$, we only need to make a bias correction for the $g_1(\sigma)$ term as the other terms, $g_2(\sigma)$ and $g_3(\sigma)$, are asymptotically unbiased. To that end, we first draw v^* from $N(0, G(\hat{\sigma}^{(1)}))$ and ϵ^* from $N(0, R)$ independently. We then create bootstrap values $y^* = X\hat{\beta} + Zv^* + \epsilon^*$. Now using the bootstrap dataset (y^*, X), we estimate the model parameters as $(\hat{\beta}^*, \hat{\sigma}^*)$ where $\hat{\sigma}^* = (\sigma^{(0)}, \hat{\sigma}^{(1)*})$. We can then obtain $g_1(\hat{\sigma}^*)$ following $g_1(\sigma)$ defined in section 2. Hence, the bootstrap bias correction of $g_1(\hat{\sigma})$ is given by $b^*(\hat{\sigma}) = E_*[g_1(\hat{\sigma}^*)] - g_1(\hat{\sigma})$, where E_* denotes the bootstrap expectation. In practice, we approximate $E_*(.)$ by drawing a large number of independent bootstrap samples. An estimator of MSPE$[t(\hat{\sigma})]$ is then given by

$$mspe_{boot}[t(\hat{\sigma})] = g_1(\hat{\sigma}) + g_2(\hat{\sigma}) + g_3(\hat{\sigma}) - b^*(\hat{\sigma}).$$

We evaluate the performance of $mspe_{boot}[t(\hat{\sigma})]$ in the data application (section 7) and simulation study (section 8).

6. An illustration: spatial Fay-Herriot model

We now spell-out our general model for the specific case of spatial FH model. Our model is $y_i = x_i^\top \beta + z_i^\top v + \epsilon_i$, where x_i is a vector of size p, z_i

is a vector of zeros (with size n) except the ith element which is one, v has a multivariate Normal distribution following CAR model with parameters $\sigma^{(1)} \equiv (\sigma_v^2, \lambda_v)$ where $\sigma_{11} \equiv \sigma_v^2$ and $\sigma_{12} = \lambda_v$, and $\epsilon_i \sim N(0, \sigma_{0i}^2)$, assuming σ_{0i}^2's are known. The small area mean for $i-$th area is:

$$\mu_i = x_i^\top \beta + v_i,$$

where $y_i = \mu_i + \epsilon_i$. If σ are known, the componentwise BLUP of $\mu = (\mu_1, ..., \mu_n)^\top$ is given by

$$t(\sigma, y) = X\tilde{\beta}(\sigma) + s(\sigma)^\top (y - X\tilde{\beta}(\sigma)), \tag{6.1}$$

$$\tilde{\beta}(\sigma) = (X^\top \Sigma^{-1} X)^{-1} (X^\top \Sigma^{-1} y),$$

where $s(\sigma) = \Sigma^{-1} G = (R + G)^{-1} G$, where G is given by (2.2) and $R = diag(\sigma_{0i}^2)$. In practice, however, $\sigma^{(1)}$ are not known. Substituting consistent estimators $\hat{\sigma}$ for σ in (6.1), we get the EBLUP given by

$$t(\hat{\sigma}, y) = X\hat{\beta} + s(\hat{\sigma})^\top (y - X\hat{\beta}), \tag{6.2}$$

where $\hat{\beta} = \tilde{\beta}(\hat{\sigma})$. Following Lemma 1, we can get the REML estimator $\hat{\sigma}^{(1)}$ as follows:

$$\hat{\sigma}^{(1)} = \sigma^{(1)} - \{E[\frac{\partial^2 \ell_R(\sigma)}{\partial \sigma^{(1)} \partial \sigma^{(1)\top}}]\}^{-1} (\frac{\partial \ell_R(\sigma)}{\partial \sigma^{(1)}}) + [o(n^{-1})]_{2\times 1}, \tag{6.3}$$

using any iterative approach such as Newton Raphson method, where

$$\frac{\partial \ell_R(\sigma)}{\partial \sigma_j^{(1)}} = \frac{1}{2}[u^\top P V_j P u - tr(P V_j)], \qquad j = 1, 2,$$

with $u = \mathrm{y} - X\beta; u \sim N(0, \Sigma); V_j = \frac{\partial \Sigma}{\partial \sigma_j^{(1)}} (j = 1, 2)$. We have $V_1 = \sigma_v^{-2} G, V_2 = \sigma_v^2 (I_n - \lambda_v D)^{-1} D (I_n - \lambda_v D)^{-1} C^{-1}$ with C and D defined in section 2. We can then have $\frac{\partial \ell_R(\sigma)}{\partial \sigma_v^2}$ and $\frac{\partial \ell_R(\sigma)}{\partial \lambda_v}$.

To get $\{E[\frac{\partial^2 \ell_R(\sigma)}{\partial\sigma^{(1)}\partial\sigma^{(1)\top}}]\}^{-1}$, we have

$$\frac{\partial^2 \ell_R(\sigma)}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}} = \frac{1}{2}[-u^\top D_{jk}u + tr(PV_kPV_j) - tr(P\frac{\partial V_j}{\partial\sigma_k^{(1)}})],$$

where $D_{jk} = P(V_kPV_j - \frac{\partial V_j}{\partial\sigma_k^{(1)}} + V_jPV_k)P$, and $\frac{\partial V_1}{\partial\sigma_v^2} = 0$, $\frac{\partial V_1}{\partial\lambda_v} = (I_n - \lambda_v D)^{-1}D(I_n - \lambda_v D)^{-1}C^{-1}$, $\frac{\partial V_2}{\partial\sigma_v^2} = (I_n - \lambda_v D)^{-1}D(I_n - \lambda_v D)^{-1}C^{-1}$, $\frac{\partial V_2}{\partial\lambda_v} = 2\sigma_v^2(I_n - \lambda_v D)^{-1}D(I_n - \lambda_v D)^{-1}D(I_n - \lambda_v D)^{-1}C^{-1}$. Hence, one can write

