The Common Structure of Parametrizations for Linear Systems*

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ABSTRACT

A general framework for parametrizations of ARMAX systems in the case of (essentially) affine restrictions is described. For this general case topological and geometrical properties of the parametrizations are derived in a unified way. Our treatment includes, for instance, echelon canonical forms, the overlapping description of the manifold of all transfer functions of order n, and structural identifiability by affine restrictions. Also the case of polynomial zero restrictions is considered.

1. INTRODUCTION

During the last twenty years a number of different parametrizations for linear systems have been proposed. Most of these parametrizations are related to state space or ARMAX systems. A large part of the literature

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concerning the properties of such parametrizations deals with the state space case only; see e.g. Glover and Willems (1974), Clark (1976), Hazewinkel and Kalman (1976), Byrnes (1977), Hazewinkel (1979), Delchamps (1983), and Helmke (1985). In this paper we restrict ourselves to the ARMAX case, which shows some genuine differences from the state space case. The properties of ARMAX parametrizations have been investigated e.g. by Hannan (1971), Deistler (1978), Deistler, Dunsmuir, and Hannan (1978), Deistler and Hannan (1981), Deistler (1983), Hannan and Kavalieris (1984), and Hannan and Deistler (1988); see also Hinrichsen and Prätzel-Wolters (1983) for a more general polynomial case.

As is well known, certain statistical and numerical properties of identification procedures depend heavily on the corresponding properties of the parametrization used (see e.g. Deistler and Hannan, 1981; Hannan and Deistler, 1988); for instance, consistency of an identification procedure depends on a certain continuity property of the parametrization used. In particular, in the multioutput case the understanding of the complex structure of the parametrization for such systems turns out to be essential for the design and understanding of the behavior of identification procedures. In analysing identification procedures, usually either special parametrizations are investigated or some desirable properties of the parametrizations are postulated in an abstract way. The purpose of this paper is to derive a number of such desirable (topological and geometrical) properties for a general class of parametrizations, using a common framework, which includes most of the special parametrizations used.

Consider an ARMAX system

$$a(z)y(t) = b(z)u(t) \tag{1.1}$$

where $y(t)$ are the outputs, $u(t) = (\varepsilon(t)', x(t)')'$ are the inputs containing an s-dimensional unobserved white-noise component $\varepsilon(t)$ and possibly containing an observed input $x(t)$; z denotes the backward shift operator as well as a complex variable; and

$$a(z) = \sum_{j=0}^{p} A(j)z^j, \qquad b(z) = \sum_{j=0}^{p} B(j)z^j$$

are polynomial matrices of dimensions $s \times s$ and $s \times m$ respectively ($m \geqslant s$ and $m > s$ for the proper ARMAX case).

Throughout we assume

(i) The transfer function $k(z)=a^{-1}(z)b(z)$ is causal in the sense that

$$k(z)=(I,0)+\sum_{j=1}^{\infty}K(j)z^{j} \tag{1.2}$$

has a convergent power series expansion in a suitable neighborhood of zero.

Tacitly we will also assume that the transfer function $k(z)$ is uniquely determined from $y(t)$ and $x(t)$ and is the only source of information concerning the external behaviour of the system (1.1) (thus, for instance, extra information contained in transients is not available). Clearly assumption (i) implies

$$B(0)=(A(0),0). \tag{1.3}$$

If the contrary is not stated explicitly, we will in addition assume

$$\det A(0)\neq 0 \tag{1.4}$$

for the system (1.1). Note that (1.4) does not exclude any transfer function satisfying (i), since any left coprime MFD (left matrix fraction description) $a^{-1}b=k$ satisfies (1.4). As is easily seen, (1.3) and (1.4) together imply (i).

Let (for s and m fixed, but p arbitrary) Θ_A denote the set of all pairs (a,b) satisfying (1.3) and (1.4), let π denote the mapping defined by $\pi(a,b)=a^{-1}b$, and let U_A denote the image of Θ_A by π. Clearly, then U_A is the set of all rational transfer functions satisfying (i).

By $\delta(a,b)$ we denote the degree of (a,b). Let Θ_p be the set of all $(a,b)\in\Theta_A$ such that $\delta(a,b)\leqslant p$. If the degree of (a,b) is bounded by p, then clearly we can identify (a,b) with

$$\theta'=\operatorname{vec}(A(0),\dots,A(p),B(1),\dots,B(p))\in\mathbb{R}^{s(s+m)p+s^2},$$

where $\operatorname{vec}(d_1',\dots,d_s')'=(d_1,\dots,d_s)$ and d_i are row vectors.

Note that the condition (1.4) may be replaced by weaker assumptions, and most results of this paper will still be valid: Instead of Θ_p we could consider a set, Θ_p' say, of pairs (a,b) satisfying $\delta(a,b)\leqslant p$; $\Theta_p'\supset\Theta_p$;

$$\det a(z)\not\equiv 0 \text{ and the causality condition (i);} \tag{1.5}$$

and

$$\text{the mapping } \pi(a,b) = a^{-1}b \text{ is continuous on } \Theta'_p \text{ (in the pointwise topology of transfer functions as described later); } \Theta'_p \text{ is open in the embedding Euclidean space } \mathbb{R}^{s(s+m)p+s^2}. \tag{1.6}$$

2. IDENTIFIABILITY

The equation

$$k = a^{-1}b \tag{2.1}$$

implies the infinite system of linear equations

$$\begin{aligned} A(0)(I,0) &= B(0), \\ A(0)K(1) + A(1)(I,0) &= B(1), \\ &\vdots \\ A(0)K(p) + \cdots + A(p)(I,0) &= B(p), \\ \sum_{j=0}^{p} A(j)K(i+p-j) &= 0, \qquad i > 0. \end{aligned} \tag{2.2}$$

