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BOUNDARY CROSSING PROBABILITY FOR BROWNIAN MOTION AND GENERAL BOUNDARIES

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Abstract

An explicit formula for the probability that a Brownian motion crosses a piecewise linear boundary in a finite time interval is derived. This formula is used to obtain approximations to the crossing probabilities for general boundaries which are the uniform limits of piecewise linear functions. The rules for assessing the accuracies of the approximations are given. The calculations of the crossing probabilities are easily carried out through Monte Carlo methods. Some numerical examples are provided.

WIENER PROCESS; FIRST PASSAGE TIME; OPTIONAL STOPPING TIME; STRUCTURAL CHANGE; LOCAL POWER; NUMERICAL COMPUTATION; MONTE CARLO SIMULATION

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1. Introduction

Let $W(t)$, $t \geqq 0$, be a standard Brownian motion with $\boldsymbol{E}W(t)=0$, $\boldsymbol{E}W(t)W(s)=\min(t, s)$ and $c(t)$ be constants with $c(0)>0$. We are concerned with the boundary crossing probability

$$Q(c(t); T)=\boldsymbol{P}(W(t) \geqq c(t), \text{ for some } t \leqq T), \tag{1}$$

where $T>0$ is fixed. Calculations of this kind of probability occur frequently in many fields of statistics. One example is the test of structural change in the regression model, where the asymptotic local power of the CUSUM test of Brown *et al.* (1975), the fluctuation test of Krämer *et al.* (1988) and some other test statistics turn out to be the probabilities of the Brownian motion crossing certain nonlinear boundaries. Other examples are the Kolmogorov–Smirnov (and similar) statistics, sequential analysis, cumulative sum techniques and optional stopping times. See, e.g., Sen (1981) and Siegmund (1986) and references therein. Many methods and techniques for the calculation of (1) have been developed for various boundaries. However, most methods use differential or integral equations and only give numerical approximations to the corresponding

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probabilities. Explicit analytic formulae have been obtained only for linear and a few special nonlinear boundaries. See, e.g., Durbin (1971, 1985), Sen (1981), Siegmund (1985, 1986), Lerche (1986) and Loader and Deely (1987). Durbin (1985) derived an asymptotic formula for a general boundary, which depends on the limit of certain conditional expectations.

In this paper we use a new and direct method to derive an explicit formula for probability (1) for piecewise linear boundaries, which can be very easily calculated, for example by using the Monte Carlo simulation method. This formula is used to obtain approximations to the crossing probabilities for general (nonlinear) boundaries which are the uniform limits of piecewise linear functions. The accuracy of the approximation may be assessed by the difference between the crossing probabilities of the two piecewise linear boundaries which approximate the general boundary from each side. An approach for determining the number and the locations of the vertices for the approximating piecewise linear functions is proposed, so that a prescribed precision is achieved. Finally, some numerical examples are presented.

2. Linear boundary

First we cite some known results concerning the linear boundary case which are needed for later development. Suppose $c(t)=at+b$, $t\in[0, T]$ and $b>0$. Then using the well-known formula of Siegmund (1986, p. 375)

$$\boldsymbol{P}(W(t)\geqq at+b, \text{ for some } t<T \mid W(T)=x)=\exp\left[-\frac{2b(aT+b-x)}{T}\right], \tag{2}$$

we have

$$\begin{aligned} Q(at+b; T) &= 1-\boldsymbol{P}(W(t)<at+b, t\leqq T) \\ &= 1-\int_{-\infty}^{aT+b} \boldsymbol{P}(W(t)<at+b, t<T \mid W(T)=x)d\boldsymbol{P}_T(x) \\ &= 1-\int_{-\infty}^{aT+b}\left(1-\exp\left[-\frac{2b(aT+b-x)}{T}\right]\right)d\boldsymbol{P}_T(x) \\ &= 1-\Phi\left(\frac{aT+b}{\sqrt{T}}\right)+\exp[-2ab]\Phi\left(\frac{aT-b}{\sqrt{T}}\right), \end{aligned} \tag{3}$$

where

$$\frac{d\boldsymbol{P}_T(x)}{dx}=\frac{1}{\sqrt{2\pi T}}\exp\left[-\frac{x^2}{2T}\right]$$

is the probability density function of $W(T)$ and $\Phi(\cdot)$ is the standard normal distribution function.