$$\frac{\partial^2 \ell_R(\sigma)}{\partial\sigma_v^2\partial\sigma_v^2} = -u^\top PV_1PV_1Pu + \frac{1}{2}tr(PV_1PV_1),$$

$$\begin{aligned}\frac{\partial^2 \ell_R(\sigma)}{\partial\lambda_v\partial\lambda_v} &= \frac{-1}{2}[u^\top PV_2PV_2Pu + u^\top PV_2PV_2Pu - u^\top P\frac{\partial V_2}{\partial\lambda_v}Pu \\ &\quad -tr(PV_2PV_2) + tr(P\frac{\partial V_2}{\partial\lambda_v})], \\ \frac{\partial^2 \ell_R(\sigma)}{\partial\sigma_v^2\partial\lambda_v} &= \frac{-1}{2}[u^\top PV_2PV_1Pu + u^\top PV_1PV_2Pu - u^\top P\frac{\partial V_1}{\partial\lambda_v}Pu \\ &\quad -tr(PV_2PV_1) + tr(P\frac{\partial V_1}{\partial\lambda_v})].\end{aligned}$$

Then,

$$E[\frac{\partial^2 \ell_R(\sigma)}{\partial\sigma_v^2\partial\sigma_v^2}] = -tr(\Sigma PV_1PV_1P) + \frac{1}{2}tr(PV_1PV_1), \tag{6.4}$$

$$E[\frac{\partial^2 \ell_R(\sigma)}{\partial\lambda_v\partial\lambda_v}] = -tr(\Sigma PV_2PV_2P) + \frac{1}{2}tr(PV_2PV_2) + \frac{1}{2}tr[(\Sigma P - I_n)\frac{\partial V_2}{\partial\lambda_v}P], \tag{6.5}$$

$$E[\frac{\partial^2 \ell_R(\sigma)}{\partial\sigma_v^2\partial\lambda_v}] = \frac{-1}{2}tr[\Sigma PV_2PV_1P + \Sigma PV_1PV_2P - PV_2PV_1 - (\Sigma P - I_n)\frac{\partial V_1}{\partial\lambda_v}P]. \tag{6.6}$$

We then have the REML estimator $\hat{\sigma}^{(1)}$ by (6.3).

To get the MSPE of EBLUP $t(\hat{\sigma}, y)$, we can have

$$MSPE[t(\hat{\sigma})] = g_1(\sigma) + g_2(\sigma) + g_3(\sigma) + [o(n^{-1})]_{n\times n},$$

where

$$g_1(\sigma) = G - G\Sigma^{-1}G,$$

$$g_2(\sigma) = [X^\top - X^\top s(\sigma)]^\top (X^\top \Sigma^{-1} X)^{-1} [X^\top - X^\top s(\sigma)],$$

$$g_3(\sigma) = -\nabla^\top s(\sigma)[A^{-1} \otimes \Sigma]\nabla s(\sigma),$$

where $\nabla s(\sigma) = col_{1\le j\le 2}[\nabla_j s(\sigma)]$ and $\nabla_j s(\sigma) = \frac{\partial s(\sigma)}{\partial \sigma_j^{(1)}} = -(R+G)^{-1}\frac{\partial G}{\partial \sigma_j^{(1)}}(R+G)^{-1}G + (R+G)^{-1}\frac{\partial G}{\partial \sigma_j^{(1)}}$, and A is the 2×2 matrix with elements (6.4)-(6.6). The MSPE of area-specific EBLUP $t(\hat{\sigma}, y_i)$ is the (i,i) element of MSPE$[t(\hat{\sigma})]$. In the case of non-spatial random effects ($G = \sigma_v^2 I_n$), our model reduces to the conventional FH model with

$$g_{1i}(\sigma) = \sigma_v^2 - \sigma_v^4/(\sigma_{0i}^2 + \sigma_v^2) = \sigma_{0i}^2\gamma_i,$$

$$g_{2i}(\sigma) = (1-\gamma_i)^2 x_i^\top (X^\top \Sigma^{-1} X)^{-1} x_i,$$

where $\gamma_i = \sigma_v^2/(\sigma_v^2 + \sigma_{0i}^2)$. To get the $g_{3i}(\sigma)$, we have $\frac{\partial G}{\partial \sigma_v^2} = I_n, s_i(\sigma) = \sigma_v^2/(\sigma_v^2 + \sigma_{0i}^2), \nabla_1 s_i(\sigma) = \sigma_{0i}^2/(\sigma_v^2 + \sigma_{0i}^2)^2$, so

$$g_{3i}(\sigma) = V(\hat{\sigma}_v^2)\Big[\frac{\sigma_{0i}^2}{(\sigma_v^2 + \sigma_{0i}^2)^2}\Big]^2 (\sigma_v^2 + \sigma_{0i}^2) = \frac{V(\hat{\sigma}_v^2)}{\sigma_v^2 + \sigma_{0i}^2}(1-\gamma_i)^2,$$

noting that $-A^{-1} = V(\hat{\sigma}_v^2)$.

