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BOUNDARY CROSSING PROBABILITY FOR BROWNIAN MOTION

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Abstract

Wang and Pötzelberger (1997) derived an explicit formula for the probability that a Brownian motion crosses a one-sided piecewise linear boundary and used this formula to approximate the boundary crossing probability for general nonlinear boundaries. The present paper gives a sharper asymptotic upper bound of the approximation error for the formula, and generalizes the results to two-sided boundaries. Numerical computations are easily carried out using the Monte Carlo simulation method. A rule is proposed for choosing optimal nodes for the approximating piecewise linear boundaries, so that the corresponding approximation errors of boundary crossing probabilities converge to zero at a rate of $O(1/n^2)$.

Keywords: First hitting time; first passage time; optimal stopping; curved boundaries; Wiener process; random walk; barrier options; cumulative sums; sequential analysis; Monte Carlo simulation

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1. Introduction

Boundary crossing probabilities, or the first hitting time distributions, of Brownian motion arise in many fields, e.g., in non-parametric statistics (Durbin (1971), Sen (1981)), sequential analysis (Sen (1981), Siegmund (1985), (1986)), CUSUM type techniques (Brown *et al.* (1975)), change point problems in econometrics (Krämer *et al.* (1988)), biology and epidemiology (Martin-Löf (1998)) and, recently, mathematical finance (Roberts and Shortland (1997), Lin (1998)).

However, explicit formulas for computing boundary crossing probabilities exist only for linear boundaries and very few, if any, nonlinear boundaries. For general one-sided nonlinear boundaries, some authors use integral equation method and propose approximate solutions, e.g., Durbin (1992), Daniels (1996) and Sacerdote and Tomassetti (1996), among others. The actual numerical computations are either intractable or give approximate solutions for which the approximation accuracies are difficult to assess (Daniels (1996), Sacerdote and Tomassetti (1996)).

Recently, Wang and Pötzelberger (1997) used a different approach to derive an explicit formula for boundary crossing probabilities for one-sided piecewise linear boundaries and use

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this formula to obtain approximations of boundary crossing probabilities for general nonlinear boundaries. The numerical computations are done using Monte Carlo simulation method.

In the present paper, we extend the results of Wang and Pötzelberger (1997) to the situation of two-sided boundaries. We also derive a sharper asymptotic upper bound for the approximation errors of the formula. We propose a rule for choosing optimal nodes for approximating piecewise linear boundaries which leads to an approximation rate of $O(1/n^2)$, which is better than the convergence rate obtained by some other authors in the recent literature, e.g., Novikov *et al.* (1999). Over and above the simplicity of our formula, this novel approach allows us to assess and control the approximation accuracy. Another advantage of our approach is that it can be applied to non-smooth or even discontinuous boundaries.

The paper is organized as follows. In Section 2, we use the approach of Wang and Pötzelberger (1997) to derive an explicit formula for boundary crossing probabilities for two-sided piecewise linear boundaries. In Section 3, we use this formula to obtain approximations to boundary crossing probabilities for general nonlinear boundaries. In Section 4, we propose a rule for choosing optimal nodes for approximating piecewise linear boundaries and derive the corresponding asymptotic upper bounds of the approximation errors. Some numerical examples using Monte Carlo simulation method are shown in Section 5. Finally, proofs of lemmas are given in Section 6.

2. Crossing probability for piecewise linear boundaries

Let $W(t),\ t \geq 0$, be a standard Brownian motion with $\mathrm{E}W(t) = 0$ and $\mathrm{E}W(t)W(s) = \min(t, s)$. We consider the following probability

$$\mathrm{P}(a, b) = \mathrm{P}[a(t) < W(t) < b(t),\ 0 \leq t \leq T], \tag{1}$$

where $T > 0$ is fixed, the real functions $a(t)$ and $b(t)$ are continuous and satisfy $a(t) < b(t)$ for all $0 < t \leq T$ and $a(0) < 0 < b(0)$. The boundary crossing probability is then given by $1 - \mathrm{P}(a, b)$. If the lower boundary $a(t) = -\infty$, then $1 - \mathrm{P}(-\infty, b)$ is also called one-sided boundary crossing probability. Throughout the paper we denote by $(t_i)_{i=1}^n$, $0 < t_1 < \cdots < t_{n-1} < t_n = T$, a partition of interval $[0, T]$ of size $n \geq 1$. Further, we let $t_0 = 0$, $\Delta t_i = t_i - t_{i-1}$, $\beta_i = b(t_i)$, $\alpha_i = a(t_i)$ and $d_i = \beta_i - \alpha_i$.

For one-sided boundary crossing probability, Wang and Pötzelberger (1997) proved the following result.

Theorem 1. *If b is a piecewise linear function on $[0, T]$ with nodes $(t_i)_{i=1}^n$, then probability (1) is given by*

$$\mathrm{P}(-\infty, b) = \mathrm{E}g[W(t_1), W(t_2), \ldots, W(t_n)],$$

where

$$g(\boldsymbol{x}) = \prod_{i=1}^{n} \mathbf{1}(x_i < \beta_i)\left\{1 - \exp\left[-\frac{2}{\Delta t_i}(\beta_{i-1} - x_{i-1})(\beta_i - x_i)\right]\right\},$$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$, $\Delta x_i = x_i - x_{i-1}$, $x_0 = 0$*, and* $\mathbf{1}$ *is the indicator function.*

For two-sided boundary crossing probabilities, Anderson (1960) derived a formula for probability (1) when a and b are linear functions on $[0, T]$. A different form of the formula was proposed by Hall (1997). In this section we use the approach of Wang and Pötzelberger (1997) to derive a formula for probability (1) when both a and b are piecewise linear functions.

To start with, it is easy to see that a similar argument to that of Theorem 1 of Wang and Pötzelberger (1997) leads to the following natural generalization of that theorem.