3. Piecewise linear boundary

In this section we generalize the formula (3) to a boundary that is a polygonal function on the interval $[0, T]$. Specifically, let $0=t_0<t_1<\cdots<t_n=T$ and $c(t)$ be linear on each of the intervals $[t_{j-1}, t_j]$, $j=1, 2,\cdots, n$ and $c(0)>0$. To avoid the trivial case we assume $n>1$. Then we derive a formula for the probability (1) for $c(t)$. The method of derivation is based on the idea that the event that $W(t)$ does not cross $c(t)$ on the interval $[0, T]$ may be split into n conditional events that $W(t)$ does not cross $c(t)$ on the interval $(t_j, t_{j+1}]$ given that $W(t)$ does not cross $c(t)$ on the interval $(t_{j-1}, t_j]$, and on each such interval the conditional probability may be calculated by (2) or (3).

Formally, denote $c_j=c(t_j)$, $j=0, 1, 2,\cdots, n$ and $c=(c_1,\cdots, c_n)'$. Then we have the first main result of this paper.

Theorem 1. *The crossing probability* (1) *for the piecewise linear boundary* $c(t)$ *is given by*

$$Q(c(t); T)=1-\boldsymbol{E}g(W(t_1),\cdots, W(t_n); c), \tag{4}$$

where

$$g(x_1,\cdots, x_n; c) = \prod_{j=1}^{n} I(x_j<c_j)\left(1-\exp\left[-\frac{2(c_{j-1}-x_{j-1})(c_j-x_j)}{t_j-t_{j-1}}\right]\right),$$

and $I(\cdot)$ *is the indicator function.*

Proof. By the strong Markovian property of $W(t)$ (Billingsley 1986, Section 37), we have

$$\begin{aligned}1-Q(c(t); T) &= \boldsymbol{P}(W(t)<c(t), t\leqq T)\\ &= \int_{-\infty}^{c_1} \boldsymbol{P}(W(t)<c(t), t_1\neq t\leqq T \mid W(t_1)=x_1)d\boldsymbol{P}_{t_1}(x_1)\\ &= \int_{-\infty}^{c_1} \boldsymbol{P}(W(t)<c(t), t<t_1 \mid W(t_1)=x_1)\\ &\qquad \times \boldsymbol{P}(W(t)<c(t), t_1<t\leqq T \mid W(t_1)=x_1)d\boldsymbol{P}_{t_1}(x_1).\end{aligned}$$

By (2) the first factor in the last integral is

$$\boldsymbol{P}(W(t)<c(t), t<t_1 \mid W(t_1)=x_1)=1-\exp\left[-\frac{2c_0(c_1-x_1)}{t_1}\right].$$

Further note that given $W(t_1)=x_1$, the process $W(t+t_1)-x_1$ is again a Brownian motion starting from 0 and therefore the second factor in the last integral is

$$\begin{aligned}
&\boldsymbol{P}(W(t) < c(t),\ t_1 < t \leqq T \mid W(t_1) = x_1) \\
&\quad = \boldsymbol{P}(W(t) < c(t+t_1) - x_1,\ t \leqq T - t_1) \\
&\quad = \int_{-\infty}^{c_2 - x_1} \left(1 - \exp\left[-\frac{2(c_1 - x_1)(c_2 - x_1 - x_2)}{t_2 - t_1}\right]\right) \boldsymbol{P}(W(t) < c(t+t_1) - x_1, \\
&\qquad\qquad t_2 - t_1 < t \leqq T - t_1 \mid W(t_2 - t_1) = x_2) d\boldsymbol{P}_{t_2 - t_1}(x_2) \\
&\quad = \int_{-\infty}^{c_2} \left(1 - \exp\left[-\frac{2(c_1 - x_1)(c_2 - x_2)}{t_2 - t_1}\right]\right) \boldsymbol{P}(W(t) < c(t+t_1) - x_1, \\
&\qquad\qquad t_2 - t_1 < t \leqq T - t_1 \mid W(t_2 - t_1) = x_2 - x_1) d\boldsymbol{P}_{t_2 - t_1}(x_2 - x_1) \\
&\quad = \int_{-\infty}^{c_2} \left(1 - \exp\left[-\frac{2(c_1 - x_1)(c_2 - x_2)}{t_2 - t_1}\right]\right) \\
&\qquad\qquad \boldsymbol{P}(W(t) < c(t+t_2) - x_2,\ t \leqq T - t_2) d\boldsymbol{P}_{t_2 - t_1}(x_2 - x_1).
\end{aligned}$$