To get the estimation of MSPE$[t(\hat{\sigma}, y)]$ using Taylor expansion, we can write

$$mspe[t(\hat{\sigma}, y)] = g_1(\hat{\sigma}) + g_2(\hat{\sigma}) + 2g_3(\hat{\sigma}) - \triangle(\hat{\sigma}), \tag{6.7}$$

where $\triangle(\sigma) = \frac{-1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} A_{jk}^{-1}(\sigma)[\frac{\partial^2 G}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}} + s^\top \frac{\partial^2 G}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}} s - 2\frac{\partial^2 G}{\partial\sigma_j^{(1)}\partial\sigma_k^{(1)}} s]$, with $\frac{\partial^2 G}{\partial\sigma_v^2\partial\sigma_v^2} = 0$, $\frac{\partial^2 G}{\partial\sigma_v^2\partial\lambda_v} = (I_n - \lambda_v D)^{-1} D (I_n - \lambda_v D)^{-1} C^{-1}$, and $\frac{\partial^2 G}{\partial\lambda_v\partial\lambda_v} = 2\sigma_v^2 (I_n - \lambda_v D)^{-1} D (I_n - \lambda_v D)^{-1} D (I_n - \lambda_v D)^{-1} C^{-1}$. Note that in the $mspe[t(\hat{\sigma}, y)]$ given by (6.7), the $\sigma^{(1)}$ are estimated using the REML method. An area-specific estimation of MSPE$[t(\hat{\sigma}, y_i)]$ is the (i, i) element of $mspe[t(\hat{\sigma}, y)]$. In the case of non-spatial random effects ($G = \sigma_v^2 I_n$), we have $\triangle(\sigma) = 0$ since $\frac{\partial^2 G}{\partial\sigma_v^2\partial\sigma_v^2} = \frac{\partial^2 G}{\partial\sigma_v^2\partial\lambda_v} = \frac{\partial^2 G}{\partial\lambda_v\partial\lambda_v} = 0$. As a result, in the case of non-spatial random effects, the spell-out of Fay-Herriot model coincides with the known result (e.g., Prasad and Rao, 1990).

7. Application

Performance of the proposed approach is evaluated by using a real dataset. We study physician visits for Total Respiratory Morbidity (TRM) conditions (a patient diagnosed with any of the following respiratory diseases: asthma, chronic or acute bronchitis, emphysema, or chronic airway obstruction, and chronic obstructive pulmonary disease) in the Canadian province of Manitoba between April 1, 2000 to March 31, 2010. The population of Manitoba was stable during the study period from 1.15 million in 2000 to 1.20 million in 2010. The province consisted of 5 Regional Health Authorities that were responsible for the delivery of health care services. These 5 regions were further sub-divided into 67 non-overlap Regional Health Authorities Districts (RHADs) and these RHADs were used as small areas in our model. Note that out of 67 RHADs, 12 RHADs belong to Winnipeg (major city and capital of province of Manitoba). Our interest is to use the normal mixed model to make an inference on the rate of physician visits for TRM in the

$n = 67$ small areas (RHADs). Let y_i be the average rate of physician visits for TRM and x_i are the corresponding covariates in the ith area. Let $y_i|v_i \sim N(\mu_i, D_i)$ and $\mu_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + v_i$, where D_i are average of variance of y_i given v_i over 10 years (2000–2010), $v = (v_1, ..., v_n)$ are spatial random effects given by (2.4) assuming weight $w_{ij} = 1$, and x_{i1}, x_{i2} are rates of indigenous and immigrants, respectively, in the ith area; noting that these covariates are extracted from 2006 Canadian census microfile data.

Table 1 Parameters estimate and corresponding standard error (SE) of Total Respiratory Morbidity study using the spatial Fay-Herriot model.

Parameter	β_0	β_1	β_2	σ_v^2	λ_v
Estimate	0.252	-0.115	-0.008	0.020	0.618
SE	0.020	0.033	0.130	0.004	0.280

The estimate of model parameters and associated standard errors are reported in Table 1. It seems that the covariate indigenous and spatial model parameters are statistically significant. We also applied the SAR spatial FH model to this dataset. Estimate of model parameters for $\beta_0, \beta_1, \beta_2, \sigma_v^2, \lambda_v$ and corresponding standard errors (in parenthesis) are $0.241(0.017), -0.110(0.030), 0.083(0.118), 0.004$, and 0.139, respectively. We used the *sae* package in R for the SAR spatial FH model where using the *sae* package, the standard errors for the spatial model parameters are not provided, note that the spatial model parameters in the CAR and SAR should be interpreted differently due to their spatial model construction. It is worth mentioning that the model parameters estimates behave differently in the CAR and SAR spatial

FH models. We further investigate this behaviour in the simulation study (section 8). The prediction of rates of physician visits for TRM, $t(\hat{\sigma}, y_i)$, and corresponding $mspe[t(\hat{\sigma}, y_i)]$ using the Taylor expansion method are also provided in Figures 1 and 2, respectively. Note that the $\triangle_i$ involved in the $mspe[t(\hat{\sigma}, y_i)]$ has the same order as g_{2i} and g_{3i}. Hence, if one ignores the term $\triangle_i$ and applies the conventional (non-spatial) $mspe$, it will not estimate the MSPE correctly; we will further investigate this issue in the simulation study (section 8). In particular, the range of ratio $\triangle/g_2$ over 67 areas is (0.14, 5.74) and this range for the ratio $\triangle/g_3$ is (0.001, 0.97). We also provide the MSPE estimation of $t(\hat{\sigma}, y_i)$ using the bootstrap approach. Figure 3 shows the box plots of MSPE estimation using the parametric bootstrap and Taylor expansion methods; it appears that the both Taylor expansion and parametric bootstrap methods behave similarly. In particular, it shows that the bias correction of g_1 term in the parametric bootstrap method (with 1000 bootstrap samples) tracks nicely the bias correction provided using the Taylor expansion.

