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Statistical Methods for Set-valued Observations*

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Abstract

Paper [1] first proposed the concept and a theoretical framework of random sets and provided experimental justification and background of practical applications. This paper studies statistical theory and methods for random sets from the statistical decision theory point of view. Some mathematical concepts such as the definition of random set are slightly different from that in [1].

1 Introduction

The traditional statistical approach is to observe a certain sample point in a sample space through statistical experiment, and then to use the information provided by this point to estimate or infer certain characteristics of the population. In real applications and scientific experiments, however, there are many statistical experiments in which the observations usually are not or cannot be points of the sample space, rather they are more general subsets. Paper [1] and [2] provided many such examples. From the statistical point of view we may regard such a subset as a realization of a certain randomly varied set ξ. The variation of ξ is governed by many factors including psychological, social and natural ones. We may regard the combined effect of all these factors on ξ as coming from a certain influence field Ω. Arbitrary determination of any point ω in Ω means the determination of ξ. In this sense ξ is a mapping from Ω to the power space $\mathcal{P}(U)$ of its image space U. Our interest is the law of variation of this random set ξ on $\mathcal{P}(U)$.

2 Random set and its shadow function

Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ and an image space $(U, \mathcal{B})$, where $\mathcal{B}$ is a σ-algebra on U. A mapping $\xi(\cdot) : \Omega \mapsto \mathcal{B}$ is called a random set on U, if for every $u \in U$, $\{\omega \mid u \in \xi(\omega)\} \in \mathcal{F}$. If ξ and η are two random sets on U and we define the mapping $(\xi \cup \eta)(\omega) = \xi(\omega) \cup \eta(\omega)$, then it is easy to see that $\xi \cup \eta$ is also a random set on U. Likewise, $\xi \cap \eta$ and $\xi \setminus \eta$ are all random sets. In particular, the entire space U and empty set $\emptyset$ are random sets on U.

Let ξ be a random set on U. Then the function $F_\xi(u) = P(u \in \xi), u \in U$ is called the shadow function of ξ on U. From the definition it is easy to see that the random set ξ has the following properties:

(1) ξ is a random set on U if and only if for every $u \in U$, $\chi(\xi, u)$ is a random variable on $(\Omega, \mathcal{F})$, where

$$\chi(\xi(\omega), u) = \begin{cases} 1, & \text{if } u \in \xi(\omega) \\ 0, & \text{if } u \notin \xi(\omega). \end{cases}$$

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(2) $E\chi(\xi,u) = F_\xi(u)$, $u \in U$, where E denotes the mathematical expectation.

(3) $E[\chi(\xi,u) - F_\xi(u)]^2 = F_\xi(u)[1 - F_\xi(u)]$, $u \in U$.

(4) $E[\chi(\xi,u) - F_\xi(u)]^2 \equiv 0$ if and only if ξ is a constant set almost everywhere. In this case $F_\xi(u) = \chi(\xi,u) \quad a.e.P.$

For the later development, we consider a class of stronger random sets. Let ξ be a random set on U. If the graph of ξ, $G_\xi = \{(u,\omega) \mid u \in \xi(\omega)\} \in \mathcal{B} \times \mathcal{F}$, then ξ is called a regular random set. If ξ and η are regular random sets, then it is easy to see that $\xi \cup \eta$, $\xi \cap \eta$ and $\xi \setminus \eta$ are all regular random sets. In particular, U and $\emptyset$ are regular random sets. For the regular random sets we have the following results.

Theorem 2.1.

(1) If U is finite or countable, $\mathcal{B} = \mathcal{P}(U)$ (i.e. the class of all subsets of U), then any random set on U is regular.

(2) If U is the real line $\mathbb{R}_1$ or a subset of $\mathbb{R}_1$, $\mathcal{B}$ is the class of all Borel subsets of U, and the random set ξ on U satisfies: for any $\omega \in \Omega$, $\xi(\omega)$ is a closed set consisting of a sequence of non-single-point intervals, then ξ is regular.

(3) If $(U,\mathcal{B})$ is as in (2) and the random set ξ satisfies: for any $\omega \in \Omega$, $\xi(\omega)$ is an open set and the distance between its any two adjacent subintervals is greater than 0, then ξ is regular.