Apply the same steps to the probability $\boldsymbol{P}(W(t) < c(t+t_2) - x_2,\ t \leqq T - t_2)$ and repeat this procedure until we obtain

$$\begin{aligned}
&\boldsymbol{P}(W(t) < c(t+t_{n-1}) - x_{n-1},\ t \leqq t_n - t_{n-1}) \\
&\quad = \int_{-\infty}^{c_n - x_{n-1}} \boldsymbol{P}(W(t) < c(t+t_{n-1}) - x_{n-1},\ t < t_n - t_{n-1} \mid W(t_n - t_{n-1}) = x_n) d\boldsymbol{P}_{t_n - t_{n-1}}(x_n) \\
&\quad = \int_{-\infty}^{c_n} \boldsymbol{P}(W(t) < c(t+t_{n-1}) - x_{n-1},\ t < t_n - t_{n-1} \mid W(t_n - t_{n-1}) = x_n - x_{n-1}) \\
&\qquad\qquad d\boldsymbol{P}_{t_n - t_{n-1}}(x_n - x_{n-1}) \\
&\quad = \int_{-\infty}^{c_n} \left(1 - \exp\left[-\frac{2(c_{n-1} - x_{n-1})(c_n - x_n)}{t_n - t_{n-1}}\right]\right) d\boldsymbol{P}_{t_n - t_{n-1}}(x_n - x_{n-1}).
\end{aligned}$$

By the definition of $W(t)$,

$$d\boldsymbol{P}_{t_j - t_{j-1}}(x_j - x_{j-1}) = \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left[-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right] dx_j,$$

$j = 1, 2, \cdots, n$, where it is assumed that $x_0 = 0$. Thus, we have finally

$$(5) \qquad Q(c(t); T) = 1 - \int_{(-\infty, c)} \prod_{j=1}^{n} \left(1 - \exp\left[-\frac{2(c_{j-1} - x_{j-1})(c_j - x_j)}{t_j - t_{j-1}}\right]\right) f(x) dx,$$

where $x = (x_1, \cdots, x_n)'$,

$$f(x)=\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\exp\left[-\frac{(x_j-x_{j-1})^2}{2(t_j-t_{j-1})}\right]$$

and the region of integration is $(-\infty, c)=(-\infty, c_1)\times(-\infty, c_2)\times\cdots\times(-\infty, c_n)$. The theorem follows from the fact that the function $f(x)$ is exactly the probability density function of $(W(t_1),\cdots, W(t_n))'$ (Billingsley 1986, p. 523).

It is easily seen that (5) is exactly (3) for $n=1$ and hence (4) is a generalization of (3).

4. General boundary

We have obtained (4) for the crossing probability for a piecewise linear boundary. In this section we use this formula to obtain an approximation to the crossing probability for a general boundary $c(t)$, provided $c(t)$ may be uniformly approximated by a sequence of piecewise linear functions. Let $0=t_0<t_1<\cdots<t_n=T$ and throughout this section denote by $c_n(t)$ the polygonal function taking the points $c_j=c(t_j)$, $j=0, 1,\cdots, n$, as vertices. Then we have the following result.

