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An Inexact Projected Gradient Method for Sparsity-Constrained Quadratic Measurements Regression*

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In this paper, we employ the sparsity-constrained least squares method to reconstruct sparse signals from the noisy measurements in high-dimensional case, and derive the existence of the optimal solution under certain conditions. We propose an inexact sparse-projected gradient method for numerical computation and discuss its convergence. Moreover, we present numerical results to demonstrate the efficiency of the proposed method.

Keywords: Quadratic measurements regression; sparsity; uniform s-regularity; uniqueness; greedy algorithm.

1. Introduction

The sparsity-constrained linear models have been intensively studied and widely applied in the literature. The main goal is to reconstruct sparse signals from the sampled measurements. Recently, the theory has been extended to nonlinear models,

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see, e.g., Beck and Eldar (2013), Blumensath (2013), Ohlsson *et al.* (2012, 2013), Shechtman *et al.* (2011, 2012) for more details. Particularly, there is a large number of real-data problems, especially in phase retrieval, localization problems, where the regression relationships are in the form of quadratic functions of unknown parameters. To deal with such problems in the high-dimensional case, Fan *et al.* (2018) introduced a unified framework for the sparsity-constrained models. In particular, they proposed a quadratic measurements regression (QMR) model as follows:

$$y_i = x^T A_i x + b_i^T x + \varepsilon_i, \quad i = 1, \ldots, m \quad \text{s.t. } \|x\|_0 \leq s, \tag{1}$$

where $y_i, b_i \in \mathbb{R}^n$ and $A_i \in \mathbb{S}^{n\times n}, i = 1, \ldots, m$ are the given real vectors and symmetrical matrices, respectively, $x \in \mathbb{R}^n$ is unknown parameter, and $\varepsilon_i \in \mathbb{R}$ is the random noise. Here $\|x\|_0$ denotes the number of nonzero elements in x and $s > 0$ is an integer smaller than m. It is obvious that when $A_i \equiv 0$, model (1) reduces to the sparse linear model,

$$y_i = x^T b_i + \varepsilon_i, \quad i = 1, \ldots, m \quad \text{s.t. } \|x\|_0 \leq s.$$

On the other hand, if $b_i \equiv 0$, model (1) becomes a purely sparse QMR (Fan *et al.* (2018))

$$y_i = x^T A_i x + \varepsilon_i, \quad i = 1, \ldots, m \quad \text{s.t. } \|x\|_0 \leq s. \tag{2}$$

We provide two real examples of model (1) as follows.

Example 1 (Phase retrieval problem). Phase retrieval problem plays an important role in X-ray crystallography, transmission electron microscopy, coherent diffractive imaging, etc. (Candès *et al.*, 2013a, 2013b; Eldar *et al.*, 2015; Shechtman *et al.*, 2014). Generally speaking, the phase retrieval problem is to recover the lost phase information through the observed magnitudes. For the real case, the sparse phase retrieval model is formulated as

$$y_i = \langle a_i, x\rangle^2 + \varepsilon_i, \quad i = 1, \ldots, m \quad \text{s.t. } \|x\|_0 \leq s.$$

For more details, see Candès *et al.* (2013a), Eldar and Mendelson (2014), Lauer and Ohlsson (2014), Li and Voroninski (2012), Ohlsson and Eldar (2014), Wang and Xu (2014).

Example 2 (Localization problem). Localization problem arises in many important applications, including mobile phone, wireless E911 calls, GPS, as well as robot localizations (Biswas and Ye, 2004; Mao *et al.*, 2007). Since a common feature in high-dimensional data analysis is the sparsity of the unknown source localization, the sparsity-constrained QMR can also be used in localization problems in high-dimensional case. Specifically, Beck *et al.* (2008a,b), Meng *et al.* (2008) and Qi *et al.* (2013) studied the single source localization problem, while Chen *et al.* (2007), Lévyleduc and Roueff (2009), Lung-Yut-Fong *et al.* (2012) and Vlassis *et al.* (2002) considered the high-dimensional localization. The known sensor position are a set of n-dimensional vectors $b_1, b_2, \ldots, b_n$ and the signal source location is unknown vector $x \in \mathbb{R}^n$. Then the measured distance y_i from the source to each sensor node is given by

$$y_i = \|b_i - x\|_2^2 + \varepsilon_i, \quad i = 1, \dots, m \quad \text{s.t. } \|x\|_0 \le s.$$

Clearly, the above relation can be written as $y_i - \|b_i\|_2^2 = x^T x - 2b_i^T x + \varepsilon_i$.

Moreover, quadratic compressive sensing, as an extension of the popular compressive sensing, aims to recover the sparse unknown signal in model (1). For example, Shechtman *et al.* (2011, 2012) used a noiseless version of this model to study sub-wavelength imaging.

To reconstruct sparse signals using (1), we employ the ℓ_0-constrained least squares method as

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} f(x) := \sum_{i=1}^m (x^T A_i x + b_i^T x - y_i)^2 \\ &\text{s.t.} \quad \|x\|_0 \le s, \end{aligned} \tag{3}$$

where $s < n$ is a positive integer. However, (3) is generally NP-hard, and hence it is challenging to design a suitable and fast algorithm to solve it. Indeed, Natarajan (1995) showed that the above minimization problem is NP-hard in linear model that is a relatively simple case of (3).

Some special cases of the optimization (3) have attracted considerable attention, such as Beck and Eldar (2013) for quadratic compressive sensing with $b_i \equiv 0, i = 1, \dots, m$, Shechtman *et al.* (2014) for phase retrieval. In this paper, we establish the existence and uniqueness of the minimizer problem (3) through the uniform s-regularity which was first introduced by Fan *et al.* (2016, 2018). As a direct result, we obtain the identifiability result of model (1) which is a general and fundamental problem as a concomitant of the scientific procedure that postulates the existence of a structure. Under the uniform s-regularity, we use majorization and variable-splitting methods to establish a fixed-point equation for the minimization (3) and consequently provide an inexact projected gradient algorithm which permits the use of the inexact gradient and then makes the algorithm more flexible. we also prove the convergence of this algorithm under appropriate conditions. Our simulation studies demonstrate that the proposed method performs well and the uniform s-regularity is an important condition for successful sparse recoveries. The results obtained generalize those in Fan *et al.* (2016). Moreover, we propose a novel inexact gradient algorithm and apply it to the application to localization problem.

Throughout the paper, we use the following notations. For any d-dimensional vector $v = (v_1, \dots, v_d)^T$, let $\|v\|_2 = (\sum_{i=1}^d v_i^2)^{\frac{1}{2}}$. For any set $\Gamma \subseteq \{1, \dots, d\}$, $|\Gamma|$ denotes its cardinality and $\Gamma^c = \{1, \dots, d\}/\Gamma$. For any $n \times d$ matrix A, denote by A_Γ the sub-matrix of A consisting of the columns of A with index in $\Gamma \subseteq \{1, \dots, d\}$, by $A^{\Gamma'}$ the sub-matrix of A consisting of the rows with index in $\Gamma' \subseteq \{1, \dots, n\}$, and by $A^{\Gamma'\Gamma}$ the sub-matrix of A consisting of the rows and columns indexed by Γ'

and Γ, respectively. Especially, we use the notation v_Γ to denote the sub-vector for either a column or a row vector v.