Theorem 2. *If boundaries $a, b \in C[0, T]$, then probability (1) is given by*

$$\mathrm{P}(a, b) = \mathrm{E}g[W(t_1), W(t_2), \ldots, W(t_n)], \tag{2}$$

where

$$g(\boldsymbol{x}) = \prod_{i=1}^{n} \mathbf{1}(\alpha_i < x_i < \beta_i) \times \mathrm{P}[a(t + t_{i-1}) - x_{i-1} < W(t) < b(t + t_{i-1}) - x_{i-1}, t < \Delta t_i \mid W(\Delta t_i) = \Delta x_i]. \tag{3}$$

Now we show that, when boundaries a and b are piecewise linear functions on $[0, T]$, the explicit form of g in (3) can be derived.

Theorem 3. *If a and b are piecewise linear functions on $[0, T]$ with common nodes $(t_i)_{i=1}^n$, then probability (1) is given by (2) with*

$$g(\boldsymbol{x}) = \prod_{i=1}^{n} \mathbf{1}(\alpha_i < x_i < \beta_i)\left[1 - \sum_{j=1}^{\infty} q(i, j)\right], \tag{4}$$

where

$$\begin{aligned} q(i, j) = {} & \exp\left\{-\frac{2}{\Delta t_i}[jd_{i-1} + (\alpha_{i-1} - x_{i-1})][jd_i + (\alpha_i - x_i)]\right\} \\ & - \exp\left\{-\frac{2j}{\Delta t_i}[jd_{i-1}d_i + d_{i-1}(\alpha_i - x_i) - d_i(\alpha_{i-1} - x_{i-1})]\right\} \\ & + \exp\left\{-\frac{2}{\Delta t_i}[jd_{i-1} - (\beta_{i-1} - x_{i-1})][jd_i - (\beta_i - x_i)]\right\} \\ & - \exp\left\{-\frac{2j}{\Delta t_i}[jd_{i-1}d_i - d_{i-1}(\beta_i - x_i) + d_i(\beta_{i-1} - x_{i-1})]\right\}. \end{aligned}$$

Proof. Since boundaries a and b are linear functions on each interval $[t_{i-1}, t_i]$, the conditional probability in (3) can be written as

$$\begin{aligned} & \mathrm{P}[a(t + t_{i-1}) - x_{i-1} < W(t) < b(t - t_{i-1}) - x_{i-1}, t < \Delta t_i \mid W(\Delta t_i) = \Delta x_i] \\ & \quad = \mathrm{P}\left[\frac{\Delta\alpha_i}{\Delta t_i}t + \alpha_{i-1} - x_{i-1} < W(t) < \frac{\Delta\beta_i}{\Delta t_i}t + \beta_{i-1} - x_{i-1}, t < \Delta t_i \,\middle|\, W(\Delta t_i) = \Delta x_i\right] \\ & \quad = 1 - \mathrm{P}_{1i} - \mathrm{P}_{2i}, \end{aligned}$$

where P_{1i} is the conditional probability that $W(t)$ crosses the upper boundary $(\Delta\beta_i/\Delta t_i)t + \beta_{i-1} - x_{i-1}$ first, given $W(\Delta t_i) = \Delta x_i$, and P_{2i} is the conditional probability that $W(t)$ crosses the lower boundary $(\Delta\alpha_i/\Delta t_i)t + \alpha_{i-1} - x_{i-1}$ first. Substituting $\Delta\beta_i/\Delta t_i$ and $\beta_{i-1} - x_{i-1}$ into the first part of equation (4.24) of Anderson (1960) we obtain the expression for P_{1i}, which can be further simplified to the sum of the first two terms in $q(i, j)$ with respect to j. Multiplying all sides of the inequalities in the above conditional probability by -1, we obtain P_{2i} similarly, which is given by the sum of the last two terms of $q(i, j)$ with respect to j. The theorem follows.

3. Approximation for general boundaries

In this section, we consider the case where boundaries a and b are general nonlinear functions on $[0, T]$. In this situation, Theorems 2 and 3 suggest a possible way to approximate $\mathrm{P}(a, b)$ by (2) and (4), provided a and b are 'sufficiently well' approximated by some piecewise linear functions. Indeed, if $a_n(t) \to a(t)$ and $b_n(t) \to b(t)$ uniformly on $[0, T]$, then it follows from the continuity property of probability measure that

$$\lim_{n\to\infty} \mathrm{P}(a_n, b_n) = \mathrm{P}(a, b). \tag{5}$$

The accuracy of the approximation (5) depends on the partition size n. Usually, larger n will give a more accurate approximation. One simple and straightforward way to assess the approximation accuracy is to use the difference of boundary crossing probabilities for two piecewise linear functions which approach $a(t)$ and $b(t)$ from each side respectively. To illustrate this idea, let us consider special cases where $a(t)$ and $b(t)$ are either concave or convex. In particular, let $b_n(t)$ be the piecewise linear function on $[0, T]$ connecting points $(t_i, b(t_i))$, $i = 1, 2, \ldots, n$, and let $\tilde{b}_n(t)$ be the piecewise linear function on $[0, T]$ taking values at t_i to be $b(t_i)$ plus or minus the maximum distance between $b(t)$ and $b_n(t)$ over two adjacent sub-intervals of t_i, depending on whether b is concave or convex. It is easy to see that the functions $b_n(t)$ and $\tilde{b}_n(t)$ thus defined are piecewise linear and continuous on $[0, T]$. Furthermore, $b_n(t)$ and $\tilde{b}_n(t)$ converge uniformly to $b(t)$ from below and above respectively, if $b(t)$ is concave, and from above and below, if $b(t)$ is convex. Hence, for the one-sided boundary b,

$$\mathrm{P}(-\infty, b_n) \le \mathrm{P}(-\infty, b) \le \mathrm{P}(-\infty, \tilde{b}_n),$$

if b is concave and

$$\mathrm{P}(-\infty, \tilde{b}_n) \le \mathrm{P}(-\infty, b) \le \mathrm{P}(-\infty, b_n),$$

if b is convex. In any case, the approximation accuracy is given by $|\mathrm{P}(-\infty, b_n) - \mathrm{P}(-\infty, \tilde{b}_n)|$.