Theorem 2. *If $c_n(t)\to c(t)$ as $n\to\infty$ uniformly on* $[0, T]$, *then*

$$Q(c(t); T)=1-\lim_{n\to\infty}\boldsymbol{E}g(W(t_1),\cdots, W(t_n); c), \tag{6}$$

where $c=(c_1,\cdots, c_n)'$ and $g(\cdot)$ is defined as in Theorem 1.

Proof. The result is an immediate consequence of the uniform convergence of $c_n(t)$ to $c(t)$ and the continuity property of the probability measure.

By Theorem 2, the probability (1) for a general boundary may be approximated through (4) with sufficiently large n. However, as in every numerical problem, a crucial issue is then to be able to assess the accuracy of the approximation. In the following we consider two approaches to achieve this.

The first approach is to approximate the general boundary $c(t)$ by piecewise linear boundaries from each side. In this case the upper and lower bounds for the crossing probability of the general boundary are given by the corresponding crossing probabilities of the approximating polygonal boundaries. For simplicity of notation let us consider a boundary $c(t)$ which is concave on the interval $[0, T]$. Then $c_n(t)$ converges to $c(t)$ from below uniformly. Now define

$$d_n(t)=c_n(t)+\sup_{0\leqq t\leqq T}|c(t)-c_n(t)|.$$

Then $d_n(t)$ converges to $c(t)$ uniformly from above. By the monotonicity of the probability measure the crossing probability for $c(t)$ lies between the crossing probabilities for $c_n(t)$ and $d_n(t)$:

$$Q(d_n(t); T)\leqq Q(c(t); T)\leqq Q(c_n(t); T),$$

and the precision of the approximation is given by

$$|Q(c_n(t); T) - Q(d_n(t); T)|.$$

Some examples of this approach are given in the last section. While this approach is intuitive and easy to implement, it is a 'passive' method in the sense that the accuracy of the approximation is simply reported and not previously controlled.

The second approach we propose is to give a rule of the choice of the number of partitions n, so that a prescribed precision is achieved. To this end we show first the following result.

Theorem 3. *For any given* $\varepsilon > 0$ *and integer* n, *if*

$$\sup_{0 \leqq t \leqq T} |c(t) - c_n(t)| \leqq \frac{\varepsilon}{2}, \tag{7}$$

then

$$|Q(c(t); T) - Q(c_n(t); T)| \leqq \psi(\varepsilon, \eta), \tag{8}$$

where $0 < \eta < \eta_0$,

$$\psi(\varepsilon, \eta) = 4\Phi\left(\frac{\varepsilon}{2\sqrt{\eta}}\right) - 2\Phi\left(\frac{c_*}{\sqrt{\eta}}\right), \tag{9}$$

$\eta_0 \in (0, T)$ *is given and* $c_* = \inf_{0 \leqq t \leqq \eta_0} c(t) > 0$. *Furthermore, there exists* $\eta = \eta(\varepsilon)$, *such that*

$$\psi(\varepsilon, \eta(\varepsilon)) = O(\varepsilon\sqrt{-\log \varepsilon}) = o(\varepsilon^{1-\alpha})$$

for any $\alpha > 0$.

Proof. Define $d(t) = c(t) - \varepsilon/2$; then we have

$$\begin{aligned}
\Delta &= |Q(c(t); T) - Q(c_n(t); T)| \\
&= |\boldsymbol{P}(W(t) < c(t), t \leqq T) - \boldsymbol{P}(W(t) < c_n(t), t \leqq T)| \\
&\leqq |\boldsymbol{P}(W(t) < d(t) + \varepsilon, t \leqq T) - \boldsymbol{P}(W(t) < d(t), t \leqq T)| \\
&\leqq |\boldsymbol{P}(W(t) < d(t) + \varepsilon, t \leqq \eta) - \boldsymbol{P}(W(t) < d(t), t \leqq \eta)| \\
&\quad + |\boldsymbol{P}(W(t) < d(t) + \varepsilon, \eta \leqq t \leqq T) - \boldsymbol{P}(W(t) < d(t), \eta \leqq t \leqq T)| \\
&:= \Delta_1 + \Delta_2.
\end{aligned} \tag{10}$$