The rest of this paper is organized as follows. In Sec. 2, we study the uniform s-regularity and discuss the existence and uniqueness of the minimizer problem (3). By using the fixed-point equation and Armijo-type line search method, we present the algorithm and further prove its convergence in Sec. 3. In Sec. 4, we give some numerical experiments.

2. Existence and Uniqueness

In this section, we will further clarify the concept of uniform s-regularity introduced by Fan *et al.* (2018) and study the existence and uniqueness of the minimizer problem (3). Especially, the corresponding results are applied to Examples 1 and 2.

2.1. *Uniform regularity*

For a sparse linear model, Beck and Eldar (2013) introduced the s-regularity of B, i.e., any s columns of B are linearly independent and showed that the uniqueness of the underlying signal x can be characterized through it. The author stated that if the components of B are independently randomly generated from a continuous distribution, then the s-regularity property will be satisfied with probability one when $s \leq m$. That means it is rather mild for the design matrix B. Note that in linear model, $-B$ is the Jacobian matrix of the residual function $R(x) = y - Bx$, where $y = (y_1, \ldots, y_n)^T$. Correspondingly, in our model (1), the Jacobian of the residual function $R(x) = (R_1(x), \ldots, R_n(x))^T$, where $R_i(x) = y_i - x^T A_i x - b_i^T x - c_i$, is $(-2A_1x - b_1, \ldots, -2A_nx - b_n)^T$. This leads to the following definitions.

Definition 1. The linear transform $\mathbf{A}(x) = (A_1x, \ldots, A_mx)^T$ is said to be *uniformly s-regular* if $\mathbf{A}(x)_\Gamma$ has full column rank for any $\Gamma \subseteq \{1, \ldots, n\}$ with $|\Gamma| = s$ and $x \in \mathbb{R}^n/\{0\}$ with $\mathrm{supp}(x) \subseteq \Gamma$.

Definition 2. The affine transform $\mathcal{A}(x) = (A_1x + b_1, \ldots, A_mx + b_m)^T$ is said to be *uniformly s-regular* if $\mathcal{A}(x)_\Gamma$ has full column rank for any $\Gamma \subseteq \{1, \ldots, n\}$ with $|\Gamma| = s$ and $x \in \mathbb{R}^n$ with $\mathrm{supp}(x) \subseteq \Gamma$.

The following proposition reveals the relationship between the uniform s-regularity of $\mathbf{A}(\cdot)$ and $\mathcal{A}(\cdot)$.

Proposition 1. *Assume* $\sum_{i=1}^m b_i \otimes A_i = 0$, *where* $\otimes$ *stands for the Kronecker product. Then* $\mathcal{A}(\cdot) = \mathbf{A}(\cdot) + B$ *is uniformly s-regular if and only if* $\mathbf{A}(\cdot)$ *is uniformly s-regular or* B *is s-regular.*

Proof. Denote by e_j the jth column of the $n \times n$ identity matrix I_n. For each $k, l = 1, 2, \ldots, n$, it follows that

$$e_k^T \left(\sum_{i=1}^n A_i x b_i^T \right) e_l = x^T \left(\sum_{i=1}^m A_i^T e_k e_l^T b_i \right) = 0,$$

where the last equality follows from the assumption that $\sum_{i=1}^m b_i \otimes A_i = 0$. Therefore, we have

$$\sum_{i=1}^{n} A_i x b_i^T = 0. \tag{4}$$

Since

$$(\mathbf{A}(x)+B)^T(\mathbf{A}(x)+B) = \mathbf{A}(x)^T\mathbf{A}(x) + \mathbf{A}(x)^T B + B^T\mathbf{A}(x) + B^T B,$$

$$\mathbf{A}(x)^T B = \sum_{i=1}^{n} A_i x b_i^T$$

and $B^T A = (A^T B)^T$, we have

$$(\mathbf{A}(x)+B)^T(\mathbf{A}(x)+B) = \mathbf{A}(x)^T\mathbf{A}(x) + B^T B.$$

For an index set $\Gamma \subseteq \{1,\ldots,n\}$ with $|\Gamma| = s$, we then conclude that for any $u \in \mathbb{R}^s/\{0\}$,

$$(\mathbf{A}_\Gamma(u)+B_\Gamma)^T(\mathbf{A}_\Gamma(u)+B_\Gamma) = \mathbf{A}_\Gamma(u)^T\mathbf{A}_\Gamma(u) + B_\Gamma^T B_\Gamma$$

is positive definite if only if either $\mathbf{A}_\Gamma(u)^T\mathbf{A}_\Gamma(u)$ or $B_\Gamma^T B_\Gamma$ is positive definite. $\text{rank}(\mathbf{A}_\Gamma(u)+B_\Gamma)^T(\mathbf{A}_\Gamma(u)+B_\Gamma) = s$. Combining this, the definition of $\mathcal{A}(\cdot)$ and $\mathbf{A}(\cdot)$ and the fact that $\text{rank}(\mathbf{A}_\Gamma(u)+B_\Gamma) = \text{rank}((\mathbf{A}_\Gamma(u)+B_\Gamma)^T(\mathbf{A}_\Gamma(u)+B_\Gamma))$, we get the desired result. □

It is easy to see that when all A_i are zero matrices, the affine transform $\mathcal{A}(x)$ reduces to the constant transform B and therefore the uniform s-regularity coincides with s-regularity. Further, by taking $x = 0$, one can see that the uniform s-regularity of the affine transform $\mathcal{A}(x)$ implies the s-regularity of B.

In the phase retrieval problem, Balan *et al.* (2006) and Bandeira *et al.* (2014) introduced the complement property and showed that it is a necessary and sufficient condition for the measurement vectors to yield injective and stable intensity measurements. For the sparse case, Ohlsson and Eldar (2014) propose the concept of s-complement property, i.e., either $\{a_i^\Gamma\}_{i\in\mathbb{K}}$ or $\{a_i^\Gamma\}_{i\in\mathbb{K}^c}$ span $\mathbb{R}^s$ for every subset $\mathbb{K} \subseteq \{1,\ldots,m\}$ and $\Gamma \subseteq \{1,\ldots,n\}$ with $|\Gamma| = s$, which is less restrictive than the complement property and then provide theoretical result on unique recovery of a s-sparse real signal. While the next result showing the uniform regularity of $\mathbf{A}(x)$ coincides with the complement property.