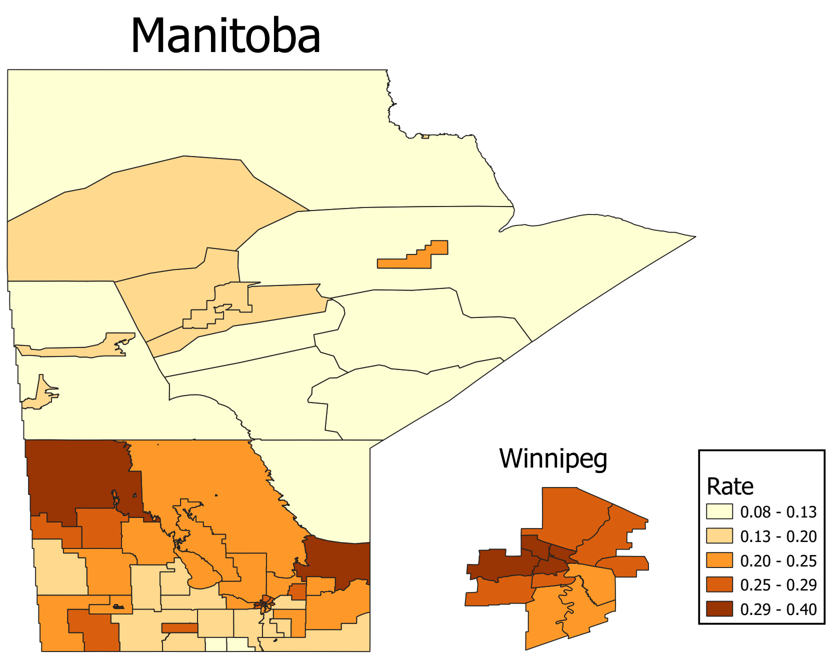

Figure 1 Prediction of Total Respiratory Morbidity visit rates in 67 health regions (small areas) in Manitoba using the spatial Fay-Herriot model.

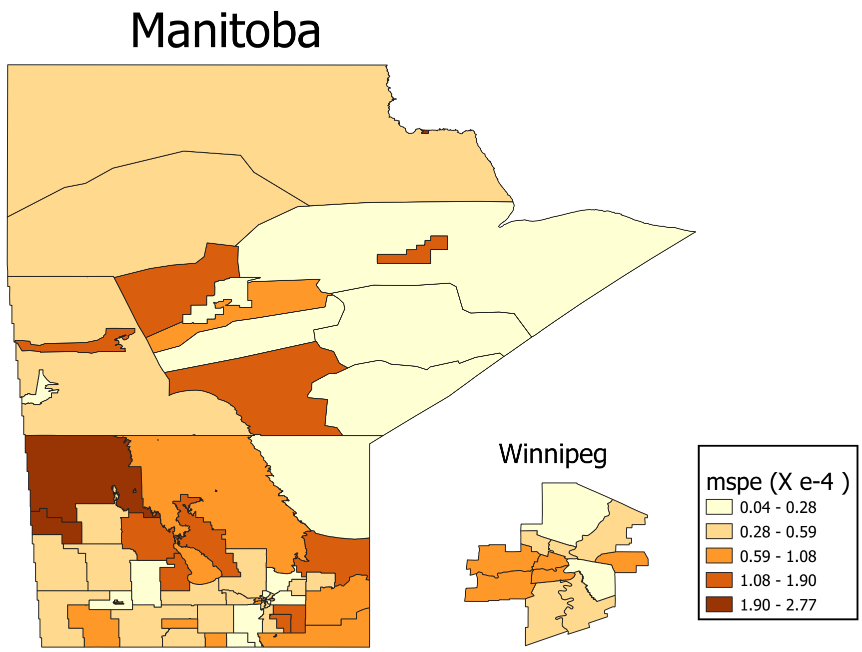

Figure 2 MSPE estimation of prediction of Total Respiratory Morbidity visit rates in 67 health regions (small areas) in Manitoba using the Taylor expansion approach based on the spatial Fay-Herriot model.

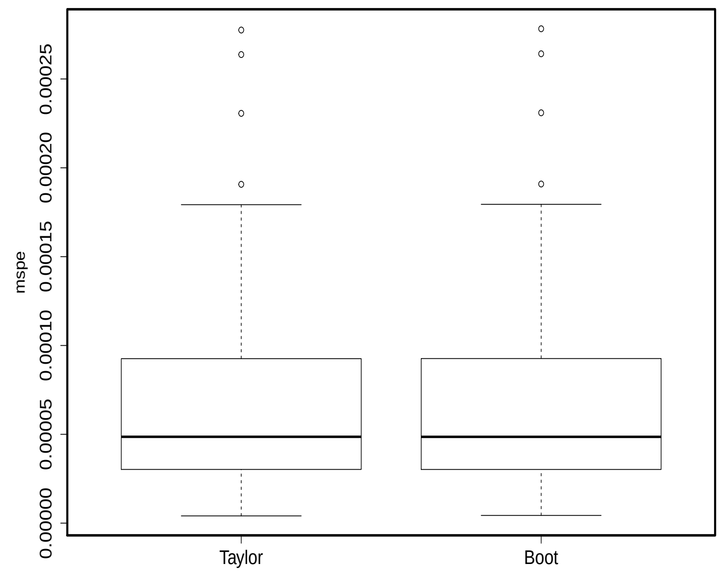

Figure 3 Box plots of MSPE estimation of prediction of Total Respiratory Morbidity visit rates in 67 health regions (small areas) in Manitoba using the Taylor expansion and parametric bootstrap based on the spatial Fay-Herriot model.