Clearly (for given k) every (a,b) satisfying (2.1) also satisfies $ak = b$ and (2.2). Conversely, if (a,b) satisfies (2.2) and (1.4), then it satisfies (2.1). Note however that solutions of (2.2) may correspond to noncausal transfer functions or may even not satisfy the condition $\det a(z) \not\equiv 0$. As has been shown, e.g. in Deistler (1983), if $(a,b) \in \Theta_p$, then for $k = a^{-1}b$, the set of all solutions of (2.2) which are contained in Θ_p is equal to the set of all solutions of the finite equation system

$$
\begin{aligned}
A(0)(I,0) &= B(0),\\
A(0)K(1)+A(1)(I,0) &= B(1),\\
&\ \vdots\\
A(0)K(p)+\cdots+A(p)(I,0) &= B(p),\\
\sum_{j=0}^{p} A(j)K(i+p-j) &= 0, \qquad i=1,2,\dots,sp,
\end{aligned}
\tag{2.3}
$$

which are contained in Θ_p.

Now if the contrary is not stated explicitly, we will only consider elements in Θ_p for a suitably chosen p. Then the information about (a,b) contained in k, i.e. in (2.3), can be written as

$$(I_s \otimes H(k))\theta = 0, \tag{2.4}$$

where $\theta' = \operatorname{vec}(A(0),\dots,A(p),B(1),\dots,B(p))$ is an $N = s[s(1+p)+mp]$-vector, where

$$H(k)' = \begin{pmatrix} F(k) & G(k) \\ -I_{mp} & 0 \end{pmatrix} \in \mathbb{R}^{(N/s)\times[m(1+s)p]}$$

with

$$F(k) = \begin{pmatrix} K(1) & K(2) & \cdots & K(p) \\ (I,0) & K(1) & \cdots & K(p-1) \\ 0 & (I,0) & \cdots & K(p-2) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & (I,0) \end{pmatrix} \in \mathbb{R}^{[s(p+1)]\times(mp)}$$

and

$$G(k) = \begin{pmatrix} K(p+1) & K(p+2) & \cdots & K(p+sp) \\ K(p) & K(p+1) & \cdots & K(p+sp-1) \\ \cdots & \cdots & \cdots & \cdots \\ K(1) & K(2) & \cdots & K(sp) \end{pmatrix} \in \mathbb{R}^{[s(p+1)]\times(msp)},$$

and finally where $I_s \otimes H(k)$ denotes the Kronecker product

$$\begin{pmatrix} H(k) & 0 & \cdots & 0 \\ 0 & H(k) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & H(k) \end{pmatrix}.$$

As has been said earlier, we will identify θ with $(a, b) \in \Theta_p$. In addition we assume that we have linear restrictions on θ of the form

$$R\theta = r \tag{2.5}$$

where $R \in \mathbb{R}^{M \times N}$, $r \in \mathbb{R}^{M \times 1}$ are known, and where, without loss of generality, we assume that R has rank $M \leqslant N$. By Θ_R we denote the set $\{\theta \in \Theta_p \mid R\theta = r\}$. We will always assume that the restrictions (2.5) do not contradict (1.4). Two MFDs $(\bar{a}, \bar{b})$ and (a, b) are called observationally equivalent if $\pi(\bar{a}, \bar{b}) = \pi(a, b)$. A subset $\Theta \subset \Theta_p$ is called *identifable* if π restricted to Θ is injective. Equations (2.4) and (2.5) together give

$$K\theta = \begin{pmatrix} r \\ 0 \end{pmatrix}, \tag{2.6}$$

where

$$K = \begin{pmatrix} R \\ I \otimes H(k) \end{pmatrix} \in \mathbb{R}^{[M + sm(1+s)p] \times N}.$$

Then, for given k, the set of all corresponding ARMAX systems satisfying (2.5) is that part of the set of all solutions of (2.6) where in addition (1.4) is fulfilled. This gives

THEOREM 1. *Let Θ be any subset of Θ_R. If*

(ii) *K has rank N for all $k = a^{-1}b \in \pi(\Theta)$,*

then Θ is identifiable. Conversely, if (ii) *is not fulfilled for one $k \in \pi(\Theta)$, then* (2.6) *has at least two different (observationally equivalent) solutions contained in Θ_R.*

REMARK 1. Note however that condition (ii) in general is not necessary for identifiability, since the intersection of Θ with the set $\ker K + \theta_0$ [where

$\ker K$ is the right kernel of K, and θ_0 satisfies (2.6)], might contain one element only.

REMARK 2. Also note that (ii) is equivalent to the condition (12) (this condition is explained below) in Deistler and Schrader (1979). To show this, let us write the matrix R in (2.5) in the form

$$R = \begin{pmatrix} R_{a1} & R_{b1} & R_{a2} & R_{b2} & \cdots & R_{as} & R_{bs} \end{pmatrix},$$

where R_{ai} and R_{bi} are $M \times [s(p+1)]$ and $M \times (mp)$ matrices corresponding to the restrictions in the ith row of $(A(0),\dots, A(p))$ and of $(B(1),\dots, B(p))$ respectively.