Proposition 2. *Let $A_i = a_i a_i^T$ for some $\{a_i\} \in \mathbb{R}^n/\{0\}$, $i = 1,\ldots,m$. Then the uniform s-regularity of $\mathbf{A}(\cdot)$ is equivalent to the s-complement property of $\{a_i\}$.*

Proof. Suppose $\{a_i\} \in \mathbb{R}^n/\{0\}$ satisfies the s-complement property. For any $\Gamma \subseteq \{1,\ldots,n\}$ with $|\Gamma| = s$ and $x \in \mathbb{R}^n/\{0\}$ with $\text{supp}(x) \subseteq \Gamma$, it is not hard to check that

$$(\mathbf{A}(x)_\Gamma)^T = ((a_{1\Gamma}^T x_\Gamma)a_{1\Gamma}, \ldots, (a_{m\Gamma}^T x_\Gamma)a_{m\Gamma})$$

$$= (a_{1\Gamma}, \ldots, a_{m\Gamma})\mathrm{diag}(a_{1\Gamma}^T x_\Gamma, \ldots, a_{m\Gamma}^T x_\Gamma),$$

where $\mathrm{diag}(a_{1\Gamma}^T x_\Gamma, \ldots, a_{m\Gamma}^T x_\Gamma)$ is a diagonal matrix and $a_{i\Gamma}^T x_\Gamma$ is the ith main diagonal entry. Denote $\mathbb{T}(x) = \{i : a_{i\Gamma}^T x_\Gamma = 0\}$ and $t = |\mathbb{T}|$. Without loss of generality, we let $\mathbb{T}(x) = \{1, 2, \ldots, t\}$ and then obtain

$$(\mathbf{A}(x)_\Gamma)^T = (0, (a_{t+1\Gamma}, \ldots, a_{m\Gamma})D),$$

where $D = \mathrm{diag}(a_{t+1\Gamma}^T x_\Gamma, \ldots, a_{m\Gamma}^T x_\Gamma)$. Based on $x \in \mathbb{R}^n/\{0\}$ and $\mathrm{supp}(x) \subseteq \Gamma$, it follows that $x_\Gamma \neq 0$ which together with the s-complement property of $\{a_i\}$ yields that $t < n$ and $\mathrm{rank}(a_{1\Gamma}, \ldots, a_{t\Gamma}) < s$. Since D is a invertible matrix, we conclude that for any $x \in \mathbb{R}^n/\{0\}$ with $\mathrm{supp}(x) \subseteq \Gamma$,

$$\mathrm{rank}(\mathbf{A}(x)_\Gamma) = \mathrm{rank}(\mathbf{A}(x)_\Gamma^T) = \mathrm{rank}(a_{t+1\Gamma}, \ldots, a_{m\Gamma}).$$

Again, the s-complement property of $\{a_i\}$ implies that $\mathrm{rank}(a_{t+1\Gamma}, \ldots, a_{m\Gamma}) = s$ since $\mathrm{rank}(a_{1\Gamma}, \ldots, a_{t\Gamma}) < s$. Then, $\mathrm{rank}(\mathbf{A}(x)_\Gamma) = s$ which means $\mathbf{A}(\cdot)$ is uniformly s-regular.

Given that $\mathbf{A}(\cdot)$ is uniformly s-regular, we now show that $\{a_i\}$ satisfy the s-complement property. To this end, it suffices to prove that either $\{a_i^\Gamma\}_{i\in\mathbb{K}}$ or $\{a_i^\Gamma\}_{i\in\mathbb{K}^c}$ span $\mathbb{R}^s$ for every subset $\mathbb{K} \subseteq \{1, \ldots, m\}$. Without loss of generality, we assume that $\mathbb{K} = \{1, \ldots, k\}$. We prove the result by contradiction. Suppose that both $a_{1\Gamma}, \ldots, a_{k\Gamma}$ and $a_{k+1\Gamma}, \ldots, a_{m\Gamma}$ cannot span $\mathbb{R}^s$ and denote $\tilde{A}_1 = (a_{1\Gamma}, \ldots, a_{k\Gamma})^T$ and $\tilde{A}_2 = (a_{k+1\Gamma}, \ldots, a_{m\Gamma})^T$. Then there exists $u_0 \in \mathbb{R}^s/\{0\}$ such that $\tilde{A}_1 u_0 = 0$ and it follows that

$$((a_{1\Gamma}^T u_0)a_{1\Gamma}, \ldots, (a_{m\Gamma}^T u_0)a_{m\Gamma})^T = (0, \tilde{A}_2^T \mathrm{diag}(a_{k+1\Gamma}^T u_0, \ldots, a_{m\Gamma}^T u_0))^T$$

and therefore

$$\mathrm{rank}((a_{1\Gamma}^T u_0)a_{1\Gamma}, \ldots, (a_{m\Gamma}^T u_0)a_{m\Gamma})^T = \mathrm{rank}(0, \tilde{A}_2^T \mathrm{diag}(a_{k+1\Gamma}^T u_0, \ldots, a_{m\Gamma}^T u_0))^T.$$

Since $a_{k+1\Gamma}, \ldots, a_{m\Gamma}$ cannot span $\mathbb{R}^s$, it follows that $\mathrm{rank}(\tilde{A}_2) < s$, which implies that

$$\begin{aligned}\mathrm{rank}(0, \tilde{A}_2^T \mathrm{diag}(a_{k+1\Gamma}^T u_0, \ldots, a_{m\Gamma}^T u_0)) &= \mathrm{rank}(\tilde{A}_2^T \mathrm{diag}(a_{k+1\Gamma}^T u_0, \ldots, a_{m\Gamma}^T u_0)) \\ &\leq \mathrm{rank}(\tilde{A}_2) \\ &< s.\end{aligned}$$

However, the uniform s-regularity of $\mathbf{A}(\cdot)$ implies that

$$\mathrm{rank}(\mathbf{A}_\Gamma(u)) = \mathrm{rank}((a_{1\Gamma}^T u_0)a_{1\Gamma}, \ldots, (a_{m\Gamma}^T u_0)a_{m\Gamma})^T$$

has full column rank, which is a contradiction. □

Note that in Example 2, $A_i = I, i = 1, \ldots, m$. Thus, from Proposition 1, we have the following proposition.

Proposition 3. *For Example* 2, $\mathcal{A}(\cdot) = \mathbf{A}(\cdot) + B$ *is uniformly* s*-regular if* $\sum_{i=1}^{m} b_i = 0$ *and* B *is* s*-regular.*

2.2. *Existence and uniqueness*

Here, we employ the concept of uniform regularity to discuss the existence and the uniqueness of problem (3). To do that, we first provide a lemma as follows.

Lemma 1. *Let* $\Gamma \subseteq \{1, 2, \ldots, n\}$ *be any index set with* $|\Gamma| = s$. *If* $\mathcal{A}(\cdot)$ *is uniformly* s*-regular, then*

$$\lim_{\|u\| \to \infty} \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u + u^T b_{i\Gamma} - y_i)^2 = \infty, \quad u \in \mathbb{R}^s.$$

Proof. It suffices to prove that

$$\lim_{\|u\| \to \infty} \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u + u^T b_{i\Gamma})^2 = \infty \quad \text{for any } x \in \mathbb{R}^n \quad \text{and} \quad \text{supp}(x) \subseteq \Gamma. \tag{5}$$

For the case $\mathcal{A}(\cdot) = \mathbf{A}(\cdot)$, i.e., $b_i \equiv 0, i = 1, 2, \ldots, m$, denote $f_{1\Gamma}(u) = \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u)^2$. Note that there exists a vector $v^* \in \mathbb{R}^s$ satisfying $\|v^*\| = 1$ and

$$f_{1\Gamma}(v^*) = \min f_{1\Gamma}(v) \quad \text{s.t. } \|v\| = 1$$

because $f_{1\Gamma}(\cdot)$ is continuous and the set $\|v\| = 1$ is compact. Then the uniform s-regularity of $\mathbf{A}(\cdot)$ implies that the matrix $\sum_{i=1}^{m} A_i^{\Gamma\Gamma} v^* v^{*T} A_i^{\Gamma\Gamma T}$ is positive definite, and therefore $f_{1\Gamma}(v^*) > 0$. For any nonzero vector $u \in \mathbb{R}^n$ with $\text{supp}(x) \subseteq \Gamma$, it follows that

$$f_{1\Gamma}(u) = \|u\|^4 \sum_{i=1}^{m} \left(\left(\frac{u}{\|u\|} \right)^T A_i^{\Gamma\Gamma} \left(\frac{u}{\|u\|} \right) \right)^2 \geq \|u\|^4 f_{1\Gamma}(v^*)$$

which implies that $f_{1\Gamma}(u) \to \infty$ as $\|u\| \to \infty$. Therefore, (5) holds for the case $\mathcal{A}(\cdot) = \mathbf{A}(\cdot)$.