8. Simulation study

We also conduct a simulation study on the relative efficiency of the EBLUP estimator $t(\hat{\sigma}, y_i)$ and the naive EBLUP estimator $\tilde{t}(\hat{\sigma}, y_i)$ which is obtained by ignoring the spatial random effects ($\lambda_v = 0$) in the following FH model:

$$y_i = \beta_0 + z_i^\top v + \epsilon_i, (i = 1, ..., n), \tag{8.1}$$

where $v \sim N(0, G)$ and $\epsilon_i \sim N(0, \sigma_{0i}^2)$. We also compare performance of the proposed approach (CAR spatial FH model) with the corresponding SAR spatial FH model in terms of estimation of the MSPE of EBLUP estimator $t(\hat{\sigma}, y_i)$. For simplicity, we consider the same spatial structure as in our data analysis (section 7). To this end, we generate $B = 1,000$ independent sets of normal variates $\{(v^{(b)}, \epsilon_i^{(b)}); i = 1, ..., n; b = 1, ..., B\}$ with mean zero and specified variances $G(\sigma_v^2, \lambda_v)$ and σ_{0i}^2, respectively, with assuming weight $w_{ij} = 1$. We then obtain $\{y_i^{(b)}; i = 1, ..., n; b = 1, ..., B\}$ from the model (8.1) by assuming that $\beta_0 = 5$, and σ_{0i}^2 are generated from uniform distribution between 0.5 and 1.5, and different values for $\sigma_v^2 (= 20, 50)$ and $\lambda_v (= -0.1, -0.25, -0.5, -0.75)$, noting that the range of possible λ_v to get the positive definite G in our set-up is between -1.44 and 1. Note that the σ_{0i}^2's are known during the simulation study. The ith small area mean of the bth simulated dataset is

$$\mu_i^{(b)} = 5 + v_i^{(b)}.$$

For each simulated dataset, we estimate the model parameters as $(\hat{\beta}_0^{(b)}, \hat{\sigma}_v^{2(b)}, \hat{\lambda}_v^{(b)})$ and then obtain $t[\hat{\sigma}^{(b)}, y_i^{(b)}]$ and $\tilde{t}[\hat{\sigma}^{(b)}, y_i^{(b)}]$.

Table 2 presents the mean and median values of the model parameters estimate, the empirical variances of model parameters estimate, and the model-based variances (using the generalized weighted least squared approach for fixed parameters and the inverse of Fisher information matrix for variance components) of the estimate parameters for the both proposed and naive methods for different values of σ_v^2 and λ_v. It seems that the estimates of model parameters in our proposed approach are reasonably unbiased, and their variances are also estimated well with comparing the model-based variances with the corresponding empirical variance values. However, the estimate of variance component σ_v^2 is heavily biased for the naive method, and the model-based variances of model parameters estimate are not comparable with the corresponding empirical values. For instance, in the case of $\sigma_v^2 = 20$ and $\lambda_v = -0.10$, the mean squared error (MSE) ($= bias^2 + var$) of model parameters for (β_0, σ_v^2) are 0.08 and 19.63 for our spatial FH model, while these values are 0.09 and 216.77 for the naive model, respectively. Note that, in the case of $\sigma_v^2 = 20$ and $\lambda_v = -0.10$, the MSE of λ_v is 0.24 for our spatial FH model while the naive model ignores this variation in the model. Performance of the naive method even gets worse with increasing σ_v^2. For instance, in the case of $\sigma_v^2 = 50$ and $\lambda = -0.1$, the MSE of σ_v^2 is 1351.94 in the case of naive method compared to 95.41 in the case of spatial method. Overall, it seems that the model parameters estimate in our proposed approach provides good point estimates and variances.

The empirical MSPE (EMSPE) of $t(\hat{\sigma}, y_i)$ and $\tilde{t}(\hat{\sigma}, y_i)$ are then calculated as

$$\text{EMSPE}[t(\hat{\sigma}, y_i)] = \frac{1}{B}\sum_{b=1}^{B}\{t[\hat{\sigma}^{(b)}, y_i^{(b)}] - \mu_i^{(b)}\}^2,$$

and

$$\text{EMSPE}[\tilde{t}(\hat{\sigma}, y_i)] = \frac{1}{B}\sum_{b=1}^{B}\{\tilde{t}[\hat{\sigma}^{(b)}, y_i^{(b)}] - \mu_i^{(b)}\}^2.$$

We only report the empirical MSPE in the case of $\sigma_v^2 = 20$ and $\lambda_v = -0.10$. Figure 4 shows that in terms of empirical MSPE, $t(\hat{\sigma}, y_i)$ is more efficient than the naive predictor $\tilde{t}(\hat{\sigma}, y_i)$ with relative efficiency, EMSPE$[\tilde{t}(\hat{\sigma}, y_i)]$/ EMSPE$[t(\hat{\sigma}, y_i)]$, ranging from 100% to 129%. Note that one can consider other set-ups such as smaller values of σ_v^2 to get even more efficient prediction of small area means compared to the naive method.

We also study performance of the proposed MSPE estimator, $mspe[t(\hat{\sigma}, y_i)]$, and compare it with the corresponding MSPE estimator through the SAR

Table 2 Mean and median values of the model parameters estimates, the empirical variances of the estimated parameters, and the model-based variances of the parameters estimates of proposed model (spatial FH model) and naive model (ignoring spatial random effects) based on 1000 simulated datasets.