Now, for any $0 < \eta \leqq \eta_0$,

$$\begin{aligned}
\Delta_1 &= |Q(d(t) + \varepsilon; \eta) - Q(d(t); \eta)| \\
&\leqq Q(d(t) + \varepsilon; \eta) \\
&\leqq \boldsymbol{P}\left(\sup_{0 \leqq t \leqq \eta} W(t) \geqq c_*\right) \\
&= 2\left[1 - \Phi\left(\frac{c_*}{\sqrt{\eta}}\right)\right],
\end{aligned} \tag{11}$$

where the last equation follows from (3). For Δ_2, note that the process $W(t)-\varepsilon$ is also a Brownian motion starting from $-\varepsilon$ and hence, if $q(u)=\boldsymbol{P}(W(t)<d(t),\ \eta<t\leqq T \mid W(\eta)=u)$, then

$$\Delta_2 \leqq \int_{-\infty}^{+\infty} |q(u)| \left| \phi\left(\frac{u}{\sqrt{\eta}}\right) - \phi\left(\frac{u+\varepsilon}{\sqrt{\eta}}\right) \right| du$$

$$\leqq \int_{-\infty}^{+\infty} \left| \phi\left(\frac{u}{\sqrt{\eta}}\right) - \phi\left(\frac{u+\varepsilon}{\sqrt{\eta}}\right) \right| du$$

$$= \int_{-3\varepsilon/2}^{-\varepsilon/2} \phi\left(\frac{u+\varepsilon}{\sqrt{\eta}}\right) du + \int_{-\varepsilon/2}^{\varepsilon/2} \phi\left(\frac{u}{\sqrt{\eta}}\right) du \tag{12}$$

$$= 2\left[\Phi\left(\frac{\varepsilon}{2\sqrt{\eta}}\right) - \Phi\left(\frac{-\varepsilon}{2\sqrt{\eta}}\right)\right]$$

$$= 2\left[2\Phi\left(\frac{\varepsilon}{2\sqrt{\eta}}\right) - 1\right].$$

Combining (10), (11) and (12) we obtain (9). To show the second part of the theorem, we first use the inequalities $1-\Phi(u) < \phi(u)/u$ and $\Phi(u)-1/2 < u\phi(0)$ and have

$$\psi(\varepsilon, \eta) \leqq \frac{2\sqrt{\eta}}{c_*} \phi\left(\frac{c_*}{\sqrt{\eta}}\right) + \frac{2\varepsilon}{\sqrt{\eta}} \phi(0)$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sqrt{\eta}}{c_*} \exp\left(-\frac{c_*^2}{2\eta}\right) + \frac{\varepsilon}{\sqrt{\eta}}\right]. \tag{13}$$

Then it is obvious that one way to choose $\eta(\varepsilon)$ is, for example, to differentiate the function $\psi(\varepsilon, \eta)$ for η and set the derivative to zero. This will lead to

$$\eta(\varepsilon) = \frac{c_*^2 - (\varepsilon/2)^2}{2\log(c_*/\varepsilon)}. \tag{14}$$

However, for simplicity of calculation we choose

$$\eta(\varepsilon) = \frac{c_*^2}{2\log(c_*/\varepsilon)},$$

which is equivalent to (14) in terms of $\varepsilon \to 0$. Inserting $\eta(\varepsilon)$ into both sides of (13) we obtain

$$\psi(\varepsilon, \eta(\varepsilon)) \leqq \frac{2\varepsilon\sqrt{\log(c_*/\varepsilon)}}{\sqrt{\pi c_*}}\left(\frac{1}{2\log(c_*/\varepsilon)}+1\right),$$

which is easily seen to be $O(\varepsilon\sqrt{-\log\varepsilon})$ and, in turn, to be $o(\varepsilon^{1-\alpha})$, for any $\alpha > 0$.

Now we turn to consider the question of how to determine the number n, so that (7) is satisfied. First we consider the case where the partition of the interval $[0, T]$ is equally spaced. For this case we have the following result.