Denote

$$D = \begin{pmatrix} R_{a1} & 0 & R_{b1} & 0 & \cdots & R_{as} & 0 & R_{bs} & 0 \\ 0 & I_{s^2p} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & I_{smp} & & & \vdots & & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ 0 & & \vdots & & & 0 & I_{s^2p} & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & I_{smp} \end{pmatrix}$$

and

$$\hat{C} = \left(\hat{C}_a, \hat{C}_b\right),$$

where

$$\hat{C}_a = \begin{pmatrix} A(0) & \cdots & A(p) & \cdots & 0 \\ \cdot & \cdot & & \ddots & \vdots \\ \cdot & & \cdot & & A(p) \\ \cdot & & & \cdot & \vdots \\ 0 & \cdot & \cdot & \cdot & A(0) \end{pmatrix} \in \mathbb{R}^{[s(1+p+sp)]\times[s(1+p+sp)]}$$

and

$$\hat{C}_b = \begin{pmatrix} B(1) & \cdots & B(p) & 0 & \cdots & 0 \\ B(0) & \cdots & B(p-1) & B(p) & \cdots & 0 \\ 0 & \cdot & & \ddots & \ddots & \vdots \\ \cdot & \cdot & \cdot & & \ddots & B(p) \\ \cdot & & \cdot & \cdot & & B(p-1) \\ \cdot & & & \cdot & \cdot & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & B(0) \end{pmatrix}$$

$$\in \mathbb{R}^{[s(1+p+sp)]\times[m(1+s)p]}.$$

Then the condition (12) in Deistler and Schrader (1979) is that $D(I_s \otimes \hat{C}')$ has full column rank. Denote further

$$E(k) = \begin{pmatrix} K(1) & K(2) & \cdots & K(p+sp) \\ (I,0) & K(1) & \cdots & K(p+sp-1) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (I,0) \end{pmatrix}.$$

Then

$$\hat{C}_a E(k) = \hat{C}_b,$$

and hence

$$D(I \otimes \hat{C}') = D\left[I \otimes \begin{pmatrix} I \\ E(k)' \end{pmatrix}\right](I \otimes \hat{C}_a')$$

As $A(0)$ and hence $I \otimes \hat{C}_a'$ are nonsingular, the column rank of $D(I \otimes \hat{C}')$ is equal to the column rank of

$$D\left[I \otimes \begin{pmatrix} I \\ E(k)' \end{pmatrix}\right]. \tag{2.7}$$

On the other hand, by elementary operations, K may be transformed to

$$\begin{pmatrix} R_{a1}+R_{b1}F(k)' & R_{b1} & \cdots & R_{as}+R_{bs}F(k)' & R_{bs} \\ 0 & -I_{mp} & \cdots & 0 & 0 \\ G(k)' & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -I_{mp} \\ 0 & 0 & \cdots & G(k)' & 0 \end{pmatrix}$$

without changing the column rank. It is easily seen that this matrix has full column rank if and only if the matrix in (2.7) has full column rank, i.e., K has full column rank N if and only if $D(I_s \otimes \hat{C}')$ has full column rank.

3. SOME IMPORTANT SPECIAL CASES

Let us now discuss some important special cases which fit into our framework.

3.1. *Canonical Forms*

First let us consider the case of echelon or reversed echelon (sometimes also called Guidorzi) forms. (See e.g. Hannan and Deistler, 1988.) Let $\alpha=(n_1,\dots,n_s)$ denote the Kronecker indices. Remember that an MFD $(\tilde{a},\tilde{b})$ of $\tilde{k}(z)=k(z^{-1})$ is in the echelon form if and only if it has the following properties:

$$(\tilde{a},\tilde{b}) \text{ is left coprime} \tag{3.1}$$

$\tilde{a}_{ii}$ are monic polynomials, and

$$\begin{aligned} &\delta(\tilde{a}_{ij}) \leqslant \delta(\tilde{a}_{ii}) = n_i, && j \leqslant i, \\ &\delta(\tilde{a}_{ij}) < \delta(\tilde{a}_{ii}), && j > i, \\ &\delta(\tilde{a}_{ji}) < \delta(\tilde{a}_{ii}), && j \neq i, \\ &\delta(\tilde{b}_{ij}) \leqslant \delta(\tilde{a}_{ii}), && j = 1,\dots,s, \\ &\delta(\tilde{b}_{ij}) < \delta(\tilde{a}_{ii}), && j = s+1,\dots,m. \\ &C_\alpha(\tilde{b}) = (C_\alpha(\tilde{a}),0), \end{aligned} \tag{3.2}$$

where $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ denote the (i, j) elements of $\tilde{a}$ and $\tilde{b}$ respectively, and $C_\alpha(\tilde{a})$ and $C_\alpha(\tilde{b})$ denote the row end matrices of $\tilde{a}$ and $\tilde{b}$ respectively corresponding to row degrees n_i.

The reversed echelon form is given by

$$(a(z), b(z)) = \operatorname{diag}(z^{n_i})\left(\tilde{a}(z^{-1}), \tilde{b}(z^{-1})\right). \tag{3.3}$$

Thus (a, b) is in reversed echelon form if and only if it satisfies

(a, b) is left coprime and the row end matrix of (a, b) corresponding to degrees $(n_1, \ldots, n_s)$ is of rank s (the latter means that (a, b) is row reduced with row degrees given by $(n_1, \ldots, n_s)$); (3.4)

$A(0)$ is lower triangular with diagonal elements equal to 1;

$$B(0) = (A(0), 0); \tag{3.5}$$

$z^{n_i - n_{ij}}$ is a divisor of a_{ij} with n_{ij} given by

$$n_{ij} = \begin{cases} \min(n_{i+1}, n_j) & \text{for} \quad j < i, \\ \min(n_i, n_j) & \text{for} \quad j \geqslant i. \end{cases}$$

It should be noted that $(\tilde{a}, \tilde{b})$ and the reversed form (a, b) are both solutions of (2.4) with the same $H(k)$; the only difference is in the restrictions (3.1), (3.2) and (3.4), (3.5) respectively. Moreover due to (3.3), the respective solution vectors, $\tilde{\theta}$ and θ say, are identical up to rearrangement of their entries, and thus an analogous statement holds for the restriction matrices in (2.5).