For two-sided boundaries, let piecewise linear functions $a(t)$ and $\tilde{a}_n(t)$ be defined in a similar way to $b(t)$ and $\tilde{b}_n(t)$ respectively. Then $b_n(t)$, $a_n(t)$ and $\tilde{b}_n(t)$, $\tilde{a}_n(t)$ approach $b(t)$, $a(t)$ from inside and outside if b is concave and a is convex. Hence,

$$\mathrm{P}(a_n, b_n) \le \mathrm{P}(a, b) \le \mathrm{P}(\tilde{a}_n, \tilde{b}_n)$$

and the corresponding approximation accuracy is therefore given by $|\mathrm{P}(a_n, b_n) - \mathrm{P}(\tilde{a}_n, \tilde{b}_n)|$. Some examples are computed in Section 5 using this approach. A sharper asymptotic upper bound of the approximation errors for (5) will be derived in the next section.

In the remaining part of this section we propose another, simpler approximation formula for probability (1) for general nonlinear boundaries. This method is based on the idea that, since $\mathrm{var}[W(t)] = t$, if $[t_{i-1}, t_i]$ is sufficiently small, say $\Delta t_i \le c \inf_{t_{i-1}\le t\le t_i} |b(t) - a(t)|$ for some positive constant c, then the probability that $W(t)$ crosses both boundaries $a(t)$ and $b(t)$ in $[t_{i-1}, t_i]$ is very small, and will vanish when $\Delta t_i \to 0$. Thus, the conditional probability in (3) can be approximated by

$$\begin{aligned}
&1 - \mathrm{P}\left(W(t) \le \frac{\Delta\alpha_i}{\Delta t_i} t + \alpha_{i-1} - x_{i-1},\ \text{for some } t < \Delta t_i \;\middle|\; W(\Delta t_i) = \Delta x_i \right) \\
&- \mathrm{P}\left(W(t) \ge \frac{\Delta\beta_i}{\Delta t_i} t + \beta_{i-1} - x_{i-1},\ \text{for some } t < \Delta t_i \;\middle|\; W(\Delta t_i) = \Delta x_i \right) \\
&\quad = 1 - \exp\left[-\frac{2}{\Delta t_i}(\alpha_{i-1} - x_{i-1})(\alpha_i - x_i) \right] - \exp\left[-\frac{2}{\Delta t_i}(\beta_{i-1} - x_{i-1})(\beta_i - x_i) \right],
\end{aligned}$$

where the last equality follows from Siegmund (1986, p. 375) or Wang and Pötzelberger (1997) and the reflection property of Brownian motion. This leads to the following result.

Theorem 4. *If* $a, b \in C[0, T]$ *and* $\max_{1\le i\le n} \Delta t_i \to 0$ *as* $n \to 0$*, then*

$$\mathrm{P}(a, b) = \lim_{n\to\infty} \mathrm{E}h[W(t_1), W(t_2), \ldots, W(t_n)], \tag{6}$$

where

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} \mathbf{1}(\alpha_i < x_i < \beta_i)\left\{1 - \exp\left[-\frac{2}{\Delta t_i}(\alpha_{i-1} - x_{i-1})(\alpha_i - x_i)\right] - \exp\left[-\frac{2}{\Delta t_i}(\beta_{i-1} - x_{i-1})(\beta_i - x_i)\right]\right\}.$$

Our previous numerical studies show that, for moderate partition sizes, e.g., $4 \le n \le 64$, formula (6) gives very similar numerical results to those in Table 1 and Table 2 of Section 5, which are computed using (2) and (4).

Remark 1. The continuity of a and b have been assumed for the convenience of notation. All results in this section remain true if one or both of a and b is discontinuous at finite points on $[0, T]$ and such that, at any point of discontinuity t^*, $\lim_{t\to t^*-} b(t) < \lim_{t\to t^*+} b(t)$ and $\lim_{t\to t^*-} a(t) > \lim_{t\to t^*+} a(t)$. In this case, we need only include the points of discontinuity in the partition nodes and to define values α_i and β_i to be the left- or right-limit respectively of a and b at t_i.

4. Convergence rate of approximation errors

The approach in the last section allows us to assess approximation errors easily. In this section we derive a more accurate asymptotic upper bound for approximation errors in terms of the partition size n. This will allow us to control approximation errors before the actual computation begins. In particular, we propose a rule for choosing an optimal partition for the approximating piecewise linear boundaries, so that the corresponding approximation errors of boundary crossing probabilities converge to zero with a rate of $O(1/n^2)$. It is worthwhile to note that, using a different approach, Novikov *et al.* (1999) achieved a rate $O((\log n/n^3)^{1/2})$ for their approximation errors, which is the best result so far known in the literature. In this paper, we are able to achieve a better rate by introducing a smooth approximating function ϕ, as shown below.

First, let us provide some motivation of our approach for the case of a one-sided boundary. Let the boundary $b : [0, T] \to \mathbb{R}$ be sufficiently smooth with $b(0) > 0$ and let $b_n : [0, T] \to \mathbb{R}$ be the piecewise linear function with nodes $(t_i)_{i=1}^n$ and such that $b_n(t_i) = b(t_i)$. Our goal is to find a partition $(t_i)_{i=1}^n$ such that

$$\Delta_n = |\mathrm{P}(-\infty, b) - \mathrm{P}(-\infty, b_n)|$$

is small. Our approach is based on the following lemma, the proof of which is given in Section 6.