Theorem 4. *Suppose the function* $c(t) \in C^1([0, T])$ *and the first derivative* $c'(t)$ *satisfies the Lipschitz condition* $|c'(t) - c'(s)| \leqq L|t-s|$, $\forall s, t \in [0, T]$. *If the polygonal function* $c_n(t)$ *takes the equidistant vertices* $t_j = jT/n$, $j = 0, 1, \cdots, n$, *then* (7) *is satisfied for any*

$$n \geqq \frac{T}{2}\sqrt{\frac{L}{\varepsilon}}. \tag{15}$$

Furthermore, if $c(t) \in C^2([0, T])$, *then the Lipschitz constant may be taken as*

$$L = \|c''(\cdot)\|_\infty = \sup_{0 \leqq t \leqq T} |c''(t)|. \tag{16}$$

Proof. First it is easily shown that

$$\sup_{0 \leqq t \leqq T} |c(t) - c_n(t)| \leqq \frac{L}{8}\left(\frac{T}{n}\right)^2. \tag{17}$$

Then (15) follows by setting the right-hand side of (17) to be less than or equal to $\varepsilon/2$. The second part of the theorem is an immediate consequence of the mean-value theorem.

Remark 1. The results of Theorem 4 are based on the approximation (17). According to the theory of approximation by splines, a rate of convergence of $O(1/n^2)$ is the best possible rate for general $g \in W^2_\infty$ (De Vore and Lorentz 1993). In fact, a rate of convergence of $O(1/n^2)$ implies that g is a polynomial of degree 1, i.e. linear on $[0, T]$.

From the proof of Theorem 4 we see that, if the polygonal function $c_n(t)$ takes equidistant vertices, then an upper bound for the approximation (7) is given by (17) with L being the sup-norm $\|c''(\cdot)\|_\infty$. In the next theorem we point out that this bound may be further improved by replacing the sup-norm by the squared L_1-norm $\|\sqrt{|c''(\cdot)|}\,\|^2_{L_1}$, where

$$\|\sqrt{|c''(\cdot)|}\,\|_{L_1} = \frac{1}{T}\int_0^T \sqrt{|c''(t)|}\,dt,$$

if the polygonal function $c_n(t)$ may take the arbitrary vertices $t_0 < t_1 < \cdots < t_n$ which are not necessarily equidistant.

Theorem 5. *Suppose* $c(t) \in C^2([0, T])$ *and is nonlinear. Then the constant* L *in* (15) *is given by* $\|\sqrt{|c''(\cdot)|}\,\|^2_{L_1}$, *if the function* $c_n(t)$ *takes the* j/n *quantile of the probability density*

$$\frac{\sqrt{|c''(t)|}}{\int_0^T \sqrt{|c''(s)|}\,ds}, \qquad 0 \leqq t \leqq T \tag{18}$$

as the vertex t_j, $j=1, 2,\cdots, n$.

Proof. We give here an outline instead of a detailed proof, which is straightforward but somewhat tedious. The basic idea is that the 'optimal' vertices $t_0 < t_1 < \cdots < t_n$ should be such that the approximation errors in all intervals are equal, i.e.

$$e_j = \sup_{t_{j-1} \leqq t \leqq t_j} |c(t) - c_n(t)| = e^*$$

for all $j=1, 2,\cdots, n$. This implies that

$$|t_j - t_{j-1}| = \frac{C}{\sqrt{|c''(t_{j-1})|}} \tag{19}$$

up to an additive term which is of order $o(|t_j - t_{j-1}|)$ for all $j=1, 2,\cdots, n$, where C is a constant. Further we may show that (19) is satisfied if t_j are chosen to be the j/n quantiles, respectively, of the probability density (18).