Proof: (1) Suppose $U = \{u_1, u_2, ...\}$. Then from

$$G_\xi = \{u \in \xi(\omega)\} = \bigcup_{n=1}^{\infty} (\{u_n\} \times \{\omega \mid u_n \in \xi(\omega)\})$$

it follows that $G_\xi \in \mathcal{B} \times \mathcal{F}$. The proof is analog when U is finite.

(2) First we construct a sequence of intervals on $\mathbb{R}_1$: The class of all open intervals with consecutive integers as two ends $(k, k+1)$ is denoted as $I_1 = \{I_{1m}, m = 1,2,...\}$. Then every open interval in I_1 is divided at the midpoint into two smaller open intervals and the class of all such intervals is denoted as $I_2 = \{I_{2m}, m = 1,2,...\}$. Repeating the above procedure we obtain a sequence of classes of sets on $\mathbb{R}_1$, $I_n = \{I_{nm}, m = 1,2,...\}, n = 1,2,...$, satisfying the conditions: (i) every interval I_{nm} in I_n has length $\frac{1}{2^{n-1}}$; (ii) I_{n+1} is finer than I_n.

In the following we use r to denote a rational number. Let

$$A_1 = \bigcup_{r \in \mathbb{R}_1} [\{r\} \times \{\omega \mid r \in \xi(\omega)\}],$$

$$A_2 = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} [I_{nm} \times \bigcup_{r \in I_{nm}} \{\omega \mid r \in \xi(\omega)\}].$$

Then $A_1, A_2 \in \mathcal{B} \times \mathcal{F}$. Next we show that under the condition of (2) we have $G_\xi = A_1 \cup A_2$ and hence $G_\xi \in \mathcal{B} \times \mathcal{F}$.

If $(u,\omega) \in G_\xi$, then $u \in \xi(\omega)$ and there is a (closed) component interval J of $\xi(\omega)$, such that $u \in J$. If $(u,\omega) \notin A_1$, then for any n, there exists an m, such that $u \in I_{nm}$. Therefore $I_{nm} \cap U \neq \emptyset$. It follows that there exists a rational number $r \in I_{nm}$, such that $r \in J \subset \xi(\omega)$, that is, $(u,\omega) \in A_2$. Conversely, if $(u,\omega) \in A_2$, then for every n, there exists m_n, such that $u \in I_{nm_n}$ and there exists rational number $r_n \in I_{nm_n}$, such that $r_n \in \xi(\omega)$. Since as $n \to \infty$, $r_n \to u$ and $\xi(\omega)$ is closed, it follows that $u \in \xi(\omega)$, that is, $(u, \omega) \in G_\xi$.

(3) The random set defined here is the compliment of that in (2) and, therefore, is also a regular random set. The proof is complete.

In the following we are given a σ–finite measure m on $(U, \mathcal{B})$.

Theorem 2.2. *Suppose ξ is a regular random set on U and define*

$$\mu(B) = Em(B \cap \xi), \; B \in \mathcal{B}. \tag{2.1}$$

Then μ is a σ–finite measure on $(U, \mathcal{B})$ and is absolutely continuous relative to m. Furthermore, the Radon-Nikodym derivative of μ with respect to m is the shadow function of ξ, that is $d\mu/dm = F_\xi(\cdot)$.

Proof: Since ξ is regular, $\chi(\xi, u)$ is a $\mathcal{B} \times \mathcal{F}$–measurable function and, therefore, the right-hand side of (2.1) is well-defined and μ is a σ–finite measure. Further by Fubini Theorem we have, for all $B \in \mathcal{B}$,

$$\mu(B) = \int_B F_\xi(u)\, dm \tag{2.2}$$

It follows that $\frac{d\mu}{dm}(u) = F_\xi(u)$. The proof is complete.

In the above proof we obtain formula (2.2). If $F_\xi(u)$ is a continuous function of u (under some given distance), then it follows from the Integral Mean Value Theorem (see [2], p.174 Exercise 2) and the Mean Value Theorem for continuous functions, there exists $u_0 \in B$, such that $\mu(B) = F_\xi(u_0)\, m(B)$. Thus we have the following result.