	Spatial FH model				FH model			
			Variance				Variance	
Parameter	Mean	Median	Model-base	Empirical	Mean	Median	Model-base	Empirical
$\beta_0 = 5$	5.00	4.99	0.08	0.07	5.00	4.99	0.09	0.08
$\sigma_v^2 = 20$	18.93	18.64	18.49	18.86	5.32	5.20	1.27	1.78
$\lambda_v = -0.10$	-0.14	-0.10	0.24	0.17	–	–	–	–
$\beta_0 = 5$	5.00	4.99	0.07	0.06	5.00	4.99	0.10	0.08
$\sigma_v^2 = 20$	18.94	18.60	18.72	19.44	5.38	5.28	1.29	1.82
$\lambda_v = -0.25$	-0.24	-0.25	0.24	0.17	–	–	–	–
$\beta_0 = 5$	5.00	4.99	0.06	0.05	5.00	4.99	0.10	0.07
$\sigma_v^2 = 20$	19.10	18.63	19.90	22.39	5.59	5.46	1.38	1.97
$\lambda_v = -0.50$	-0.40	-0.50	0.25	0.13	–	–	–	–
$\beta_0 = 5$	5.00	4.99	0.06	0.05	5.00	4.99	0.10	0.07
$\sigma_v^2 = 20$	19.03	18.71	20.24	25.78	5.95	5.81	1.53	2.28
$\lambda_v = -0.75$	-0.58	-0.75	0.24	0.10	–	–	–	–
$\beta_0 = 5$	4.99	4.98	0.16	0.14	5.00	4.99	0.21	0.19
$\sigma_v^2 = 50$	48.00	46.98	91.41	93.08	13.32	12.94	6.52	9.59
$\lambda_v = -0.10$	-0.14	-0.12	0.18	0.16	–	–	–	–
$\beta_0 = 5$	4.99	4.98	0.15	0.13	5.00	5.00	0.22	0.18
$\sigma_v^2 = 50$	47.98	46.97	92.40	94.99	13.47	13.13	6.66	9.84
$\lambda_v = -0.25$	-0.26	-0.25	0.19	0.15	–	–	–	–
$\beta_0 = 5$	4.99	4.98	0.13	0.11	5.00	5.00	0.22	0.16
$\sigma_v^2 = 50$	47.97	47.25	94.51	100.55	13.98	13.61	7.14	10.71
$\lambda_v = -0.50$	-0.43	-0.50	0.19	0.12	–	–	–	–
$\beta_0 = 5$	4.99	4.99	0.11	0.10	5.00	5.00	0.24	0.15
$\sigma_v^2 = 50$	48.00	47.18	98.29	116.56	14.89	14.53	8.04	12.54
$\lambda_v = -0.75$	-0.61	-0.75	0.18	0.08	–	–	–	–

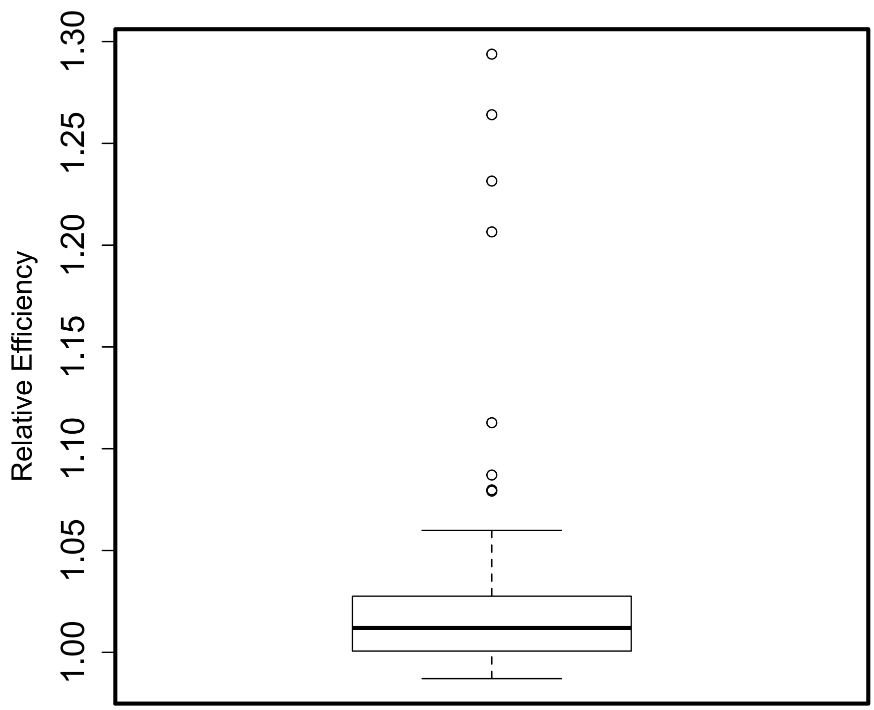

Figure 4 Box plot of relative efficiency of the proposed EBLUP estimator $t(\hat{\sigma}, y_i)$ (spatial FH) to the corresponding naive EBLUP estimator $\tilde{t}(\hat{\sigma}, y_i)$ (FH) over 67 health regions (small areas).

approach. Figure 5 shows the percent relative bias (%RB) of the MSPE estimators where RB of a MSPE estimator, $mspe[t(\hat{\sigma}, y_i)]$, is given by

$$RB_i = \frac{E\{mspe[t(\hat{\sigma}, y_i)]\}}{EMSPE[t(\hat{\sigma}, y_i)]} - 1,$$

where $E\{mspe[t(\hat{\sigma}, y_i)]\} = \frac{1}{B}\sum_{b=1}^{B} mspe[t(\hat{\sigma}^{(b)}, y_i^{(b)})]$. It is clear from Figure 5 that the proposed MSPE estimator of the EBLUP estimator $t(\hat{\sigma}, y_i)$ performs well in terms of RB for different values of σ_v^2 and λ_v for the both Taylor expansion and parametric bootstrap approaches. On the other hand, the SAR spatial FH model has much larger RB compared to our proposed CAR spatial FH model in all scenarios, and in particular for small σ_v^2.