Lemma 1. *Let* $u_1, u_2 : [0, T] \to \mathbb{R}$ *be piecewise differentiable with* $u_1(0) = u_2(0)$*. Let* $X_i(t) = W(t) - u_i(t)$ *and denote the law of the process* $(X_i(t))_{t\in[0,T]}$ *by* P_i*. Then* $\|\mathrm{P}_1 - \mathrm{P}_2\| = 2\Phi(\frac{1}{2}\sigma) - 1$*, where* Φ *is the standard normal distribution function and*

$$\sigma^2 = \int_0^T (u_1'(t) - u_2'(t))^2 \,\mathrm{d}t.$$

In view of Lemma 1, let us fix a differentiable function $\phi : [0, T] \to [0, \infty)$ with $\phi(0) = 0$, and a constant $\varepsilon = \varepsilon_n(\phi) \in [0, \infty)$. We are to find nodes $(t_i)_{i=1}^n$ and to construct b_n, such that for all $t \in [0, T]$,

$$|b(t) - b_n(t)| \leq \varepsilon\phi(t). \tag{7}$$

To this end, let $u_1(t) = b(t) - \varepsilon\phi(t)$, $u_2(t) = b(t) + \varepsilon\phi(t)$, $X_i(t) = W(t) - u_i(t)$ and let P_i be the law of X_i. Then, by Lemma 1 we have, for the boundary b_n satisfying (7),

$$\begin{aligned}
\Delta_n &\leq |\mathrm{P}(-\infty, u_1) - \mathrm{P}(-\infty, u_2)| \\
&= |\mathrm{P}(\exists t \in [0, T] : W(t) > u_1(t)) - \mathrm{P}(\exists t \in [0, T] : W(t) > u_2(t))| \\
&= |\mathrm{P}(\exists t \in [0, T] : X_1(t) > 0) - \mathrm{P}(\exists t \in [0, T] : X_2(t) > 0)| \\
&\leq \|\mathrm{P}_1 - \mathrm{P}_2\| = 2\Phi(2/\sigma) - 1 \\
&\leq \frac{\sigma}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\varepsilon\left(\int_0^T \phi'(t)^2 \,\mathrm{d}t\right)^{1/2},
\end{aligned} \tag{8}$$

where the last equality follows from

$$\begin{aligned}
\sigma^2 &= \int_0^T (u_1'(t) - u_2'(t))^2 \,\mathrm{d}t \\
&= 4\varepsilon^2 \int_0^T \phi'(t)^2 \,\mathrm{d}t.
\end{aligned}$$

The choice of the nodes $(t_i)_{i=1}^n$ is crucial in our approach. We will use the symbol $\approx$ to mean 'up to terms of smaller order'. Further, let b be twice differentiable. Then, since for $t \in [t_{i-1}, t_i]$,

$$\begin{aligned}
|b(t) - b_n(t)| &\leq \tfrac{1}{8}|b''(t_{i-1})|\Delta t_i^2 + o(\Delta t_i^2) \\
&\approx \tfrac{1}{8}|b''(t_{i-1})|\Delta t_i^2,
\end{aligned}$$

we have

$$\tfrac{1}{8}|b''(t_{i-1})|\Delta t_i^2 \leq \varepsilon\phi(t_{i-1}) + o(\Delta t_i^2),$$

and therefore

$$\Delta t_i \approx \left(\frac{8\varepsilon\phi(t_{i-1})}{|b''(t_{i-1})|}\right)^{1/2}. \tag{9}$$

The nodes $(t_i)_{i=1}^n$ may be chosen as quantiles of some distribution F with a density f, so that $t_i = F^{-1}(i/n)$ and hence

$$\Delta t_i \approx \frac{1}{nf(t_{i-1})}.$$

Combined with (9) this leads to

$$\frac{1}{n^2 f^2(t_{i-1})} \frac{|b''(t_{i-1})|}{\phi(t_{i-1})} \approx 8\varepsilon,$$

which implies that

$$f(t) \propto \sqrt{\frac{|b''(t)|}{\phi(t)}},$$

that is,

$$f(t) = \frac{\sqrt{|b''(t)|/\phi(t)}}{\int \sqrt{|b''(u)|/\phi(u)}\,\mathrm{d}u}, \tag{10}$$

and hence

$$\varepsilon \approx \frac{1}{8n^2}\left(\int_0^T \sqrt{\frac{|b''(u)|}{\phi(u)}}\,\mathrm{d}u\right)^2.$$

It then follows from (8) that

$$\Delta_n \le \frac{1}{n^2}\frac{A(\phi)}{4\sqrt{2\pi}} + o(1/n^2),$$

where

$$A(\phi) = \left(\int_0^T \sqrt{\frac{|b''(u)|}{\phi(u)}}\,\mathrm{d}u\right)^2 \left(\int_0^T \phi'(t)^2\,\mathrm{d}t\right)^{1/2}. \tag{11}$$

Before we state our result formally, let us introduce the following concepts.

Definition 1. Let f denote a probability density function on $[0, T]$, F its cumulative distribution function and F^{-1} its quantile function. A partition of interval $[0, T]$ with nodes $t_i = F^{-1}(i/n),\ i = 1, 2, \ldots, n$, is called a *regular partition* of size n generated by f. A regular partition is called a *uniform partition*, if it is generated by the uniform distribution on $[0, T]$.

Note that a uniform partition is the one having equally spaced nodes $t_i = iT/n,\ i = 1, 2, \ldots, n$.

Theorem 5. *Let $b : [0, T] \to \mathbb{R}$ be twice continuously differentiable with $b''(0) \neq 0$, $\phi : [0, T] \to [0, \infty)$ differentiable with $\phi(0) = 0$ and let $|b''/\phi|^{1/2}$ be integrable. Furthermore, suppose there exist $t^* \in (0, T]$ and $B > 0$ such that ϕ is non-decreasing on $[0, t^*]$ and $\phi \ge B$ on $[t^*, T]$.*

If $b'' \neq 0$ on $[0, T]$, then for the regular partition $(t_i)_{i=1}^n$ generated by $f \propto |b''/\phi|^{1/2}$,

$$\limsup_{n\to\infty} n^2 \Delta_n \le \frac{A(\phi)}{4\sqrt{2\pi}} \tag{12}$$

with $A(\phi)$ defined by (11).