Theorem 2.3. *Suppose U is a metric space, ξ is a regular random set on U and $F_\xi(u)$ is continuous. Then for any $B \in \mathcal{B}$, $0 < m(B) < \infty$, there exists $u_0 \in B$, such that*

$$F_\xi(u_0) = \frac{\mu(B)}{m(B)}.$$

Next we discuss some properties of a sequence of independent random sets. Let ξ_1, ξ_2, ..., ξ_n be random sets on U. If for any subsequence ξ_{i_1}, ξ_{i_2}, ..., ξ_{i_k} and any u_{11}, u_{12}, ..., u_{1m_1}; ... ; u_{k1}, u_{k2}, ..., $u_{km_k} \in U$, it holds

$$\begin{aligned} &P(\{u_{11}, ..., u_{1m_1}\} \subset \xi_{i_1}, ..., \{u_{k1}, ..., u_{km_k}\} \subset \xi_{i_k}) \\ &= P(\{u_{11}, ..., u_{1m_1}\} \subset \xi_{i_1}) \cdots P(\{u_{k1}, ..., u_{km_k}\} \subset \xi_{i_k}), \end{aligned}$$

then the random sets are said to be strongly independent. If the above equation holds for $m_1 = m_2 = ... = m_k = 1$, then the random sets are said to be (mutually) independent. A class of random sets $\{\xi_t, t \in T\}$ is said to be (strongly) independent, if any finite subclass of it is (strongly) independent. For independent random sets we have the following result.

Theorem 2.4. *Suppose ξ_1, ξ_2, ... is a sequence of independent random sets with identical shadow function. Then for any $u \in U$, as $n \to \infty$, we have $\frac{1}{n}\sum_{i=1}^{n} \chi(\xi_i, u) \to F_{\xi_1}(u)$, a.e.P.*

Proof: Under the Theorem's condition, $\chi(\xi_1, u)$, $\chi(\xi_2, u)$, ... is a sequence of independent and identically distributed random variables. The result follows then from the Kolmogorov Law of Large Numbers.

Theorem 2.5. *Suppose ξ_1, ξ_2, ... is a sequence of strongly independent random sets with identical shadow function. Then under any one of the following conditions, it holds: for any $B \in \mathcal{B}$, as $n \to \infty$,*

$$\frac{1}{n}\sum_{i=1}^{n} m(\xi_i \cap B) \to Em(\xi_1 \cap B), \; a.e.P.$$

(1) U is finite or countable, $\mathcal{B} = \mathcal{P}(U)$ and m is the count measure.

(2) $U = \mathbb{R}_1$ or a subset of $\mathbb{R}_1$, $\mathcal{B}$ is the class of all Borel subsets of U and m is the Lebesgue measure. In addition, ξ_1, ξ_2, ... are regular random sets and are such that for every ξ_i and $\omega \in \Omega$, $\chi(\xi_i(\omega), u)$ is almost everywhere continuous on U.

Proof: By the generalized law of large numbers of Loéve [3], §33.4, we need only to show that under any one condition $m(\xi_1 \cap B)$, $m(\xi_2 \cap B)$, ... is a sequence of independent and identically distributed random variables.

In the case of (1), $m(\xi_i \cap B) = \sum_{u_j \in B} \chi(\xi_i, u_j)$, $i = 1, 2, \ldots$. By the assumption of strong independence, it is easily seen that any class of random vectors $(\chi(\xi_{i_1}, u_{11}), \ldots, \chi(\xi_{i_1}, u_{1m_1}))$, ..., $(\chi(\xi_{i_k}, u_{k1}), \ldots, \chi(\xi_{i_k}, u_{km_k}))$ is independent. It follows that $m(\xi_1 \cap B)$, $m(\xi_2 \cap B)$, ... are independent and identically distributed.

In the case of (2), since B is a Borel set, there exists a sequence of open sets $O_n \subset B$, such that $m(B - O_n) \to 0$ (as $n \to \infty$). Let $\{O_{nj}\}$ be the sequence of component intervals of O_n. Then

$$m(\xi_i \cap B) = \lim_{n\to\infty} \int_{O_n} \chi(\xi_i, u)\, dm = \lim_{n\to\infty} \sum_j \int_{O_{nj}} \chi(\xi_i, u)\, dm.$$

Since $\chi(\xi_i, u)$ is Riemann-integrable on every O_{nj}, it follows that

$$\int_{O_{nj}} \chi(\xi_i, u)\, dm = \lim_{k\to\infty} \sum_l \chi(\xi_i, u_l)\, \Delta^k u_l.$$

Therefore $m(\xi_1 \cap B), m(\xi_2 \cap B), \ldots$ are independent and identically distributed. The proof is complete.