9. Concluding remarks

There are extensive literature in small area estimation for linear mixed models, assuming small areas are independent from each other. However, assuming the independency of small areas may not be a valid assumption in many applications. For instance, health agencies (e.g., policy making) may need to know the spatial pattern of a rare disease (e.g., chronic disease or cancer) to identify small areas with high risk of disease to implement the prevention.

We have proposed a unified approach for Normal response with spatial patterns in the context of small area estimation. In particular, we have provided prediction of small area parameters and rigorously derived second order approximation to the mean squared prediction error (MSPE) of small area parameters. We have also rigorously obtained second-order MSPE estimation of small area predictors by Taylor expansion and parametric bootstrap

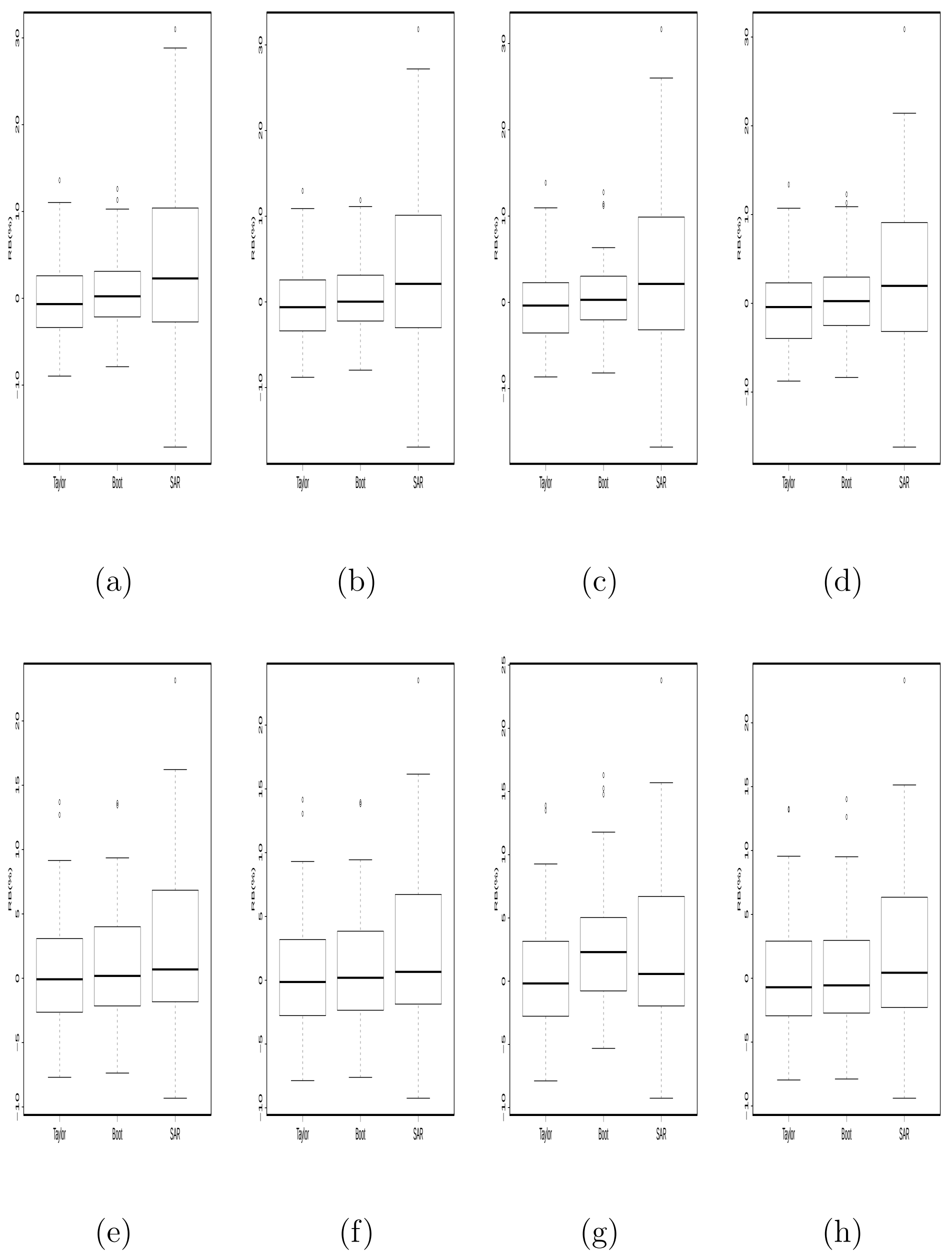

Figure 5: Box plots of percent relative bias of estimator of MSPE of the EBLUP $t(\hat{\sigma}, y_i)$ using the Taylor expansion, parametric bootstrap (with 100 bootstrap samples), and SAR spatial FH models: (a) $\sigma_v^2 = 20, \lambda_v = -0.10$, (b) $\sigma_v^2 = 20, \lambda_v = -0.25$, (c) $\sigma_v^2 = 20, \lambda_v = -0.50$, (d) $\sigma_v^2 = 20, \lambda_v = -0.75$, (e) $\sigma_v^2 = 50, \lambda_v = -0.10$, (f) $\sigma_v^2 = 50, \lambda_v = -0.25$, (g) $\sigma_v^2 = 50, \lambda_v = -0.50$, and (h) $\sigma_v^2 = 50, \lambda_v = -0.75$.

methods. We have shown by simulation study (and a real data application) that the proposed approach works very well in terms of small area predictors and their precisions.

Supplementary Materials

The supplementary materials contain two section. The first section provides proofs of Lemma 1, Theorem 1, and Theorem 2. The second section provides R codes and corresponding "readme" files for the simulation and application conducted in this paper.