If b'' has only a finite number of roots in $(0, T]$, then for every $\delta > 0$, the density f may be modified in a neighbourhood of the roots of b'', such that for the sequence of regular partitions generated by the modified density,

$$\limsup_{n\to\infty} n^2 \Delta_n \le \frac{A(\phi)}{4\sqrt{2\pi}} + \delta.$$

Remark 2. Theorem 5 may be extended to piecewise linear boundaries which are not continuous. Let (t_i) be a regular partition of size n and b_n a function which is linear on $[t_{i-1}, t_i]$ such that for all $t \in [0, T]$,

$$|b(t) - b_n(t)| \le \sup_{t\in[0,T]} \left| b(t) - b(t_{i-1}) - (t - t_{i-1})\frac{\Delta b_i}{\Delta t_i} \right|.$$

Then (12) holds.

The proof of Theorem 5 is based on the following lemma, the proof of which is given in Section 6.

Lemma 2. *Let b be twice continuously differentiable, $\phi : [0, T] \to [0, \infty)$ continuous with $\phi(0) = 0$ and f a density on $[0, T]$. Let $(t_i)_{i=1}^n$ be the regular partition generated by f. Assume that there exist constants A_1, A_2, t^* and $B > 0$, such that $b'' \neq 0$ on $[0, t^*]$, ϕ is non-decreasing on $[0, t^*]$, $\phi \geq B$ on $[t^*, T]$, $f \geq A_1$ on $[0, T]$ and $f^2 \geq A_2|c''|/\phi$ on $[0, t^*]$. Let $\Delta t_i = t_i - t_{i-1}$ and $\Delta b_i = b(t_i) - b(t_{i-1})$. Then*

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \frac{n^2}{\phi(t)} \left| b(t) - b(t_{i-1}) - (t - t_{i-1}) \frac{\Delta b_i}{\Delta t_i} \right| < \infty.$$

Proof of Theorem 5. Assume that $b'' \neq 0$ on $[0, T]$. It follows from Lemma 2 that there exist a constant $M > 0$ and a n_0 such that for $n > n_0$ and all $t \in [0, T]$,

$$|b(t) - b_n(t)| \leq \phi(t) \frac{M}{n^2}.$$

Moreover, for all $t \in (0, T]$,

$$\limsup_{n\to\infty} n^2 |b(t) - b_n(t)| \leq \frac{\phi(t)}{8} \left(\int_0^T \sqrt{\frac{|b''(u)|}{\phi(u)}} \, \mathrm{d}u \right)^2.$$

Let $\delta > 0$, $t^{**} \leq t^*$ and define

$$\varepsilon(t) = \begin{cases} \dfrac{M}{n^2} & \text{if } t \leq t^{**}, \\[2ex] \dfrac{1}{n^2} \left[\frac{1}{8} \left(\displaystyle\int_0^T \sqrt{\frac{|b''(u)|}{\phi(u)}} \, \mathrm{d}u \right)^2 + \delta \right] & \text{if } t > t^{**}. \end{cases}$$

Then, for n large enough,

$$|b(t) - b_n(t)| \leq \phi(t) \varepsilon(t).$$

Therefore,

$$\limsup_{n\to\infty} n^2 \Delta_n \leq \frac{\sqrt{2/\pi}2}{n^2} \left[\left(\frac{1}{8} \left(\int_0^T \sqrt{\frac{|b''(t)|}{\phi(t)}} \, \mathrm{d}t \right)^2 + \delta \right)^2 \int_{t^{**}}^T \phi'(t)^2 \, \mathrm{d}t + M^2 \int_0^{t^{**}} \phi'(t)^2 \, \mathrm{d}t \right]^2.$$

Choosing t^{**} small enough leads to (12).

The case when b'' has only a finite number of roots which are in $(0, T]$ requires only the obvious modification of f, which leads to a density $\tilde{f}$ with $\inf_{t\in[0,T]} \tilde{f}(t) > 0$.

Remark 3. For practical purposes it turns out that the choice of ϕ with small $A(\phi)$ leads to sufficiently good approximations. In the light of Theorem 5 we may even want to find ϕ that minimizes $A(\phi)$. A straightforward computation shows that if a twice differentiable function ϕ minimizes $A(\phi)$, then it satisfies the first order condition

$$\phi''(t) = -\gamma \phi^{-3/2}(t) |b''(t)| \tag{13}$$

with

$$\gamma = \frac{\int_0^T \phi'(t)^2 \, \mathrm{d}t}{\int_0^T |b''(t)/\phi(t)|^{1/2} \, \mathrm{d}t} \tag{14}$$

and $\phi(0) = 0$. Note that if ϕ solves (13), (14) and $\phi(0) = 0$, then so does $c\phi$, for any $c > 0$. Moreover, if ϕ solves (13) and (14), then $\phi(T)\phi'(T) = 0$. Typically this differential equation has to be solved numerically.

Approximating two-sided boundaries by piecewise linear functions gives no additional difficulties. Let functions a and b be given with $a(0) < 0 < b(0)$. Let us now define

$$\Delta_n = |\mathrm{P}(a, b) - \mathrm{P}(a_n, b_n)|.$$

Our goal is to find nodes $(s_i)_{i=1}^n$ and $(t_i)_{i=1}^n$ and compute functions a_n and b_n which are linear on $[s_{i-1}, s_i]$ and on $[t_{i-1}, t_i]$ respectively. Obviously, a_n and b_n are linear on $[r_{i-1}, r_i]$, where $(r_i)_{i=1}^{2n}$ are the nodes $\{s_i, t_i \mid i \leq n\}$ in increasing order.