Let $\Theta_\alpha^{(1)} \subset \Theta_p$ denote the set of all (a, b) in reversed echelon form with Kronecker indices α $(n_i \leqslant p)$, and let R, r correspond to the restrictions (3.5) and to the restrictions that the row degrees of (a, b) are smaller than or equal to n_i $(i = 1, \ldots, s)$ [except for $B(0) = (A(0), 0)$, which has already been taken into account in the definition of θ]. We will now show that the conditions (3.4) are equivalent to (ii) in Theorem 1 in the sense that $\Theta_\alpha^{(1)}$ is the largest subset of $\Theta_R = \{\theta \in \Theta_p \mid R\theta = r\}$ such that (ii) holds.

First, if (ii) holds, then the equation (2.6) has a unique solution, say (a, b). Then $(\tilde{a}, \tilde{b})$ defined by (3.3) is the unique solution of

$$\tilde{a}(z) k(z^{-1}) = \tilde{b}(z),$$

which in addition satisfies (3.2). Thus the rows of the Hankel matrix

$$\begin{pmatrix} K(1) & K(2) & K(3) & \cdots \\ K(2) & K(3) & K(4) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

of $k = a^{-1}b$ which correspond to the structure index $(n_1, \dots, n_s)$ form a basis of the row space, and therefore $(\tilde{a}, \tilde{b})$ is left coprime. Thus (a, b) satisfies (3.4).

Conversely, if (a, b) satisfies (3.4) and (3.5), then this (a, b) is the unique solution of (2.6). Thus (ii) holds.

The same results may be obtained in an analogous way for all other canonical ARMAX forms based on selection of basis rows by nice multiindices, as for instance for the (reversed) Hermite form. The only difference then is that (1.4) does not necessarily hold, and we have to replace (1.4) by (1.5) and (1.6) (i.e. Θ_p by Θ_p').

3.2. *The Overlapping Parametrizations of the Manifold of all Systems of Order n*

Now consider the *overlapping* ARMAX parametrization of the manifold $M(n)$ of all transfer functions of order n (see e.g. Deistler and Hannan 1981, Corrêa and Glover 1984, and Hannan and Deistler 1988). In this case some difficulties arise (which are specific to the ARMAX as opposed to the state space case). The following can be found in more detail e.g. in Hannan and Deistler (1988, Section 2.6). Let $\alpha = (n_1, \dots, n_s)$ be a structure index of k (i.e., α corresponds to a nice selection of basis rows in the Hankel matrix of k), and let U_α be the set of all transfer functions in U_A with structure index α. Then using the linear dependence relations in the Hankel matrix, we can define an $\tilde{a}$ and thus an MFD $(\tilde{a}, \tilde{b})$ which are unique with respect to U_α and which have the following properties:

$$(\tilde{a}, \tilde{b}) \text{ is left coprime,} \tag{3.6}$$

$$\tilde{a}_{ii} \text{ are monic,} \tag{3.7a}$$

$$\delta(\tilde{a}_{ji}) < \delta(\tilde{a}_{ii}) = n_i, \qquad j \neq i \tag{3.7b}$$

A problem arises, since the row degrees of $\tilde{a}$ (m_i say) may be higher than the corresponding column degrees n_i, and since $\tilde{a}$ may not be row-reduced and thus the conditions $\delta(\tilde{b}_{ij}) \leqslant m_i$, $i = 1, \ldots, s$, do not guarantee that $\tilde{a}^{-1} \cdot \tilde{b}$ will be a proper transfer function. As a consequence of this, nonlinear restrictions in the parameters of $(\tilde{a}, \tilde{b})$ may occur due to the causality requirement for k, and the reversed form (a, b) [defined by (3.3) with n_i replaced by m_i] does not necessarily satisfy (1.4). Clearly $\delta(\tilde{a}, \tilde{b}) \leqslant \max m_i$.

A vector $\tau \in \mathbb{R}^{(\Sigma n_i)(s+m)}$ of free parameters for $\tilde{a}^{-1} \cdot \tilde{b} \in U_\alpha$ can be defined as consisting of the following entries:

$$\tilde{A}_{ij}(u), \qquad u = 0, 1, \ldots, n_j - 1, \quad i, j = 1, \ldots, s, \tag{3.8}$$

$$\tilde{B}_{ij}(u), \qquad u = 0, 1, \ldots, n_i - 1, \quad i = 1, \ldots, s \quad j = 1, \ldots, m, \tag{3.9}$$

where $\tilde{A}_{ij}(u)$ and $\tilde{B}_{ij}(u)$ are the coefficients corresponding to the power z^u of $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ respectively. Let $\Theta_\alpha^{(2)}$ denote the set of all such τ which correspond to U_α. The relation between τ and $(\tilde{a}, \tilde{b})$ turns out to be a homeomorphism between $\Theta_\alpha^{(2)}$ and the set of all $(\tilde{a}, \tilde{b})$ corresponding to U_α, and this homeomorphism can be extended to $\mathbb{R}^{(\Sigma n_i)(s+m)}$.