For the general case, denote $f_{2\Gamma}(u) = \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u + u^T b_{i\Gamma})^2$. Then we have

$$\begin{aligned} f_{2\Gamma}(u) &= \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u + u^T b_{i\Gamma})^2 \\ &= \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u)^2 + 2 \sum_{i=1}^{m} (u^T A_i^{\Gamma\Gamma} u)(u^T b_{i\Gamma}) + \sum_{i=1}^{m} (u^T b_{i\Gamma})^2 \end{aligned}$$

for any vector $u \in \mathbb{R}^s$. For the case $\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2$ is bounded for any $u \in \mathbb{R}^s$, we conclude from Cauchy's inequality that

$$f_{2\Gamma}(u) \geq \sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2 - 2\sqrt{\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2}\sqrt{\sum_{i=1}^m (u^T b_{i\Gamma})^2} + \sum_{i=1}^m (u^T b_{i\Gamma})^2.$$

Note that the uniform s-regularity of $\mathcal{A}(\cdot)$ implies that $\sum_{i=1}^m (u^T b_{i\Gamma})^2 \to \infty$ as $\|u\| \to \infty$. Then we have

$$\begin{aligned} f_{2\Gamma}(u) \ &\geq \left(\frac{\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2}{\sum_{i=1}^m (u^T b_{i\Gamma})^2} - 2\sqrt{\frac{\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2}{\sum_{i=1}^m (u^T b_{i\Gamma})^2}} + 1 \right) \sum_{i=1}^m (u^T b_{i\Gamma})^2 \\ &\to \infty. \end{aligned}$$

If $\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2$ is unbounded as $\|u\| \to \infty$, then

$$f_{1\Gamma}(u) = \|u\|^4 \sum_{i=1}^m \left(\left(\frac{u}{\|u\|} \right)^T A_i^{\Gamma\Gamma} \left(\frac{u}{\|u\|} \right) \right)^2 = \|u\|^4 f_{1\Gamma}(v)$$

and $f_{1\Gamma}(v) > 0$ where $v \in \mathbb{R}^s$ and $\|v\| = 1$. By Cauchy's inequality, we have

$$\begin{aligned} f_{2\Gamma}(u) &\geq \sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2 - 2\sum_{i=1}^m |u^T A_i^{\Gamma\Gamma} u||u^T b_{i\Gamma}| + \sum_{i=1}^m (u^T b_{i\Gamma})^2 \\ &\geq \sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2 - 2\|u\| \max_{1 \leq i \leq m} \|b_{i\Gamma}\| \sum_{i=1}^m |u^T A_i^{\Gamma\Gamma} u| + \sum_{i=1}^m (u^T b_{i\Gamma})^2 \\ &\geq \sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2 - 2\|u\| \max_{1 \leq i \leq m} \|b_{i\Gamma}\| \sqrt{m} \sqrt{\sum_{i=1}^m (u^T A_i^{\Gamma\Gamma} u)^2} + \sum_{i=1}^m (u^T b_{i\Gamma})^2 \\ &\geq \|u\|^4 \left(f_{1\Gamma}(v) - 2\|u\|^{-1} \max_{1 \leq i \leq m} \|b_{i\Gamma}\| \sqrt{m} \sqrt{f_{1\Gamma}(v)} \right) \\ &\to \infty \end{aligned}$$

as $\|u\| \to \infty$. Then we complete the proof of (5). □

Theorem 1. *If $\mathcal{A}(\cdot)$ is uniformly s-regular, then there exists a vector $\hat{x} \in \mathbb{R}^n$ that is a minimizer of problem* (3)*. Furthermore, if $\mathcal{A}(\cdot)$ is uniformly $2s$-regular and the optimal value of* (3) *is zero, then the problem* (3) *has a unique solution $\hat{x}$, i.e., $\hat{x}$ is a unique vector satisfying* (1) *in the noiseless case.*

Proof. (i) Based on Lemma 1, it is easy to show that $f(x) \to \infty$ as $\|x\| \to \infty$ and $\|x\|_0 \leq s$. It follows that there exists a positive constant $\hat{r}$ such that the problem

(3) is equivalent to

$$\min_{x\in\mathbb{R}^n} f(x) \quad \text{s.t. } \|x\|_0 \le s, \quad \|x\| \le \hat{r}.$$

Since f is continuous and the constrained set is compact, it follows that the problem (6) has a minimizer $\hat{x}$ which is also a solution of (3).

(ii) We now prove the second result for the purely sparse quadratic measurements model (2) by contradiction. Assume that $\tilde{x} \neq \pm x^*, \|\tilde{x}\|_0 \le \|x^*\|_0, \mathcal{M}(\tilde{x}) = \mathcal{M}(\pm x^*), \tilde{x} \in \mathbb{R}^n$, where the operator $\mathcal{M}$ is defined as $\big(\mathcal{M}(x)\big)(i) = x^T A_i x$. Denote $\Gamma = \mathrm{supp}(\tilde{x}) \cup \mathrm{supp}(x^*)$. Then, $|\Gamma| \le 2s$. For any $x \in \mathbb{R}^n$, it follows that $(\mathcal{M}(\tilde{x}))(i) = (x^\Gamma)^T A_i^{\Gamma\Gamma} x^\Gamma$ and therefore

$$\begin{aligned} 0 &= \mathcal{M}(\tilde{x}) - \mathcal{M}(x^*) \\ &= ((\tilde{x}^\Gamma - x^{*\Gamma})^T A_1^{\Gamma\Gamma}(\tilde{x}^\Gamma + x^{*\Gamma}), \ldots, (\tilde{x}^\Gamma - x^{*\Gamma})^T A_m^{\Gamma\Gamma}(\tilde{x}^\Gamma + x^{*\Gamma}))^T \\ &= A_\Gamma(\tilde{x}^\Gamma + x^{*\Gamma})(\tilde{x}^\Gamma - x^{*\Gamma}). \end{aligned}$$

Since $0 \neq \tilde{x}^\Gamma + x^{*\Gamma}$ and $\mathbf{A}(\cdot)$ is uniformly $2s$-regular, it follows that $\mathbf{A}_\Gamma(\tilde{x}^\Gamma + x^{*\Gamma})$ has full column rank, which implies $\tilde{x}^\Gamma = x^{*\Gamma}$. This is a contradiction.