We choose ϕ_a, ϕ_b, ε_a and ε_b, with $|a(t) - a_n(t)| \leq \varepsilon_a \phi_a(t)$, $|b(t) - b_n(t)| \leq \varepsilon_b \phi_b(t)$ and

$$\sigma_a = 2\varepsilon_a \left(\int_0^T \phi_a'(t)^2 \, \mathrm{d}t \right)^{1/2}, \qquad \sigma_b = 2\varepsilon_b \left(\int_0^T \phi_b'(t)^2 \, \mathrm{d}t \right)^{1/2}$$

and get

$$\Delta_n \leq \left(2\Phi\left(\frac{2}{\sigma_a} \right) - 1 \right) + \left(2\Phi\left(\frac{2}{\sigma_b} \right) - 1 \right).$$

Again, we choose a regular partition (s_i) generated by a density f_a and (t_i) generated by f_b. The following result is self-evident.

Theorem 6. *Let the assumptions of Theorem 5 hold for a, b, ϕ_a and ϕ_b. Let $(s_i)_{i=1}^n$ and $(t_i)_{i=1}^n$ denote the regular partitions generated by $f_a \propto |a''/\phi_a|^{1/2}$ and by $f_b \propto |b''/\phi_b|^{1/2}$ respectively. Then*

$$\limsup_{n\to\infty} n^2 \Delta_n \leq \frac{A(\phi_a) + A(\phi_a)}{4\sqrt{2\pi}}.$$

Remark 4. Our approach to deriving upper bounds for approximation errors is not limited to the case of boundary crossing probabilities. It may be applied whenever probabilities have to be calculated which can be expressed in the form $\mathrm{P}(X_i \in A)$, with $X_i(t) = W(t) - u_i(t)$. Consider for instance the pricing of time-dependent barrier options. In that case for $K \in \mathbb{R}$ the probability $\mathrm{P}(W(T) > K \text{ and } W(t) < u_1(t),\ 0 \leq t \leq T)$ has to be computed. Suppose for an approximating boundary u_2, say a piecewise linear boundary, the corresponding probability can be computed. Let $u_1(0) = u_2(0)$ and $u_1(T) = u_2(T)$ and define $A = \{\omega \in C[0, T] \mid \omega(t) < 0 \text{ for all } t \text{ and } \omega(T) > K - u_1(T)\}$, then

$$\mathrm{P}(W(T) > K \text{ and } W(t) < u_i(t),\ 0 \leq t \leq T) = \mathrm{P}(X_i \in A).$$

The difference of the two probabilities is therefore bounded by $\|\mathrm{P}_1 - \mathrm{P}_2\|$.

5. Numerical examples

In this section we compute some examples of probability (1) for some nonlinear boundaries, using the approximation methods proposed in Sections 3 and 4. In particular, we consider the following four one-sided boundaries: $(1+t)^{1/2}$, $\exp(-t)$, $1+t^2$, $1+t-t^2$ and four two-sided symmetric boundaries: $\pm(1+t)^{1/2}$, $\pm\exp(-t)$, $\pm(1+t^2)$ and $\pm(1+t-t^2)$.

For probabilities $\mathrm{P}(a, b)$ for these boundaries we compute the upper bounds and lower bounds, which are given by $\mathrm{P}(a_n, b_n)$ or $\mathrm{P}(\tilde{a}_n, \tilde{b}_n)$ as described in Section 3. For these piecewise

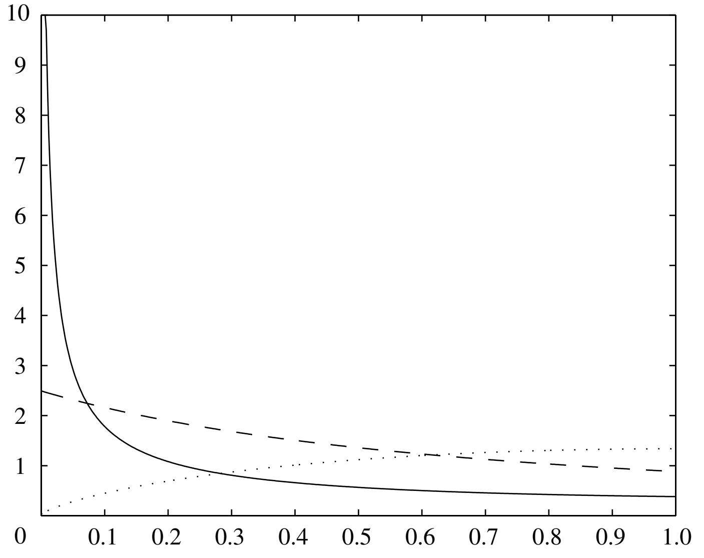

FIGURE 1: Functions $10|b''|$ (dashed line), ϕ (dotted line) and the corresponding density f (solid line) for the boundary $b(t) = (1 + t)^{1/2}$.

linear boundaries, we use the regular partition generated by the density (10) corresponding to the minimal $A(\phi)$. This is done by first solving the differential equation (13) numerically to obtain ϕ, and then compute the nodes $(t_i)_{i=1}^n$ using (10). As an example, a numerical solution ϕ and the corresponding density f for the square-root boundary are shown in Figure 1. Furthermore, approximation errors are computed as the differences between upper and lower bounds.