Let (using the old symbol in a slightly new way) θ denote the vector $\theta = (\theta_A', \theta_B')' \in \mathbb{R}^{s^2(p+1)+(\Sigma n_i)m}$ where [using the notation $\tilde{a}(z) = \Sigma \tilde{A}(j) z^j$] $\theta_A' = \operatorname{vec}(\tilde{A}(0), \ldots, \tilde{A}(p))$ with $p = \max m_i$, and θ_B consists of the free parameters in $\tilde{b}$ given in (3.9). Further let (R, r) correspond to the linear restrictions on $\tilde{a}$ given in (3.7). Then, by omitting those columns in K which correspond to the nonfree parameters in $\tilde{b}$ and by suitably reordering the columns of K, using the old symbol for the modified matrix, we can write

$$K\theta = \begin{pmatrix} r \\ 0 \end{pmatrix}. \tag{3.10}$$

As is easily seen from the structure of K [compare the formulas below (2.4)], K has full column rank $s^2(p+1) + (\Sigma n_i)m$ if and only if the selection of rows in the Hankel matrix of k corresponding to α gives basis rows, or equivalently, if and only if the corresponding $(\tilde{a}, \tilde{b})$ is left coprime. Clearly θ as defined above is trivially isomorphic to $\tau \in \mathbb{R}^{(\Sigma n_i)(s+m)}$. Now with the modifications on θ and (R, r) as described above and with Θ_R modified accordingly as $\Theta_R = \{\theta \in \mathbb{R}^{s^2(p+1)+(\Sigma n_i)m} \mid R\theta = r\}$, Theorems 1 and 2 hold for this case too. Note that due to $\det \tilde{A}(0) \neq 0$, a statement analogous to Remark 2 in the case considered here is valid.

3.3. *The Case of General Linear Restrictions*

Consider the case where the conditions (2.5) are general linear "structural" cross-equation restrictions (the word structural, in the proper sense, means that the restrictions are coming from "physical" *a priori* knowledge; of course, from the point of view of our mathematical analysis such an interpretation is not required).

Now, we want to discuss the relation between our condition (ii) and condition (iv) in Deistler (1978), i.e. the condition that $D(I\otimes\tilde{C}')$ has full column rank, where $\tilde{C}$ is the matrix consisting of the first $s(1+sp)$ rows of the $\hat{C}$ defined in Remark 2. Note that left coprimeness together with condition (iv) in Deistler (1978) implies that the solution of (2.6) is unique in $\mathbb{R}^N$ and thus (ii) holds. The converse statement does not necessarily hold, since (ii) does not necessarily imply left-coprimeness, as easily can be seen from the following example: Consider the case where $s=m=p=1$. Let $\theta=(A(0),A(1),B(1))'$, and (2.5) be given by $R=I_3$ and $r'=(1,1,1)$. Then for $k=1$, Equation (2.6) has the unique solution $(1+z,1+z)$, which is clearly not left-coprime.

If however the restrictions (2.5) are such that every observational equivalence class contains at least one left-coprime MFD [as e.g. in the case when the restrictions in R correspond to the prescriptions of column degrees of (a,b) and to $A(0)=I$; see Deistler (1983)], then (ii) implies left-coprimeness [and of course also (iv) in Deistler (1978)].

4. TOPOLOGICAL AND GEOMETRICAL PROPERTIES OF PARAMETRIZATIONS

If $\Theta\subset\Theta_p$ is identifiable, then there exists a mapping $\psi:U=\pi(\Theta)\to\Theta$ attaching to every transfer function the corresponding unique $(a,b)\in\Theta$. This mapping is called an ARMAX parametrization of U. For the investigation of statistical and numerical properties of identification procedures some properties of ψ as well as of the boundaries of U and Θ turn out to be important. These properties are summarized in the following theorem for the general case discussed in Section 2. They have been derived in special cases e.g. in Deistler, Dunsmuir, and Hannan (1978), Deistler and Hannan (1981), Deistler (1983), Hannan and Kavalieris (1984), and Hannan and Deistler (1988).

We endow U_A with the topology corresponding to the relative topology in the product space $(\mathbb{R}^{s\times m})^{\mathbf{Z}_+}$ of the power series coefficients $(K(i))_{i\in\mathbf{Z}_+}$ of the transfer functions $k\in U_A$. As convergence of transfer functions then corresponds to the pointwise convergence of the power series coefficients, we call this topology the pointwise topology T_{pt}. Parameter space like Θ_p are always endowed with the relative Euclidean topology. If A is a set in a topological space, its closure is denoted by $\bar{A}$.

By Θ_R^I we denote the largest subset of Θ_R such that condition (ii) in Theorem 1 holds, and we let $U_R^I = \pi(\Theta_R^I)$.

Now, let us add additional restrictions to the equations (2.5), which do not contradict (1.4) and (2.5); denote by $Q\theta = q$ the resulting new (extended) equation system. Here it is assumed that the rows of Q are linearly independent again. In such a case we say that (Q, q) is an extension of (R, r).

We consider $R^{-1}(r) = \{\theta \in \mathbb{R}^n \mid R\theta = r\}$; then $R^{-1}(r)$ is an affine subset of $\mathbb{R}^N$ of dimension $N - M$ and can clearly be identified with $\mathbb{R}^{N-M}$.

THEOREM 2.

(1) *If Θ_R^I is not empty, then Θ_R^I is open and dense in $R^{-1}(r) \triangleq \mathbb{R}^{N-M}$.*

(2) *For every $k \in \pi(\Theta_R)$, the closure of the observational equivalence class $\pi^{-1}(k) \cap \Theta_R$ is an affine subset of dimension N -rank K (where K corresponds to k).*

(3) *There exist a finite number of extensions (Q, q) of (R, r) such that the union of the corresponding sets $\pi(\Theta_Q^I)$ [including $\pi(\Theta_R^I)$] is equal to $\pi(\Theta_R)$.*