Similar to the above proof, we can get the desired result for model (1). So, we omit the details. □

Clearly, Theorem 1 can be applied to the purely sparse quadratic model (2) since the affine transform $\mathcal{A}(\cdot)$ reduces to $\mathbf{A}(\cdot)$ when $B = 0$. Particularly, note that x and $-x$ are not distinguishable from the observed data in model (2). To deal with it, when referring to a unique solution for model (2), it is always understood that it is up to a global sign change. This method is widely used in the phase retrieval literature such as Balan *et al.* (2006), Bandeira *et al.* (2014) and Ohlsson and Eldar (2014).

We now use Theorem 1 to study the identifiability problem for the model (1). Roughly speaking, we say that a model is identifiable if all parameters have a unique solution within the full domain of the parameter space, given specific observation points within the model. Especially, in signal process, identifiability analysis is a critical step which addresses whether it is possible to uniquely recover the model parameter from a given set of data.

Corollary 1. *Let x^* satisfy the model* (1) *with noiseless. If $\mathcal{A}(\cdot)$ is uniformly $2s$-regular, then x^* is a unique vector satisfying the system.*

Note $A_i = a_i a_i^T$ in Example 1. One can use the equivalence between the uniform s-regularity of $\mathbf{A}(\cdot)$ and the s-complement property of $\{a_i\}$ to reduce Corollary 1 to Theorem 4 in Ohlsson and Eldar (2014). Theorem 1 and Corollary 1 also can be used to study Example 2.

Corollary 2. *Assume $\sum_{i=1}^m b_i = 0$ in Example* 2*. If $B = (b_1, \ldots, b_m)^T$ is s-regular, then there exists a vector $\hat{x} \in \mathbb{R}^n$ that is a minimizer of problem* (3)*. Further, if B is*

2s-regular and the optimal value of (3) *is zero, then the minimizer of* (3) *is unique. Especially, let x^* be the true signal source location satisfying* (3) *with noiseless. If $\sum_{i=1}^m b_i = 0$ and B is 2s-regular, then x^* is the unique vector satisfying the system.*

3. Optimization Algorithm

In this section, we discuss the numerical computation of problem (3). To this end, we define $S = \{x \in \mathbb{R}^n : \|x\|_0 \le s\}$ for a positive integer s and $P_S(x)$ to be the nonlinear operator that sets all but the largest (in magnitude) s elements of x to zero. If there is no unique such set, a set can be selected either randomly or based on a predefined ordering of the elements.

We first establish a fixed point equation for the optimization problem (3), which is used to construct a projected gradient algorithm.

Theorem 2. *If $\mathcal{A}(\cdot)$ is uniformly s-regular, then there exists a positive constant $\hat{L}$ related to $\hat{x}$, a minimizer of the optimization* (3), *such that*

$$\hat{x} \in P_S(\hat{x} - \tau \nabla f(\hat{x})) \tag{6}$$

for any $\tau \in (0, \min\{\hat{L}^{-1}, 1\}]$ where $\nabla f(x) = 2\sum_{i=1}^m (x^T A_i x + b_i^T x - y_i)(2A_i x + bi)$.

Proof. From Theorem 1, one can see that the optimization (3) has a minimizer $\hat{x}$. Let $\hat{r}$ be a positive constant with $\|\hat{x}\| \le \hat{r}$. For any $\tau > 0$, define $F_\tau(x, \hat{x}) := f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x}\rangle + \frac{1}{2\tau}\|x - \hat{x}\|_2^2$ and consider the following auxiliary problem:

$$\begin{aligned} \min \quad & F_\tau(x, \hat{x}) \\ \text{s.t.} \quad & \|x\|_0 \le s, \\ & x \in \mathbb{R}^n. \end{aligned} \tag{7}$$

Denote $B_{\hat{r},s} = \{x \in \mathbb{R}^n : \|x\|_2 \le \hat{r}, \|x\|_0 \le s\}$ and $B_{\hat{r},2s} = \{x \in \mathbb{R}^n : \|x\|_2 \le \hat{r}, \|x\|_0 \le 2s\}$. It is clear that there exists a positive constant $\hat{L}$ such that $\hat{L} = \sup_{x \in B_{\hat{r},2s}} \|\nabla^2 f(x)\|_2$. Note that for any $x, y \in B_{\hat{r},s}$, the line segment $[x, y] \in B_{\hat{r},2s}$. Therefore, for any $\tau \in (0, \hat{L}^{-1}]$ and $x \in B_{\hat{r},s}$, we have

$$\begin{aligned} f(x) &= f(\hat{x}) + \langle \nabla f(\hat{x}), x - \hat{x}\rangle + \frac{1}{2}(x - \hat{x})^T \nabla^2 f(\xi)(x - \hat{x}) \\ &= F_\tau(x, \hat{x}) + \frac{1}{2}(x - \hat{x})^T \nabla^2 f(\xi)(x - \hat{x}) - \frac{1}{2\tau}\|x - \hat{x}\|_2^2 \\ &\le F_\tau(x, \hat{x}) + \frac{1}{2}\|\nabla^2 f(\xi)\|_2 \|x - \hat{x}\|_2^2 - \frac{1}{2\tau}\|x - \hat{x}\|_2^2 \\ &\le F_\tau(x, \hat{x}) + \frac{\hat{L}}{2}\|x - \hat{x}\|_2^2 - \frac{1}{2\tau}\|x - \hat{x}\|_2^2 \\ &\le F_\tau(x, \hat{x}), \end{aligned} \tag{8}$$

where $\xi = \hat{x} + \alpha(x - \hat{x})$ for some $\alpha \in (0, 1)$ and the second inequality follows from the fact that $\xi \in B_{\hat{r},2s}$ and hence $\|\nabla^2 f(\xi)\|_2 \le \hat{L}$.

Further, let

$$\bar{x} \in \arg\min_{x\in\mathbb{R}^n} F_\tau(x,\hat{x}) \quad \text{s.t. } \|x\|_0 \le s,$$

where $\tau \in (0, \hat{L}^{-1}]$. Since $f(\hat{x}) = F_\tau(\hat{x},\hat{x})$, one can conclude from the inequality (8) that for any $\tau \in (0, \hat{L}^{-1}]$,

$$F_\tau(\bar{x},\hat{x}) \le F_\tau(\hat{x},\hat{x}) = f(\hat{x}) \le f(\bar{x}) \le F_\tau(\bar{x},\hat{x}),$$

which leads to $F_\tau(\hat{x},\hat{x}) = F_\tau(\bar{x},\hat{x})$. Therefore, $\hat{x}$ is also a minimizer of the problem (7).