For the actual simulation, we use the procedure of Wang and Pötzelberger (1997). First, a random sample $w_1, w_2, \ldots, w_N$ is generated from the multivariate normal distribution of $W(t_1), W(t_2), \ldots, W(t_n)$. Then, the probability $\mathrm{P}(-\infty, b_n)$ or $\mathrm{P}(a_n, b_n)$ is estimated by the corresponding sample mean

$$\hat{\mathrm{P}} = \sum_i \frac{g(w_i)}{N}.$$

The corresponding simulation standard error is then given by

$$\hat{S} = \sqrt{\sum_i \frac{(g(w_i) - \hat{\mathrm{P}})^2}{N(N-1)}}.$$

In this paper, one million repetitions are carried out in each simulation. Table 1 contains the simulated probability bounds for one-sided boundaries, whereas Table 2 contains the results for two-sided boundaries. In both tables standard errors are given in the parentheses. Both tables show a clear pattern of converging upper and lower bounds along with the increasing partition size n. The approximation errors given by the difference also show a clear pattern of decreasing at the theoretical rate of $O(1/n^2)$. The simulation errors show that the numerical results are reliable to three significant figures. These simulation errors can be further reduced by increasing the number of repetitions in simulation, or by improving the simulation methods, e.g., by using importance sampling techniques.

TABLE 1: Upper (UB) and lower (LB) bounds of one-sided boundary crossing probabilities, and differences (DF) of the bounds (DF = UB − LB), for various partition sizes n. In parentheses are simulation standard errors.

		$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$\sqrt{1+t}$	UB	0.804505	0.803730	0.803406	0.804314	0.804202
		(0.000340)	(0.000359)	(0.000372)	(0.000379)	(0.000384)
	LB	0.803045	0.803357	0.803315	0.804291	0.804197
		(0.000341)	(0.000359)	(0.000372)	(0.000379)	(0.000384)
	DF	0.001460	0.000373	0.000091	0.000022	0.000006
$\exp(-t)$	UB	0.442504	0.440556	0.439636	0.438672	0.439200
		(0.000440)	(0.000457)	(0.000469)	(0.000477)	(0.000483)
	LB	0.434597	0.438526	0.439134	0.438548	0.439169
		(0.000439)	(0.000457)	(0.000469)	(0.000477)	(0.000483)
	DF	0.007907	0.002029	0.000502	0.000124	0.000031
$1+t^2$	UB	0.860349	0.854139	0.852180	0.851697	0.851813
		(0.000294)	(0.000317)	(0.000330)	(0.000338)	(0.000344)
	LB	0.844179	0.850320	0.851275	0.851477	0.851759
		(0.000308)	(0.000321)	(0.000331)	(0.000339)	(0.000344)
	DF	0.016170	0.003819	0.000906	0.000219	0.000053
$1+t-t^2$	UB	0.753880	0.746641	0.743366	0.743442	0.743366
		(0.000385)	(0.000403)	(0.000414)	(0.000421)	(0.000426)
	LB	0.733761	0.741439	0.742075	0.743121	0.743287
		(0.000395)	(0.000405)	(0.000415)	(0.000421)	(0.000426)
	DF	0.020118	0.005202	0.001291	0.000321	0.000079

6. Proofs of lemmas

Proof of Lemma 1. Let P denote the law of the Brownian motion $(W(t))_{t\in[0,T]}$. Then $\mathrm{P}_i \ll \mathrm{P}$ and the theorem of Girsanov identifies the Radon–Nikodym derivative as

$$\frac{\mathrm{dP}_i}{\mathrm{dP}} = \exp\biggl(\int_0^T u_i'(t)\,\mathrm{d}W(t) - \int_0^T \tfrac{1}{2}u_i'(t)^2\,\mathrm{d}t\biggr).$$

Let $Z_i = \int_0^T u_i'(t)\,\mathrm{d}W(t)$ and $\sigma_i^2 = \int_0^T u_i'(t)^2\,\mathrm{d}t$. Then Z_1 and Z_2 are jointly normally distributed with means 0, variances σ_i^2 and covariance $\sigma_{12} = \int_0^T u_1'(t)u_2'(t)\,\mathrm{d}t$. It is straightforward to verify that

$$\|\mathrm{P}_1 - \mathrm{P}_2\| = \tfrac{1}{2}\mathrm{E}(|\exp(Z_1 - \tfrac{1}{2}\sigma_1^2) - \exp(Z_2 - \tfrac{1}{2}\sigma_2^2)|) = 2\Phi(\tfrac{1}{2}\sigma) - 1$$

with $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{12} = \int_0^T (u_1'(t) - u_2'(t))^2\,\mathrm{d}t$.

Proof of Lemma 2. It is straightforward to verify that there exist an integer n_0 and a constant A, such that for all $n \ge n_0$ and $t \in [t_{t-1}, t_i]$ with $t_i \ge t^*$,

$$\frac{n^2}{\phi(t)}\left|b(t) - b(t_{i-1}) - (t - t_{i-1})\frac{\Delta b_i}{\Delta t_i}\right| \le A.$$

The corresponding statement for $t \in [0, t^*]$ is a little more involved. Note that there exists a function ω with $\omega(0) = 1$, which is continuous in 0, such that for all $u, v \in [t_{i-1}, t_i] \subseteq [0, t^*]$,

$$\frac{|b''(u)|}{|b''(v)|} \le \omega(\Delta t_i).$$

TABLE 2: Upper (UB) and lower (LB) bounds of two-sided boundary crossing probabilities, and differences (DF) of the bounds (DF = UB − LB), for various partition sizes n. In parentheses are simulation standard errors.