5. Numerical computation

In this section we consider the aspect of numerical calculation of probability (1). First note that the calculation of (4) involves evaluation of a multiple integral $J(c) = \boldsymbol{E}g(W(t_1),\cdots, W(t_n); c)$, which cannot always be expressed by an explicit formula and hence must be calculated numerically. In this case there is another approximation besides (7) in calculating probability (4). This approximation may be done straightforwardly using Monte Carlo simulation. The accuracy of this approximation can then be assessed through standard procedures for Monte Carlo simulations (Ripley 1987, Niederreiter 1992).

It turns out to be more convenient in computer programming to use a transformed version of (4), instead of (4) itself. In fact, $J(c)$ may be written in the form of an expectation with respect to a multivariate normal distribution. To see this note that, through variable substitution, the integral $J(c)$ in (4) may be written as

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{(0,+\infty)} \prod_{j=1}^{n} \left(1 - \exp\left[-\frac{2x_{j-1}x_j}{t_j - t_{j-1}}\right]\right) \frac{1}{\sqrt{t_j - t_{j-1}}} \exp\left[-\frac{(x_j - c_j - x_{j-1} + c_{j-1})^2}{2(t_j - t_{j-1})}\right] dx$$

$$= \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int_{(0,+\infty)} \prod_{j=1}^{n} \left(1 - \exp\left[-\frac{2x_{j-1}x_j}{t_j - t_{j-1}}\right]\right) \exp[-\tfrac{1}{2}(x-c)'\Sigma^{-1}(x-c)]dx,$$

where $x_0 = c_0$, $\Sigma = MDM'$, $D = \text{diag}(t_1 - t_0, t_2 - t_1,\cdots, t_n - t_{n-1})$ and M is the lower triangular matrix with all non-zero elements equal to one:

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

Therefore, if

$$h(x) = \prod_{j=1}^{n} I(x_j > 0)\left(1 - \exp\left[-\frac{2x_{j-1}x_j}{t_j - t_{j-1}}\right]\right) \tag{20}$$

and the random vector $X = (X_1, \cdots, X_n)'$ has the Normal distribution $N(c, \Sigma)$, then $J(c) = \boldsymbol{E}h(X)$. This suggests the following steps to calculate the crossing probability (4).

(1) Generate an i.i.d. sample $u = (u_1, u_2, \cdots, u_n)'$ from the standard Normal distribution $N(0, 1)$.

(2) Compute the transformation $x = c + MD^{1/2}u$, where $D^{1/2} = \mathrm{diag}(\sqrt{t_1 - t_0}, \cdots, \sqrt{t_n - t_{n-1}})$.

(3) Calculate $h(x)$ by (20).

(4) Repeat steps (1)–(3) N times and then calculate the frequencies $\hat{J}(c) = \Sigma\, h(x)/N$. The probability $Q(c(t); T)$ is then estimated by $1 - \hat{J}(c)$. The standard error of this estimator is given by (Niederreiter 1992)

$$S(c) = \sqrt{\frac{\Sigma\,[h(x) - \hat{J}(c)]^2}{N(N-1)}}.$$

In the remaining part of the paper we calculate some examples for general boundaries using the approaches proposed in Section 4. For simplicity of notation we consider three boundaries $c(t)$ which are either concave or convex on the interval $[0, T]$. Thus the vertices for the function $d_n(t)$ may be calculated by

$$d_n(t_j) = c_n(t_j) + \max_j |c(\tilde{t}_j) - c_n(\tilde{t}_j)|,$$

when $c(t)$ is concave and

$$d_n(t_j) = c_n(t_j) - \max_j |c(\tilde{t}_j) - c_n(\tilde{t}_j)|,$$

when $c(t)$ is convex, where $t_{j-1} < \tilde{t}_j \leqq t_j$ satisfies $c'(\tilde{t}_j) = c_n'(\tilde{t}_j) = n(c_j - c_{j-1})$. Without loss of generality we take $T = 1$ and the equidistant partition $t_j = jT/n$, $j = 0, 1, 2, \cdots, n$. To apply Theorems 3 and 4, we take $\eta_0 = 1/2$. For the three boundaries in Table 5.1, the $c_* = 1, 0.6, 1$ respectively and the Lipschitz constants (by (16)) are $L = 0.25, 1, 2$. For various ε, the minimum number of partitions n_ε is calculated by the right-hand side of (15) and the $\psi(\varepsilon, \eta(\varepsilon))$ by (9) with $\eta(\varepsilon)$ given by (14). To use Table 5.1, for example, for the boundary $c(t) = \sqrt{1+t}$, at least $n = 25$ partitions are needed to achieve the approximation accuracy $\psi = 0.00036$.