On the other hand, it is easy to check that the problem (7) is equivalent to the following minimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x - (\hat{x} - \tau\nabla f(\hat{x}))\|_2^2 \quad \text{s.t. } \|x\|_0 \le s$$

which together with the definition of projection operator $P_S(\cdot)$ leads to the desired result. □

Beck and Eldar (2013) studied the sparsity-constrained optimization problem

$$\min_{x\in\mathbb{R}^n} h(x) \quad \text{s.t. } \|x\|_0 \le s, \tag{9}$$

where $h(\cdot)$ is a continuously differentiable function, and introduced the so-called τ-stationary point x that satisfies

$$x \in P_S(x - \tau\nabla h(x)), \quad \tau > 0.$$

Furthermore, it is shown in Beck and Eldar (2013) that under the assumption that the gradient of the objective function ∇h is Lipschitz with constant L_h, the minimizer of problem (9) must be an τ-stationary point when $\tau < 1/L_h$. The author pointed out that the famous Iterative Hard Thresholding (IHT) algorithm (Blumensath and Davies, 2009) can be viewed as a fixed-point method for solving the condition for τ-stationarity. For the nonnegativity and sparsity-constrained optimization problem, Pan *et al.* (2017) used the classical Armijo rule in IHT and then presented an improved IHT algorithm where it was permitted that the objective function $h(\cdot)$ does not have Lipschitz gradient but must be $2s$-restricted strongly smooth with parameter $L_{2s} > 0$. It is worth mentioning that the constants L_h and L_{2s} play an important role in their algorithms because the step size depends on them. Unfortunately, our function $f(\cdot)$ does not satisfy such conditions. Recall Theorem 2 which states that the minimizer of problem (3) is still an L-stationary point with the help of the uniform regularity. Inspired by the fixed point iterative algorithm in Fan *et al.* (2018), we use the inexact gradient information and the projection operator $P_S(\cdot)$ to propose a greedy algorithm, called inexact sparse projected gradient (ISPG) algorithm as follows.

Algorithm (ISPG):

Step 0. Given $\lambda > 0, \epsilon \geq 0, \gamma, \alpha \in (0,1), \beta \in [0,1), \delta > 2\beta$, choose an arbitrary x^0 and set $x^{-1} = x^0$ and $k = 0$.

Step 1. (a) Compute $\nabla f(x^k)$ and $z^k = \nabla f(x^k) + e^k$, where $e^k = \beta_k(x^k - x^{k-1})$ and $\beta_k \in [0, \beta]$.

(b) Compute $x^{k+1} = P_S(x^k - \tau_k z^k)$, where $\tau_k = \gamma\alpha^{j_k}$ and j_k is the smallest nonnegative integer such that

$$f(x^k) - f(x^{k+1}) \geq \frac{\delta}{2}\|x^k - x^{k+1}\|_2^2 + \langle x^{k+1} - x^k, e^k\rangle. \tag{10}$$

Step 2. Stop if $\|x^{k+1} - x^k\|_2 \leq \epsilon \max\{1, \|x^k\|_2\}$. Otherwise, replace k by $k+1$ and go to Step 1.

It is clear that ISPG reduces to the algorithm in Fan *et al.* (2016) if one take $\beta_k = 0$. For the choice of step size, the Armijo-type line search (10) is adopted to achieve an adequate reduction. A key point is to find the smallest nonnegative integer j_k such that (10) holds, which can be done successfully by Lemmas 2 and 3.

Lemma 2. *Let* $g_k = \|\nabla f(x^k)\|_2$ *and* $G_k = \sup_{\beta \in B_k} \|\nabla^2 f(x)\|_2$, *where* $B_k = \{x \in \mathbb{R}^n : \|x\|_2 \leq \|x^k\|_2 + \|x^{k-1}\| + g_k\}$. *For any* $\delta > 0, \gamma, \alpha \in (0,1)$, *define*

$$j_k = \begin{cases} 0, & \text{if } \gamma(G_k + \delta) \leq 1; \\ -[\log_\alpha \gamma(G_k + \delta)] + 1, & \text{otherwise.} \end{cases}$$

Then (10) *holds.*

Proof. Based on the update formula of x^{k+1}, it is clear that x^k, g_k and G_k are finite for each k. From the definition of τ_k and j_k, it is easy to verify that

$$G_k - \frac{1}{\tau_k} \leq -\delta. \tag{11}$$

Indeed, by taking $\tau_k = \gamma$, we have

$$G_k - \frac{1}{\tau_k} = \frac{\gamma G_k - 1}{\gamma} \leq -\delta,$$

when $\gamma(G_k + \delta) \leq 1$. If $\gamma(G_k + \delta) > 1$, then

$$\tau_k = \gamma\alpha^{j_k} \leq \gamma\alpha^{\log_\alpha \gamma(G_k+\delta)} = \frac{1}{G_k + \delta}$$

which also leads to (11).

Note that

$$\begin{aligned}
\|x^{k+1}\|_2 &\le \|x^k - \tau_k(\nabla f(x^k) + e^k)\|_2 \\
&= \|(1-\tau_k\beta_k)x^k - \tau_k \nabla f(x^k) - \tau_k\beta_k x^{k-1}\|_2 \\
&\le |1-\tau_k\beta_k|\|x^k\|_2 + \tau_k\|\nabla f(x^k)\|_2 + \tau_k\beta_k\|x^{k-1}\|_2 \\
&\le \|x^k\|_2 + g_k + \|x^{k-1}\|_2
\end{aligned}$$

which implies that $x^{k+1} \in B_k$. Recall the definition of $F_\tau(\cdot,\cdot)$ in the proof of Theorem 2. Similar to (8), we conclude from (11) that

$$\begin{aligned}
f(x^{k+1}) &\le F_{\tau_k}(x^{k+1}, x^k) + \frac{1}{2}\|x^{k+1} - x^k\|_2^2 \left(\|\nabla^2 f(\xi_k)\|_2 - \frac{1}{\tau_k}\right) \\
&\le F_{\tau_k}(x^{k+1}, x^k) + \frac{1}{2}\|x^{k+1} - x^k\|_2^2 \left(G_k - \frac{1}{\tau_k}\right) \\
&\le F_{\tau_k}(x^{k+1}, x^k) - \frac{\delta}{2}\|x^{k+1} - x^k\|_2^2, \qquad (12)
\end{aligned}$$

where $\xi_k = x^k + \varrho(\tilde{x}^{k,t} - x^k)$ for some $\varrho \in (0,1)$ and hence the second inequality follows from $\xi_k \in B_k$.

By simple calculation, we have

$$\begin{aligned}
&\min_{x\in\mathbb{R}^n, \|x\|_0 \le s} \frac{1}{2}\left\|x - \left(x^k - \tau_k(\nabla f(x^k) + e^k)\right)\right\|^2 \\
&\quad = \min_{x\in\mathbb{R}^n, \|x\|_0 \le s} \frac{1}{2}\|x - x^k\|^2 + \tau_k\langle x - x^k, \nabla f(x^k)\rangle + \tau_k\langle x - x^k, e^k\rangle \\
&\quad = \min_{x\in\mathbb{R}^n,\ \|x\|_0 \le s} f(x^k) + \langle x - x^k, \nabla f(x^k)\rangle + \frac{1}{2\tau_k}\|x - x^k\|^2 + \langle x - x^k, e^k\rangle \\
&\quad = \min_{x\in\mathbb{R}^n,\ \|x\|_0 \le s} F_{\tau_k}(x, x^k) + \langle x - x^k, e^k\rangle
\end{aligned}$$

which together with the update formula of x^{k+1} implies that

$$x^{k+1} \in \arg\min_{x\in\mathbb{R}^n, \|x\|_0 \le s} F_{\tau_k}(x, x^k) + \langle x - x^k, e^k\rangle.$$

Combining this and (12), we have

$$\begin{aligned}
f(x^k) - f(x^{k+1}) &= F_{\tau_k}(x^k, x^k) + \langle x^k - x^k, e^k\rangle - f(x^{k+1}) \\
&\ge F_{\tau_k}(x^{k+1}, x^k) + \langle x^{k+1} - x^k, e^k\rangle - f(x^{k+1}) \\
&\ge \frac{\delta}{2}\|x^{k+1} - x^k\|_2^2 + \langle x^{k+1} - x^k, e^k\rangle,
\end{aligned}$$

which completes the proof. □

Lemma 3. *Let* $\{x^k\}$ *and* $\{\tau_k\}$ *be generated by ISPG. Assume that* $\mathcal{A}$ *is uniformly* s*-regular. Then,*

(i) $\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2^2 \leq 2(\delta - 2\beta)^{-1} f(x^0)$;
(ii) $\{x^k\}$ *is bounded*;
(iii) *there is a nonnegative integer* $\bar{j}$ *such that* $\tau_k \in [\gamma\alpha^{\bar{j}}, \gamma]$.