3 Nonparametric estimation of shadow function

This section discusses several nonparametric estimation methods for shadow functions, derives the best estimators under various optimal criteria and evaluates the performance of the commonly used weighted mean estimators under these criteria.

Suppose that the probability space $(\Omega, \mathcal{F}, P)$ and image space $(U, \mathcal{B})$ are given, where the probability measure P is unknown or partially unknown. We consider the random set ξ and its shadow function $F(u)$. For the sake of notational convenience we will not explicitly distinguish the random set ξ and its observed sample $x = \xi(\omega)$ and denote it as x. Further we assume that $F(u) \not\equiv 0$. A mapping $\delta : \mathcal{B} \times U \mapsto [0, 1]$ is called an estimating function of F, if for every $u \in U$, $\delta(x, u)$ is a $\mathcal{F}$-measurable function. In the following discussion we take the quadratic function to be the loss function. Then the risk function of an estimator δ is

$$R(\delta, u) = E[\delta(x, u) - F(u)]^2 = \int_\Omega [\delta(x, u) - F(u)]^2\, dP.$$

3.1 Best unbiased estimators

Suppose $U' \subset U$ and δ is an estimator. If $E\delta(x, u) = F(u)$, for every $u \in U'$, then δ is said to be a U'-unbiased estimator for F. Further if δ is a U'-unbiased estimator for F and for any U'-unbiased estimator g, it holds $R(\delta, u) \le R(g, u)$ for all $u \in U'$, then δ is said to be a U'-best unbiased estimator for F. In particular, when U' equals $U_1 = \{u \in U \mid F(u) = 1\}$ or $U_0 = \{u \in U \mid F(u) = 0\}$, the U'-best unbiased estimator has practical importance. Let us consider the following example. Suppose

$x_1, x_2, ..., x_n$ is an independent random sample with identical shadow function. Consider the class of linear estimators

$$\mathcal{L} = \left\{ \sum_{i=1}^{n} a_i(u)\, \chi(x_i, u) \mid a_i(u) \text{ are real functions on } U \right\}.$$

For any estimator in $\mathcal{L}$ it is easy to calculate

$$E\left[\sum_{i=1}^{n} a_i(u)\, \chi(x_i, u)\right] = F(u) \sum_{i=1}^{n} a_i(u), \tag{3.1}$$

$$E\left[\sum_{i=1}^{n} a_i(u)\, \chi(x_i, u) - F(u)\right]^2 = F(u)\,[1 - F(u)] \sum_{i=1}^{n} a_i(u)^2 + F^2(u) \left[1 - \sum_{i=1}^{n} a_i(u)\right]^2. \tag{3.2}$$

It follows that, 1) every estimator $\sum_{i=1}^{n} a_i(u)\, \chi(x_i, u)$ is a U_0-best unbiased estimator; 2) every estimator satisfying the condition $\sum_{i=1}^{n} a_i(u) = 1$ when $F(u) \neq 0$ is a U_1-best unbiased estimator; 3) if there exists $u_0 \in U$, such that $0 < F(u_0) < 1$, then being unbiased implies $\sum_{i=1}^{n} a_i(u) = 1$, for all $u \in \{F(u) > 0\}$, and being the best implies $a_1(u) = \cdots = a_n(u) = \frac{1}{n}$, $\forall u \in \{0 < F(u) < 1\}$. Therefore all estimators satisfying the above two conditions are U-best unbiased estimators in $\mathcal{L}$. One of them is the commonly used estimator $\frac{1}{n} \sum_{i=1}^{n} \chi(x_i, u)$.

3.2 Admissibility of the estimators

Let δ and g be two estimators. If for every $u \in U$, $R(\delta, u) \leq R(g, u)$ and there exists $u_0 \in U$ such that the inequality holds, then δ is said to be better than g. Let $\mathcal{L}$ be a class of estimators. Then $\delta \in \mathcal{L}$ is said to be admissible in $\mathcal{L}$, if no estimator in $\mathcal{L}$ is better than δ. The class of estimators $\mathcal{M} \subset \mathcal{L}$ is called a complete class in $\mathcal{L}$, if for any $g \in \mathcal{L}$, there exists $\delta \in \mathcal{M}$, such that δ is better than g. Obviously the class of all admissible estimators in $\mathcal{L}$ is contained in every complete class in $\mathcal{L}$.