(4) *$\psi: U_R^I = \pi(\Theta_R^I) \to \Theta_R^I$ is a T_{pt}-homeomorphism.*

(5) *U_R^I is (T_{pt}-)open in $\bar{U}_R^I$.*

(6) *If Θ_R^I is not empty, then $\pi(\Theta_R) \subset \bar{U}_R^I$.*

Proof. (1): By Remark 2, Θ_R^I is just the set of all elements of $R^{-1}(r)$ such that $A(0)$ and $D(I \otimes \hat{C}')$ have full rank. Since both conditions define an open set, Θ_R^I is open in $R^{-1}(r)$. Next we show that Θ_R^I is also dense in $R^{-1}(r)$. If Θ_R^I were not dense in $R^{-1}(r)$, then $R^{-1}(r) - \Theta_R^I$ would contain a nonvoid open subset of $R^{-1}(r)$ such that for all elements in this open set either $\det A(0) = 0$ or certain minors of $D(I \otimes \hat{C}')$ are zero. Now, the determinant of a matrix is a polynomial in several variables in the elements of the matrix. Therefore at least one of the two conditions mentioned above would hold everywhere in $R^{-1}(r)$, in contradiction to the assumption that Θ_R^I is nonvoid.

(2) is straightforward to see from the fact that the closure of $\pi^{-1}(k) \cap \Theta_R$ in $R^{-1}(r)$ is just the set of all solutions of Equation (2.6).

(3): We now construct a finite class of extensions (Q, q) of (R, r) such that for every $k \in \pi(\Theta_R) - U_R^I$ there is a (Q, q) from this class such that

$$\begin{pmatrix} Q \\ I \otimes H(k) \end{pmatrix} \theta = \begin{pmatrix} q \\ 0 \end{pmatrix}$$

has a unique solution. By definition and by Theorem 1, the matrix

$$K = \begin{pmatrix} R \\ I \otimes H(k) \end{pmatrix}$$

has rank less than N, and hence the solution of (2.6), say θ, is not unique. If θ_j is a component of θ which is not uniquely determined by (2.6), then θ_j may be fixed in the following way without violating (1.4): Let Q_j be the matrix obtained by adding to R one unit row vector with unity in the position corresponding to the position of θ_j in θ. In order to construct the vector q_j corresponding to Q_j, we proceed as follows: If θ_j occurs in $(A(1),\dots, A(p), B(1),\dots, B(p))$, then we add a zero in the corresponding position; if θ_j occurs in $A(0)$, then as det $A(0)$, when viewed as a function of θ_j, is a nonzero polynomial of order less than or equal to s [note that there may be linear restrictions between the entries of $A(0)$], there is at least one choice of θ_j in the set of $s+1$ suitably prescribed numbers $\{d_1^{(j)}, d_2^{(j)},\dots, d_{s+1}^{(j)}\}$ $(1 \leqslant j \leqslant N-M)$, independent of k, such that det $A(0) \neq 0$, and this element will be added to r in the corresponding position. Then θ_j is uniquely determined by

$$\begin{pmatrix} Q_j \\ I \otimes H(k) \end{pmatrix} \theta = \begin{pmatrix} q_j \\ 0 \end{pmatrix}. \tag{4.1}$$

If the matrix

$$\begin{pmatrix} Q_j \\ I \otimes H(k) \end{pmatrix}$$

still has not full column rank N, then the above procedure is repeated until all components of θ are uniquely determined. As θ has N components, in a finite number of steps we can obtain an extension (Q, q) of (R, r) such that

$$\begin{pmatrix} Q \\ I \otimes H(k) \end{pmatrix} \theta = \begin{pmatrix} q \\ 0 \end{pmatrix}$$

has a unique solution in $\mathbb{R}^N$ and this unique solution satisfies (1.4), i.e., $k \in U_Q^I$. It is easily seen that there are at most

$$(s+1)^{N-M} \sum_{i=1}^{N-M} \binom{N-M}{i} = (s+1)^{N-M}(2^{N-M}-1)$$

possible extensions of (R, r) of the form described above. Clearly $\pi(\Theta_Q^I) \subset \pi(\Theta_R)$, which completes the proof of (3).

(4): Clearly ψ is bijective, and since π is obviously continuous, so is ψ^{-1}. It remains to prove the continuity of ψ. If $k_t, k \in U_R^I$ are such that $k_t \to k$ (in T_{pt}), then using an obvious notation $K_t \to K$ (in the usual Euclidean topology). Thus by condition (ii) the solutions $\theta_t = \psi(k_t)$ of (2.6) converge to $\theta = \psi(k)$.

(5): If $k_0 \in U_R^I$, then k_0 has a neighborhood, O say, in $\overline{U}_R^I$, such that for all $k \in O$, the corresponding K have rank N. As for every element in U_R^I, due to (2.6) the matrix

$$\left(K, \begin{pmatrix} r \\ 0 \end{pmatrix}\right)$$

cannot have full column rank, the same must hold for the elements of $\overline{U}_R^I$. Thus for every $k \in O$ the equation (2.6) has exactly one solution. Since the solutions of (2.6) continuously depend on $k \in O$, O can be chosen such that (1.4) is satisfied. Thus every $k \in U_R^I$ has a neighborhood in $\overline{U}_R^I$, which is contained in U_R^I, i.e., U_R^I is open in $\overline{U}_R^I$.

(6) follows from (1) and the continuity of π. ■

Remark 3. The set $\overline{U}_R^I - \pi(\Theta_R)$ consists of the transfer functions which cannot be expressed by systems (a, b) satisfying (2.5). We have not been able to characterize the set of restrictions corresponding to these transfer functions. As is well known, for some special cases [as e.g. for Echelon forms or for the overlapping parametrizations of $M(n)$] such characterizations are available; see e.g. in Hannan and Deistler (1988).