Proof. By (10), we obtain that

$$f(x^0) - f(x^{K+1}) = \sum_{k=0}^{K} (f(x^k) - f(x^{k+1}))$$

$$\geq \frac{\delta}{2} \sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2 + \sum_{k=0}^{K} \langle x^{k+1} - x^k, e^k \rangle$$

which together with Cauchy's inequality yields that

$$f(x^0) - f(x^{K+1}) \geq \frac{\delta}{2} \sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2 - \sqrt{\sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2} \sqrt{\sum_{k=0}^{K} \|e^k\|^2}.$$

Combining this and

$$\sqrt{\sum_{k=0}^{K} \|e^k\|^2} = \beta \sqrt{\sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2 - \|x^{K+1} - x^K\|_2^2}$$

$$\leq \beta \sqrt{\sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2},$$

we have

$$f(x^0) - f(x^{K+1}) \geq \left(\frac{\delta}{2} - \beta\right) \sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2$$

and then

$$\sum_{k=0}^{K} \|x^{k+1} - x^k\|_2^2 \leq \frac{2}{\delta - 2\beta} (f(x^0) - f(x^{K+1})) \tag{13}$$

since $\delta > 2\beta \geq 0$. By $f(\cdot) \geq 0$, we get result (i) immediately.

We prove result (ii) by contradiction. Suppose $\{x^k\}$ is unbounded, which implies that there exists a subsequence $\{x^{k_j}\}$ tending to infinity as $j \to \infty$. By the uniform s-regularity of $\mathcal{A}(\cdot)$ and Lemma 1, we have $f(x^{k_j}) \to \infty$ as $j \to \infty$. On the other hand, we conclude from (13) and result (i) that $\limsup_{k\to\infty}\{f(x^k)\} < \infty$, which is a contradiction.

To show (iii), we note that since $f(\cdot)$ is a twice continuous differentiable function, it follows from the boundedness of $\{x^k\}$ that there exist two positive constants $\bar{g}$ and

$\bar{G}$ such that $\sup_{k\geq 0}\{g_k\} \leq \bar{g}$ and $\sup_{k\geq 0}\{G_k\} \leq \bar{G}$. Define $\bar{j} = \max(0, [-\log_\alpha \gamma(\bar{G} + \delta)] + 1)$. Therefore, $0 \leq j_k \leq \bar{j}$ and it follows from the definition of τ_k that $\tau_k \in [\gamma\alpha^{\bar{j}}, \gamma]$. □

Remark 1. Another key point is the choice of the sparsity parameter s which may not be known *a priori* in some applications. A popular and efficient method for choosing the penalty parameter in ℓ_1-regularized minimization is cross-validation which can be applied to problems such as compressed sensing. In the next section, we will calculate some numerical examples using five-fold cross-validation to determine the sparsity parameter s. The numerical results demonstrate the efficiency of this method. Moreover, one can use the similar method in Beck and Eldar (2013) to get the convergence of the proposed algorithm.

Now, we consider the convergence of ISPG.

Theorem 3. *Let $\{x^k\}$ be the sequence generated by ISPG. Assume that $\mathcal{A}$ is uniformly s-regular. Then,*

(i) $\lim_{k\to\infty} \|x^{k+1} - x^k\|_2 = 0$;
(ii) *any accumulation point of $\{x^k\}$ is a stationary point of the minimization problem* (3).

Proof. (i) By result (i) of Lemma 3, we get result (i) immediately.

(ii) Since $\{x^k\}$ is bounded, it has at least one accumulation point. Let $\tilde{x}$ be an accumulation point and suppose that the subsequence $\{x^{k_j}\}$ tends to $\tilde{x}$. We now prove it satisfies (6) for some $\tau > 0$. By result (i), the notation of e^k and $\beta_k \in [0, \beta]$, we have

$$\|e^k\| \to 0 \quad \text{as } k \to \infty. \tag{14}$$

Denote $\tilde{\Gamma} = \text{supp}(\tilde{x})$ and let x_i be the ith element of a vector x.

For each $i \in \tilde{\Gamma}$, there exists a positive integer j_0 such that $x_i^{k_j} \neq 0$ for $j \geq j_0$. Based on the property of the projection $P_S(\cdot)$ and the algorithm, it follows that

$$x_i^{k_j+1} = x_i^{k_j} - \tau_{k_j} \nabla f(x^{k_j})_i + e_i^{k_j}.$$

Since $x_i^{k_j} \to \tilde{x}_i$ and ∇f is continuous, the first result, the limit (14) and the second of Lemma 3 imply that

$$\nabla f(\tilde{x})_i = 0 \quad \text{for each } i \in \tilde{\Gamma}. \tag{15}$$

We now consider the case that $i \notin \tilde{\Gamma}$. If there exists an infinite number of indices k_{j_l} such that $x_i^{k_{j_l}+1} \neq 0$. As before, it follows that

$$x_i^{k_{j_l}+1} = x_i^{k_{j_l}} - \tau_{k_{j_l}} \nabla f(x^{k_{j_l}})_i + e^{k_{j_l}}$$

which yields that $\nabla f(\tilde{x})_i = 0$ and then

$$|\nabla f(\tilde{x})_i| \leq \tau M_s(x^*) \quad \text{for any } \tau > 0. \tag{16}$$

If there exists a positive integer j_0' such that $x_i^{k_j+1} = 0$ for $j \geq j_0'$. Based on the projection operator $P_S(\cdot)$, it follows that

$$|x_i^{k_j} - \tau_{k_j}(\nabla f(x^{k_j})_i + e_i^{k_j})| \leq M_s(x^{k_j} - \tau_{k_j}(\nabla f(x^{k_j}) + e^{k_j})$$

which together with the first result, the limit (14) and the second of Lemma 3 yields that

$$\begin{aligned}
\limsup_{j\to\infty} \tau_{k_j} |(\nabla f(\tilde{x}))_i| &= \limsup_{j\to\infty} |x_i^{k_j} - \tau_{k_j}(\nabla f(x^{k_j})_i + e_i^{k_j})| \\
&\leq \limsup_{j\to\infty} M_s(x^{k_j} - \tau_{k_j}(\nabla f(x^{k_j}) + e^{k_j})) \\
&= \limsup_{j\to\infty} M_s(x^{k_j+1}) \\
&= M_s(\tilde{x}).
\end{aligned}$$

Combining this, (15) and (16), we conclude from Lemma 2.2 in Beck and Eldar (2013) that

$$\tilde{x} \in P_S\left(\tilde{x} - \limsup_{j\to\infty} \tau_{k_j} \nabla f(\tilde{x})\right).$$

We then complete the proof. □

4. Numerical Examples

In this section, we demonstrate the efficiency of our proposed algorithm by calculating Examples 1 and 2. Both in these examples, let the true value x^* be generated randomly with s nonzero components from the standard Gaussian distribution and the noise $\varepsilon_i \sim N(0, \sigma^2)$. To recover x^*, we use the ℓ_0-constrained least squares method (3). To evaluate the performance of our method, we carry out 100 Monte Carlo runs in each simulation and report the successful recovery (SR) rate using the criterion $\|\hat{x} - x^*\|_2 \leq 0.01$.