Acknowledgements

Constructive comments and suggestions of associate editor and two referees, which led to an improved version of this article, are greatly appreciated. The research of Mahmoud Torabi is supported by a grant from the Natural Sciences and Engineering Research Council of Canada (NSERC). The research of Jiming Jiang is partially supported by the NSF grant SES-1121794.

References

[1] Baldermann, C., Salvati, N. and Schmid, T. (2018). Robust small area estimation under spatial non-stationarity. *Intern. Statist. Rev.* **86**, 136–159.

[2] Banerjee, S., Carlin, B.P. and Gelfand, A.E. (2014). *Hierarchical Modeling and Analysis for Spatial Data*, 2nd Edition, Chapman & Hall/CRC, Boca Raton.

[3] Besag, J.E. (1974). Spatial interaction and the statistical analysis of lattice systems (with Discussion). *J. Roy. Statist. Soc. Ser. B* **35**, 192–236.

[4] Chandra, H., Salvati, N. and Chambers, R. (2015). A spatially nonstationary Fay-Herriot model for small area estimation. *J. Surv. Statist. Methd.* **3**, 109–135.

[5] Cressie, N. (1991). Small-area prediction of undercount using the general linear model. *Proceedings of Statistics Symposium 90: Measurement and Improvement of Data Quality, Ottawa, Canada: Statistics Canada*, pp. 93–105.

[6] Cressie, N. A. and Chan, N. (1989). Spatial modeling of regional variables. *J. Amer. Statist. Assoc.* **84**, 393–401.

[7] Das, K., Jiang, J. and Rao, J.N.K. (2004). Mean squared error of empirical predictor. *Ann. Statist.* **32**, 818–840.

[8] Fay, R.E. and Herriot, R.A. (1979). Estimates of income for small places: an application of James-Stein procedures to census data. *J. Amer. Statist. Assoc.* **74**, 269–277.

[9] Henderson, C.R. (1975). Best linear unbiased estimation and prediction under a selection model. *Biometrics* **31**, 423–447.

[10] Jiang, J. (2017). *Asymptotic Analysis of Mixed Effects Models: Theory, Applications, and Open Problems*, Chapman & Hall/CRC, Portland.

[11] Jiang, J. and Lahiri, P. (2006). Mixed model prediction and small area estimation. *Test* **15**, 1–96.

[12] Kackar, R.N. and Harville, D. A. (1984). Approximations for standard errors of estimators of fixed and random effects in mixed linear models. *J. Amer. Statist. Assoc.* **79**, 853–862.

[13] Maiti, T., Sinha, S. and Zhong, P-S. (2016). Functional mixed effects model for small area estimation. *Scand. J. Statist.* **43**, 886–903.

[14] Marhuenda, Y., Molina, I. and Morales, D. (2013). Small area estimation with spatio-temporal Fay-Herriot models. *Comput. Statist. & Data Anal.* **58**, 308–325.

[15] Molina, I., Salvati, N. and Pratesi, M. (2009). Bootstrap for estimating the MSE of the Spatial EBLUP. *Comput. Statist.* **24**, 441–458.

[16] Petrucci, A. and Salvati, N. (2006). Small area estimation for spatial correlation in watershed erosion assessment. *J. Agri. Biolog. Environ. Statist.* **11**, 169–182.

[17] Porter, A.T., Holan, S.H., Wikle, C.K. and Cressie, N. (2014). Spatial Fay-Herriot model for small area estimation with functional covariates. *Spatial Statist.* **10**, 27–42.

[18] Porter, A.T., Wikle, C.K. and Holan, S.H. (2015). Multivariate spatial hierarchical Bayesian empirical likelihood methods for small area estimation. *Statist.* **4**, 108–116

[19] Prasad, N.G.N. and Rao, J.N.K. (1990). The estimation of mean squared error of small-area estimators. *J. Amer. Statist. Assoc.* **85**, 163–171.

[20] Pratesi, M. and Salvati, N. (2008). Small area estimation: the EBLUP estimator based on spatially correlated random area effects. *Statist. Method. Appl.* **17**, 113–141.

[21] R Development Core Team. (2018). *R: A language and environment for statistical computing.* Vienna, Austria: R Foundation for Statistical Computing. Available from URL: http://www.R-project.org/

[22] Rao, J.N.K. and Molina, I. (2015). *Small Area Estimation*, 2nd Edition, Wiley, Hoboken, New Jersey.

[23] Rao, J.N.K. and Yu, M. (1994). Small area estimation by combining time series and cross-sectional data. *Canad. J. of Statist.* **22**, 511–528.

[24] Schabenberger, O. and Gotway, C.A. (2004). *Statistical Methods for Spatial Data Analysis*, Chapman and Hall/CRC, Boca Raton.

[25] Schmid, T. and Münnich, R.T. (2014). Spatial robust small area estimation. *Statist. Paper.* **55**, 653–670.

[26] Singh, B.B., Shukla, G.K. and Kundu, D. (2005). Spatio-temporal models in small area estimation. *Surv. Method.* **31**, 183–195.

[27] Sinha, S.K. and Rao, J.N.K. (2009). Robust small area estimation. *Canad. J. Statist.* **37**, 381–399.

[28] Stern, H. and Cressie, N. A. (1999). Inference for extremes in disease mapping. In A. Lawson et al (Eds.), *Disease Mapping and Risk Assessment for Public Health* (pp. 63–84), Chichester: Wiley.

[29] Torabi, M. and Shokoohi, F. (2012). Likelihood inference in small area estimation by combining time-series and cross-sectional data. *J. Multi. Anal.* **111**, 213–221.