Remark 4. For the cases mentioned in Remark 3 and also for the case discussed in Deistler (1983), equality in (6) holds for the scalar output case $(s = 1)$. The following example shows, however, that this may not be true in general. Suppose that $s = m = 1$ and $p = 2$, and that the restrictions (2.5) are such that the general form of $(a, b) \in R^{-1}(r)$ is $(1 + cz, 1 + dz + z^2)$. Consider the sequence

$$(a_t, b_t) = \left(1 + tz, 1 + \left(t + \frac{1}{t}\right)z + z^2\right) \in \Theta_R^I, \qquad t = 1, 2, \ldots.$$

Clearly

$$k_t = a_t^{-1} b_t = 1 + \frac{1}{t} z \to 1 \qquad \text{as} \quad t \to \infty .$$

Thus $1 \in \overline{\pi(\Theta_R^I)}$ but is not in $\pi(\Theta_R)$.

Remark 5. One may think of generalizing the condition (1.4) to

$$\det a(z) \not\equiv 0, \tag{4.2}$$

which is indeed part of the definition of ARMAX systems. It should be noted, however, that even if we restrict ourselves to the causal transfer functions satisfying $K(0) = (I, 0)$, the mapping π is not necessarily continuous in general. Consider the following example given for $s = m = 1$. Let

$$a_t(z) = \frac{1}{t} - z, \qquad b_t(z) = \frac{1}{t} - \left(1 - \frac{1}{t}\right) z .$$

Then as $t \to \infty$, $(a_t, b_t) \to (-z, -z)$. But the sequence

$$k_t = a_t^{-1} b_t = 1 + z + t z^2 + t^2 z^3 + \cdots$$

does not converge (in T_{pt}). Of course, there are cases where (1.4) is not satisfied and we still have continuity of π. One example is the overlapping parametrization of the reversed form which has been discussed in Section 3.2. A simpler example is $y(t-1) = B(1) u(t-1)$.

5. THE CASE OF POLYNOMIAL ZEROS

Here we consider the case where certain polynomial elements in (a, b) are prescribed to be zero, i.e., where certain outputs are not influenced by certain inputs at any lag. For this case every class of ARMAX systems satisfying

(1.4) and

$$\text{in every row of } (a,b) \text{ there are at least } s-1 \text{ elements which are prescribed to be zero,} \tag{5.1}$$

$$\text{the diagonal elements of } A(0) \text{ are equal to 1,} \tag{5.2}$$

$$\text{if } c_i \text{ denotes the matrix consisting of those columns of } (a,b)\text{, where we have a zero prescribed in the } i\text{th row, then } c_i \text{ is assumed to have rank } s-1,\ i=1,\dots,s, \tag{5.3}$$

$$\text{every row of } (a,b) \text{ is relatively prime} \tag{5.4}$$

is identifiable (Hannan, 1971).

Let Θ_H denote the set of all ARMAX systems in Θ_A satisfying (5.1)–(5.4), where in (5.1) the positions of the polynomial zeros are fixed. In this set the maximum lags of the systems are not bounded. As has been shown in Hannan (1971), $\pi(\Theta_H) = U_A$ holds for a particular arrangement of the zero polynomials.

Due to a result of Hazewinkel (1979), there is no continuous parametrization for U_A [and thus, at least, for the $\pi(\Theta_H)$ mentioned before], in general. This is also demonstrated in the next example: Consider the case $s = m = 2$ where a_{12} and a_{21} are prescribed to be zero. In particular consider the sequence

$$(a_t, b_t) = \begin{pmatrix} 1 & 0 & 1 & z \\ 0 & 1+z & \frac{1}{t}z & 1+z \end{pmatrix}, \qquad t = 1,2,\dots,$$

and

$$(a_0, b_0) = \begin{pmatrix} 1 & 0 & 1 & z \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Clearly $(a_t, b_t) \in \Theta_H$ for all $t = 0,1,\dots,$ and

$$k_t = a_t^{-1} b_t = \begin{pmatrix} 1 & z \\ \dfrac{z}{t(1+z)} & 1 \end{pmatrix} \to \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = a_0^{-1} b_0.$$

However, (a_t, b_t) does not converge to (a_0, b_0).

Therefore we have to search for a cover of $\pi(\Theta_H)$ such that each element of this cover has a continuous parametrization and the corresponding parameter spaces are finite dimensional. In the next theorem such a cover is described by prescribing the (actual) row degrees, $n_1,\dots,n_s$ say.

Now given $\alpha=(n_1,\dots,n_s)$ and p such that $n_i \leqslant p$, $i=1,\dots,s$, let (R,r) correspond to the restrictions (5.1) (for fixed but arbitrary positions of polynomial zeros), (5.2), and the prescription of the maximal row degrees α. By Θ_α we denote the set of all $(a,b)\in\Theta_H$ with row degrees α attained. Then clearly Θ_α is an identifiable class in Θ_R. Let $U_\alpha=\pi(\Theta_\alpha)$, and $\psi_\alpha: U_\alpha\to\Theta_\alpha$ be the mapping attaching to every $k\in U_\alpha$ the unique MFD $(a,b)\in\Theta_\alpha$. Let $\beta=(m_1,\dots,m_s)$. Then we will write $\beta\leqslant\alpha$ if $m_i\leqslant n_i$, $i=1,\dots,s$. We use $\beta<\alpha$ if in addition $m_i<n_i$ for at least one i.

THEOREM 3. *Denote $\Theta_R^1=\{\theta\in\Theta_R \mid (5.3)\text{ holds}\}$. Then*

(1) $\Theta_\alpha\subset\Theta_R^I$, *and equality holds for $s=1$. For $s>1$, every $(a,b)\in\Theta_R^I-\Theta_\alpha$ has the property* (5.4), *and its row degrees are described by α, but it does not satisfy* (5.3).