Example 1 (Continued). Consider the sparse phase retrieval problem where the sparsity s is assumed to be unknown. For the noise, let its standard deviation $\sigma = 0.01$. As mentioned in Remark 1, we use cross-validation method to choose the sparsity s and report the mean and standard error (SE) of $\|\hat{x}\|_0$ to demonstrate its efficiency. The vectors $\{a_i\} \in \mathbb{R}^n$ are generated from the standard Gaussian distribution. Similar to Beck and Eldar (2013), we consider the cases $n = 120$ and $m = 80$ with $s = 3, 4, \ldots, 10$ respectively. In view of Ohlsson and Eldar (2014), $\{a_i\}$ satisfies the $2s$ complement property with probability 1.

Table 1. The average results of 100 simulations with $n = 120$ and $m = 80$.

$\|x\|_0$	3	4	5	6	7	8	9	10
$\|\hat{x}\|_0$	3.7	4.9	5.8	6.9	8.0	9.1	10.1	10.7
SE	0.078	0.127	0.099	0.115	0.113	0.130	0.111	0.087
SR	0.93	0.97	0.93	0.97	0.98	0.96	0.95	0.90

The numerical results are given in Table 1. These results show that the averages of $\|\hat{x}\|_0$ are fairly close to the corresponding true values $\|x^*\|_0$ and the SE are very small overall. They also confirm that the cross-validation method works well in choosing the right sparsity parameter. The rates of SR are over 90% which show that the ℓ_0-constrained least squares method and corresponding algorithm perform well for the lower dimensional case.

To assess the efficiency of our method in the situation of high-dimensional signal recovery with low sample size, we also run the simulations with $m = 3n/4$, $s = 0.05n$ and $n = 100$, 200, $300, 400$, 500, respectively. The numerical results are given in Table 2 which demonstrate further that cross-validation is an appropriate method for the choice of sparsity used in the projection operator. While the last column in Table 2 show that our method can recover the unknown signal with higher success rates even in relatively high-dimensional cases.

Example 2 (Continued). Consider the localization problem (3) for the cases $n = 2000, 3000, m = 800, 1200$ and $s = 2, 3, \ldots, 20$. For the sensors $b_1, \ldots, b_m \in \mathbb{R}^n$, we consider Case 1, $b_i = a_i - \sum_{i=1}^m a_i$ for $i = 1, \ldots, m$ where a_i is generated randomly and independently generated from a Gaussian random vector with nonzero mean; Case 2, the sensors are generated randomly and independently generated from a Gaussian random vector with nonzero mean. We use the ℓ_0-constrained least squares method (3) to find x^*. Here, we assume the sparsity s is given, otherwise, it can be chosen by cross-validation method like the next example. Based on the sample method and $\sum_{i=1}^m b_i = 0$ for Case 1, the conditions of Corollary 1 hold for Case 1 with high probability but Case 2 does not guarantee those. Indeed, both Figs. 1 and 2 show that the SR for Case 1 is far higher than that for Case 2 and the error $\|\hat{x} - x^*\|_2^2$ for Case 1 is far smaller than that for Case 2. Comparing to SR and the error $\|\hat{x} - x^*\|_2^2$ for the two cases, one can see the importance of the uniform regularity. By the way, we think it is interesting to extend this concept to symmetric

Table 2. The average results of successful recoveries.

n	$\|x^*\|_0$	($\|\hat{x}\|_0$, SE)	SR
100	5	(5.9, 0.120)	0.90
200	10	(10.6, 0.103)	0.96
300	15	(15.67, 0.111)	0.98
400	20	(20.6, 0.133)	0.97
500	25	(26.2, 0.233)	0.94

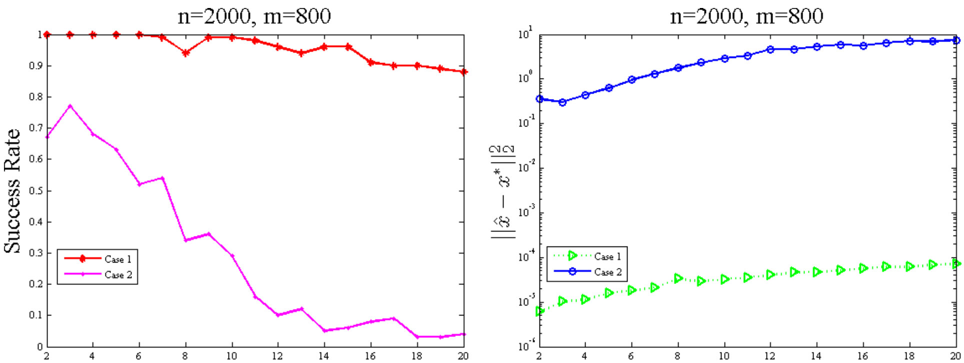

Fig. 1. Simulation results for Example 2 with $\sigma = 0.1$.

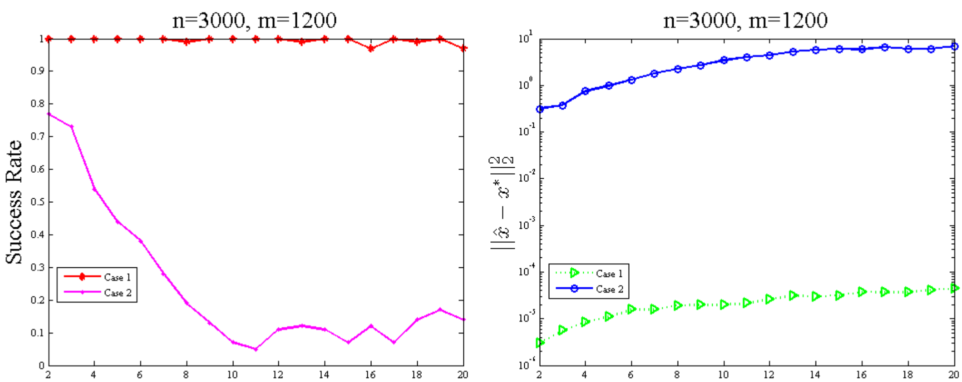

Fig. 2. Simulation results for Example 2 with $\sigma = 0.1$.

tensor in Wang *et al.*, (2009, 2015a) and Zhang and Wang (2016) or tensor sparse decompositions in Wang *et al.* (2015b).

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