In this subsection we consider the admissible estimators in the class

$$\mathcal{L}_1 = \left\{ \sum_{i=1}^{n} a_i \chi(x_i, u) \mid a_i \in I\!R_1 \right\}.$$

First we establish the following results.

Theorem 3.1. *For any estimator $g = \sum_{i=1}^{n} a_i \chi(x_i, u)$ in $\mathcal{L}_1$, there exists an estimator $\delta = c \sum_{i=1}^{n} \chi(x_i, u)$, such that for every $u \in U$, $R(\delta, u) \leq R(g, u)$. Further if $\{u \in U \mid 0 < F(u) < 1\} \neq \emptyset$, then the equality holds if only if $a_1 = a_2 = \cdots = a_n = c$.*

Proof: For the $g = \sum_{i=1}^{n} a_i \chi(x_i, u)$, take $c = \frac{1}{n} \sum_{i=1}^{n} a_i$. Then the results follow from (3.2) and Cauchy-Schwarz inequality.

In the following we assume that $\{u \in U \mid 0 < F(u) < 1\} \neq \emptyset$. Then by Theorem 3.1 the class

$$\mathcal{L}_0 = \left\{ a \sum_{i=1}^{n} \chi(x_i, u) \mid a \in I\!R_1 \right\}$$

is a complete class in $\mathcal{L}_1$. Therefore we need only to consider the estimators in $\mathcal{L}_0$. For any estimator $a \sum_{i=1}^{n} \chi(x_i, u) \in \mathcal{L}_0$, to simplify notation we denote its risk as $R(a, u)$. Let $b \sum_{i=1}^{n} \chi(x_i, u)$ be another estimator in $\mathcal{L}_0$. Then by (3.2) we have

$$R(a, u) - R(b, u) = nF(u)\,[nF(u) - F(u) + 1]\,(a - b) \left[a + b - \frac{2F(u)}{(n-1)\,F(u) + 1}\right]. \tag{3.3}$$

Denote $F_0 = \inf_{u\in U} F(u)$, $F_1 = \sup_{u\in U} F(u)$ and

$$\alpha = \frac{F_0}{(n-1)\,F_0+1},\ \beta = \frac{F_1}{(n-1)\,F_1+1}.$$

Then from (3.3) it is easy to see that: if $a > \beta$, then $\beta\sum_{i=1}^n \chi(x_i,u)$ is better than $a\sum_{i=1}^n \chi(x_i,u)$; if $a < \alpha$, then $\alpha\sum_{i=1}^n \chi(x_i,u)$ is better than $a\sum_{i=1}^n \chi(x_i,u)$; if $\alpha \le b < a \le \beta$, then there exist $u_1, u_2 \in U$, such that $R(a,u_1) < R(b,u_1)$ and $R(a,u_2) > R(b,u_2)$. Thus we obtain the following result.

Theorem 3.2. *If $\{u\in U \mid 0 < F(u) < 1\} \neq \emptyset$, then the class of estimators*

$$\overline{\mathcal{L}}_0 = \left\{ a\sum_{i=1}^n \chi(x_i,u) \mid \alpha \le a \le \beta \right\}$$

is a minimal complete class in $\mathcal{L}_1$, i.e. the class of all admissible estimators.

From the above theorem it is easily seen that under the theorem's condition the class $\left\{a\sum_{i=1}^n \chi(x_i,u) \mid 0 \le a \le \frac{1}{n}\right\}$ is a complete class in $\mathcal{L}_1$. In general F is unknown, and so are the values of α and β. However, the above class provides a reference class of estimators. Furthermore, from Theorem 3.2 we know that, when $F_1 < 1$, the commonly used estimator $\frac{1}{n}\sum_{i=1}^n \chi(x_i,u)$ is inadmissible. For example, take $0 < c < (1-F_1)/F_1$, then $\frac{1}{n+c}\sum_{i=1}^n \chi(x_i,u)$ is better than $\frac{1}{n}\sum_{i=1}^n \chi(x_i,u)$.