Next we use the Monte Carlo method of this section to calculate the lower and upper bounds of the crossing probabilities for the three boundaries. In order to obtain some idea about the performance of (4), we calculate the probabilities for the partitions $n =$

TABLE 5.1
Minimum numbers of partitions n_ε and bounds of approximations $\psi(\varepsilon, \eta(\varepsilon))$ in Theorems 3 and 4, with $\eta_0 = 1/2$

		$\varepsilon=0.01$	$\varepsilon=0.001$	$\varepsilon=0.0001$	$\varepsilon=0.00001$	$\varepsilon=0.000001$
$c(t)=\exp(-t)$	n_ε	5	16	50	159	500
	$\psi(\varepsilon, \eta(\varepsilon))$	0.041855	0.005053	0.000579	0.000064	0.000007
$c(t)=t^2+1$	n_ε	8	23	71	224	708
	$\psi(\varepsilon, \eta(\varepsilon))$	0.026620	0.003167	0.000360	0.000040	0.000004
$c(t)=\sqrt{1+t}$	n_ε	3	8	25	80	250
	$\psi(\varepsilon, \eta(\varepsilon))$	0.026620	0.003167	0.000360	0.000040	0.000004

TABLE 5.2
Upper and lower bounds of the crossing probability (1) for various boundaries calculated by (4) with sample size $N=200000$. Standard errors are in parentheses

		$n=2$	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$c(t)=\exp(-t)$	UB	0.567730	0.565738	0.562922	0.562033	0.561497	0.560832
		(0.000972)	(0.001017)	(0.001045)	(0.001065)	(0.001079)	(0.001088)
	LB	0.555087	0.562113	0.561946	0.561783	0.561433	0.560816
		(0.000978)	(0.001018)	(0.001046)	(0.001065)	(0.001079)	(0.001088)
	DFF	0.012643	0.003626	0.000976	0.000251	0.000064	0.000016
$c(t)=t^2+1$	UB	0.154917	0.150689	0.148237	0.147289	0.147106	0.148678
		(0.000660)	(0.000704)	(0.000729)	(0.000748)	(0.000761)	(0.000774)
	LB	0.132847	0.144938	0.146799	0.146929	0.147016	0.148656
		(0.000623)	(0.000693)	(0.000726)	(0.000747)	(0.000761)	(0.000774)
	DFF	0.022070	0.005750	0.001438	0.000360	0.000090	0.000022
$c(t)=\sqrt{1+t}$	UB	0.196704	0.195823	0.196073	0.195001	0.197196	0.195485
		(0.000756)	(0.000801)	(0.000828)	(0.000845)	(0.000861)	(0.000866)
	LB	0.194510	0.195193	0.195901	0.194957	0.197184	0.195482
		(0.000753)	(0.000800)	(0.000828)	(0.000845)	(0.000861)	(0.000866)
	DFF	0.002194	0.000630	0.000171	0.000045	0.000012	0.000003

2, 4, 8, 16, 32, 64. In each simulation $N=200000$ samples have been drawn. Table 5.2 contains the simulated upper and lower bounds (UB and LB) and the differences (DFF) of the bounds. These examples have been calculated before by Loader and Deely (1987).

From these results we see that the precision of the approximations becomes significantly better when the partitions of the intervals become finer. We see also that the standard errors of the estimators do not change significantly across n. These variabilities may be reduced by either increasing the sample size or using some more advanced techniques such as importance sampling (see, e.g., Ripley 1987, Niederreiter 1992). However, one advantage of our approach in this paper is that the computations are much easier to implement than for previously derived methods in the literature.

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