(2) *If Θ_α is not empty then Θ_α is open and dense in $R^{-1}(r)$.*

(3) *For every $\theta\in\Theta_R^1$, the observational equivalence class of θ in Θ_R is contained in Θ_R^1.*

(4) $\pi(\Theta_R^1)=\bigcup_{\beta\leqslant\alpha}U_\beta$.

(5) $\psi_\alpha: U_\alpha\to\Theta_\alpha$ *is a $(T_{pt}$-$)$homeomorphism.*

(6) U_α *is $(T_{pt}$-$)$open in $\bar{U}_\alpha$.*

(7) *If Θ_α is not empty, then $\pi(\Theta_R)\subset\bar{U}_\alpha$ and equality holds for $s=1$.*

(8) $\{U_\alpha \mid \alpha\in\mathbb{Z}_+^s\}$ *is a disjoint partition of $\pi(\Theta_H)$.*

Proof. (1): Let $(a,b)\in\Theta_\alpha$ and $k=\pi(a,b)$. Then any solution of (2.6) satisfies $\bar{a}a^{-1}b=\bar{b}$, and thus $(\bar{a},\bar{b})=u(a,b)$ for the rational matrix $u=\bar{a}a^{-1}$. Since (a,b) satisfies (5.1) and (5.3) and $(\bar{a},\bar{b})$ satisfies (5.1), u must be diagonal, and since (5.2) and (5.4) hold and since (a,b) has row degrees α, u must be the identity matrix. Thus (2.6) has unique solution in $\mathbb{R}^N$, and hence rank $K=N$, i.e., $(a,b)\in\Theta_R^I$. Now any $(a,b)\in\Theta_R^I$ must satisfy (5.4) and have row degrees α, as otherwise an extraction of a suitable common divisor of a and b or a multiplication by a suitable polynomial would yield another observationally equivalent MFD and hence another solution of (2.6). For $s=1$, (5.3) is trivial and hence $\Theta_R^I=\Theta_\alpha$. For $s>1$, clearly no $(a,b)\in\Theta_R^I-\Theta_\alpha$ can satisfy (5.3).

(2): Let $\theta\in\Theta_\alpha$, and let $c_i(z)$ be the matrix described in (5.3). By (5.3), then, there is a complex number z_0 such that at least one $(s-1)$-dimensional minor of $c_i(z_0)$ is unequal to zero. As for given z_0 such a minor is a continuous function of θ, there exists a neighborhood, O say, of θ in $R^{-1}(r)$, such that the statement above holds for all i and for all $\theta_1\in O$. As by

Theorem 2(1) Θ_R^l is open in $R^{-1}(r)$, $O_1 = O \cap \Theta_R^l$ is a neighborhood of θ in $R^{-1}(r)$. By (1) all $\theta_1 \in O_1$ in addition satisfy (5.4) and have row degrees α, i.e., $O_1 \subset \Theta_\alpha$. Thus Θ_α is open in $R^{-1}(r)$.

In order to prove that Θ_α is also dense in $R^{-1}(r)$, we proceed as in the proof of Theorem 2(1).

(3): Let $(a,b) \in \Theta_R^1$, and $(\bar{a},\bar{b})$ be any observationally equivalent MFD of (a,b). Then, as has been shown in the proof of (1), there is a diagonal rational matrix u such that $(\bar{a},\bar{b}) = u(a,b)$. As is easily seen, $(\bar{a},\bar{b})$ must satisfy (5.3) too.

(4): If $(a,b) \in \Theta_R^1 - \Theta_\alpha$ and some rows of (a,b) are not relatively prime, then rowwise extraction of common divisors yields an observationally equivalent. MFD $(\bar{a},\bar{b})$ satisfying (5.3) and (5.4). Let $\beta = (m_1,\dots,m_s)$ be the row degrees of $(\bar{a},\bar{b})$; then $\beta < \alpha$ and $\pi(\bar{a},\bar{b}) \in U_\beta$. The other direction is obvious.

(5) is straightforward to see from (1) and Theorem 2, as ψ_α is the restriction of ψ to U_α.

(6): For any $k \in U_\alpha$, analogously to the proof of Theorem 2(5), k has a neighborhood $O \subset U_R^l \cap \overline{U}_\alpha$. Since all minors of $c_i(z)$ for a given z are continuous functions of $\theta = \psi(k)$, O may be chosen such that (5.3) holds for all MFDs in $\pi^{-1}(O)$. As $\pi^{-1}(O) \subset \Theta_R^l$, by (1) all MFDs in $\pi^{-1}(O)$ in addition satisfy (5.4) and have row degrees α, i.e., $O \subset U_\alpha$ and hence U_α is open in $\overline{U}_\alpha$.

(7): First, $\pi(\Theta_R) \subset \overline{U}_\alpha$ because of (2) and the continuity of π. For $s=1$, as mentioned before, the conditions (5.1) and (5.3) are trivial and hence $\Theta_R^1 = \Theta_R$. For any $k \in \overline{U}_\alpha$, the matrix

$$\left(K, \begin{pmatrix} r \\ 0 \end{pmatrix}\right)$$

cannot have full column rank. Now because of the special structure of K, Equation (2.6) always has at least one solution which satisfies $A(0) = 1$. Thus $k \in \pi(\Theta_R)$, i.e., $\overline{U}_\alpha \subset \pi(\Theta_R)$.

(8) is easily seen from the definition of U_α. ∎

REMARK 6. The example before Theorem 3 shows also that the partition $\{U_\alpha \mid \alpha \in \mathbf{Z}_+^s\}$ of $\pi(\Theta_H)$ is minimal in general in order to obtain continuity of parametrizations ψ_α.

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