		$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$\pm\sqrt{1+t}$	UB	0.609137	0.608912	0.608708	0.608519	0.608438
		(0.000395)	(0.000426)	(0.000446)	(0.000459)	(0.000468)
	LB	0.606235	0.608170	0.608526	0.608475	0.608427
		(0.000395)	(0.000426)	(0.000446)	(0.000459)	(0.000468)
	DF	0.002901	0.000742	0.000182	0.000045	0.000011
$\pm\exp(-t)$	UB	0.017899	0.016140	0.015658	0.015564	0.015494
		(0.000045)	(0.000054)	(0.000070)	(0.000085)	(0.000096)
	LB	0.013801	0.015062	0.015390	0.015498	0.015477
		(0.000036)	(0.000051)	(0.000069)	(0.000085)	(0.000096)
	DF	0.004098	0.001078	0.000268	0.000066	0.000017
$\pm(1+t^2)$	UB	0.720547	0.707756	0.704492	0.705054	0.703570
		(0.000366)	(0.000389)	(0.000418)	(0.000429)	(0.000438)
	LB	0.688258	0.700133	0.702685	0.704617	0.703463
		(0.000377)	(0.000401)	(0.000419)	(0.000430)	(0.000439)
	DF	0.032289	0.007624	0.001806	0.000437	0.000106
$\pm(1+t-t^2)$	UB	0.510083	0.493958	0.489774	0.488222	0.488195
		(0.000419)	(0.000444)	(0.000461)	(0.000472)	(0.000481)
	LB	0.470248	0.483656	0.487221	0.487590	0.488039
		(0.000414)	(0.000443)	(0.000460)	(0.000472)	(0.000481)
	DF	0.039835	0.010303	0.002554	0.000632	0.000157

Let $t \in [t_{i-1}, t_i] \subseteq [0, t^*]$. Then there exist $\theta_i \in [t_{i-1}, t_i]$ and $\tilde{\theta}_i = \tilde{\theta}_i(v) \in [t_{i-1}, t_i]$, such that

$$\begin{aligned} b(t) - b(t_{i-1}) - (t - t_{i-1})\frac{\Delta b_i}{\Delta t_i} &= \int_{t_{i-1}}^{t} \left(b'(v) - \frac{\Delta b_i}{\Delta t_i}\right) \mathrm{d}v \\ &= \int_{t_{i-1}}^{t} (b'(v) - b'(\theta_i))\, \mathrm{d}v \\ &= \int_{t_{i-1}}^{t} (v - \theta_i) b''(\tilde{\theta}_i)\, \mathrm{d}v. \end{aligned}$$

Therefore,

$$\left| b(t) - b(t_{i-1}) - (t - t_{i-1})\frac{\Delta b_i}{\Delta t_i} \right| \le \int_{t_{i-1}}^{t} |v - \theta_i| |b''(v)|\, \mathrm{d}v \omega(\Delta t_i).$$

Furthermore, since

$$\begin{aligned} \frac{1}{n} &= \int_{t_{i-1}}^{t_i} f(v)\, \mathrm{d}v \\ &\ge A_2^{1/2} \int_{t_{i-1}}^{t_i} \frac{|b''(v)|^{1/2}}{\phi(v)^{1/2}}\, \mathrm{d}v \\ &\ge \frac{A_2^{1/2}}{\phi(t)^{1/2}} \int_{t_{i-1}}^{t_i} |b''(v)|^{1/2}\, \mathrm{d}v, \end{aligned}$$

for $t \in [t_{i-1}, t_i] \subseteq [0, t^*]$,

$$\frac{n^2}{\phi(t)}\left|b(t) - b(t_{i-1}) - (t - t_{i-1})\frac{\Delta b_i}{\Delta t_i}\right| \le A_2^{-1}\omega(\Delta t_i)\frac{\int_{t_{i-1}}^{t_i} |v - \theta_i||b''(v)|\,\mathrm{d}v}{(\int_{t_{i-1}}^{t_i} |b''(v)|^{1/2}\,\mathrm{d}v)^2}$$
$$\le A_2^{-1}\omega(\Delta t_i)\frac{\sup_{v\in[0,t^*]} |b''(v)|}{\inf_{v\in[0,t^*]} |b''(v)|}.$$

This completes the proof.

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References

ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31,** 165–197.

BROWN, R. L., DURBIN, J. AND EVANS, J. M. (1975). Techniques for testing the constancy of regression relationships over time. *J. R. Statist. Soc. B* **37,** 149–192.

DANIELS, H. E. (1996). Approximating the first crossing-time density for a curved boundary. *Bernoulli* **2,** 133–143.

DURBIN, J. (1971). Boundary crossing probabilities for the Brownian motion and the poisson processes and techniques for computing the power of the Kolmogorov–Smirnov test. *J. Appl. Prob.* **8,** 431–453.

DURBIN, J. (1992). The first-passage density of the Brownian motion process to a curved boundary. *J. Appl. Prob.* **29,** 291–304.

HALL, W. J. (1997). The distribution of Brownian motion on linear stopping boundaries. *Sequential Anal.* **16,** 345–352.

KRÄMER, W., PLOBERGER, W. AND ALT, R. (1988). Testing for structural change in dynamic models. *Econometrica* **56,** 1355–1369.

LIN, X. S. (1998). Double barrier hitting time distributions with applications to exotic options. *Insurance: Math. Econ.* **23,** 45–58.

MARTIN-LÖF, A. (1998). The final size of a nearly critical epidemic, and the first passage time of a Wiener process to a parabolic barrier. *J. Appl. Prob.* **35,** 671–682.

NOVIKOV, A., FRISHLING, V. AND KORDZAKHIA, N. (1999). Approximations of boundary crossing probabilities for a Brownian motion. *J. Appl. Prob.* **99,** 1019–1030.

ROBERTS, G. O. AND SHORTLAND, C. F. (1997). Pricing barrier options with time-dependent coefficients. *Math. Finance* **7,** 83–93.

SACERDOTE, L. AND TOMASSETTI, F. (1996). On evaluations and asymptotic approximations of first-passage-time probabilities. *Adv. Appl. Prob.* **28,** 270–284.

SEN, P. K. (1981). *Sequential Nonparametrics: Invariance Principles and Statistical Inference*. John Wiley, New York.

SIEGMUND, D. (1985). *Sequential Analysis: Tests and Confidence Intervals*. Springer, New York.

SIEGMUND, D. (1986). Boundary crossing probabilities and statistical applications. *Ann. Statist.* **14,** 361–404.

WANG, L. AND PÖTZELBERGER, K. (1997). Boundary crossing probability for Brownian motion and general boundaries. *J. Appl. Prob.* **34,** 54–65.