3.3 Minimax estimators

Let $\mathcal{L}$ be a class of estimators of F. Then $\delta \in \mathcal{L}$ is called a minimax estimator, if for any $g \in \mathcal{L}$, it holds $\sup_{u\in U} R(\delta,u) \le \sup_{u\in U} R(g,u)$. In this subsection we consider again the class $\mathcal{L}_1$. For notational simplicity we assume without loss of generality that the range of $F(u)$ is the interval $[0,1]$. By Theorem 3.2

$$\overline{\mathcal{L}}_0 = \left\{ a\sum_{i=1}^n \chi(x_i,u) \mid 0 \le a \le \frac{1}{n} \right\}$$

is a minimal complete class in $\mathcal{L}_1$. Therefore a minimax estimator in $\mathcal{L}_1$ (if exists) must be contained in $\overline{\mathcal{L}}_0$. For any estimator $a\sum_{i=1}^n \chi(x_i,u)$, by (3.2) its risk function is

$$R(a,u) = F^2(u)\,(na-1)^2 + F(u)\,[1-F(u)]\,na^2.$$

Denote $Q(a,F) = F^2(na-1)^2 + F(1-F)\,na^2$. Then $\max_{u\in U} R(a,u) = \max_{0\le F\le 1} Q(a,F)$. Therefore it is not difficult to calculate that $\max_{u\in U} R(a,u)$ has the unique minimizer on the interval $\left[0,\frac{1}{n}\right]$

$$a_0 = \frac{3n-\sqrt{n^2+8n}}{2n\,(n-1)}.$$

Thus we have the following result.

Theorem 3.3. *Suppose the range of F is $[0,1]$. Then in $\mathcal{L}_1$ there exists unique minimax estimator*

$$a_0\sum_{i=1}^n \chi(x_i,u) = \frac{3n-\sqrt{n^2+8n}}{2n\,(n-1)}\sum_{i=1}^n \chi(x_i,u).$$

From the above result we see that the usually used best unbiased estimator $\frac{1}{n}\sum_{i=1}^{n}\chi(x_i,u)$ is not minimax estimator. In fact, direct calculation shows that its maximum risk is even greater than that of the estimator $\frac{1}{n+1}\sum_{i=1}^{n}\chi(x_i,u)$. However, in the class of unbiased estimators $\{\sum_{i=1}^{n}a_i\chi(x_i,u)\mid\sum_{i=1}^{n}a_i=1\}$, $\frac{1}{n}\sum_{i=1}^{n}\chi(x_i,u)$ is the unique minimax estimator, and also the unique admissible estimator.

3.4 Consistency of the estimators

Let $x_1,x_2,...$ be a sequence of random sets and $\delta(x_1,...,x_n;u)$ a sequence of estimators. If for every $u\in U$, as $n\to\infty$, $\delta(x_1,...,x_n;u)\to F(u)\,,a.e.P$, then $\delta(x_1,...,x_n;u)$ is called a consistent estimator for $F(u)$. If $x_1,x_2,...$ is a sequence of independent random sets with identical shadow function, then from Theorem 2.4 we know that $\frac{1}{n}\sum_{i=1}^{n}\chi(x_i,u)$ is a consistent estimator for $F(u)$.

Furthermore, if $F(u)$ is continuous on U, then motivated by Theorem 2.3 of §2 we can construct the following estimation procedure:

First partition the image space U as $U=\bigcup_{i=1}^{\infty}U_i$, where $U_i\subset U$ are mutually disjoint and $0<m(U_i)<\infty$, $i=1,2,....$ Define the step function $\hat{F}(u)=Em(\xi\cap U_i)/m(U_i)$, when $u\in U_i$. Then along with partitions $U_i,i=1,2,...$ becoming finer, $\widehat{F}(u)$ gets closer to $F(u)$. Now if $x_1,x_2,...$ is a sequence of strongly independent and regular random sets with identical shadow function, and satisfies the condition of Theorem 2.5, then by Theorem 2.5, it holds

$$\frac{1}{n}\sum_{j=1}^{n}m(x_j\cap U_i)\xrightarrow{a.e.}Em(x_1\cap U_i)\,.$$

Let

$$\delta(x_1,...,x_n;u)=\frac{1}{n}\sum_{j=1}^{n}\frac{m(x_j\cap U_i)}{m(U_i)},\ u\in U_i.$$

Then $\delta(x_1,...,x_n;u)\xrightarrow{a.e.}\widehat{F}(u)\,,\ \forall u\in U.$

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