MATH 1300 Mid-Term Solutions for February 22, 2010

1.

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 2 \\ 0 & 2 & 4 & -6 & 3 \\ 2 & 4 & -2 & 6 & 4 \end{bmatrix} \xrightarrow{R3-2R_2} \begin{bmatrix} 1 & 1 & -1 & 3 & 2 \\ 0 & 2 & 4 & -6 & 3 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -1 & 3 & 2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \times \frac{1}{2}} \begin{bmatrix} 1 & 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & -6 & 3 \end{bmatrix} \xrightarrow[R_3-2R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \times \frac{1}{4}} \begin{bmatrix} 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{3}{4} \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{11}{4} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{3}{4} \end{bmatrix}$$

Set $x_4 = t$, free. Then $x_1 = \frac{11}{4} - \frac{3}{2}t$, $x_2 = 0$, and $x_3 = \frac{3}{4} + \frac{2}{3}t$.

2. (a) $a = 0$ and $4 - 3b \neq 0$, so b is any number except $4/3$.

(b) $a = 0$ and $b = 4/3$.

(c) $a \neq 0$ and b any number. Then $x = -2 + 2b$ and $ay = 4 - 3b$, so the solution is $x = -2 + 2b$, $y = \frac{4-3b}{a}$.

3. (a) DNE[1], because $2AB$ will be a 3×2 matrix but C is 2×3. Cannot subtract.

(b) $CD^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -8 & 0 \end{bmatrix}$, so $CD^T + E = \begin{bmatrix} 9 & 5 \\ -7 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 \end{bmatrix}$.

(d) DNE. CB is a 2×2 matrix, so has no third column.

(e) DNE. B is not square, so B^3 is not defined.

4. (a) A is invertible if and only if $A \to I$, so the RREF is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{x} = \mathbf{0}$ is the unique solution.

(c) $\det(A^2) = (\det(A))^2$, so $\det(A)^2 = 3\det(A)$. But also, A is invertible, so $\det(A) \neq 0$. Therefore we are allowed to cancel, giving $\det(A) = 3$.

[1] "Does Not Exist"

5. (a) FALSE. $(AB)^T = B^T A^T$.

(b) FALSE. If $AB = AC$ and A is invertible, then $B = C$.

(c) FALSE. It is true for any $a \neq 0$, $b \neq 0$, and $c \neq 0$.

(d) FALSE. It's $A \xrightarrow{R_2+3R_4} A'$.

6. (a) $\det(2AB) = 2^5 \det(A)\det(B) = (32)(-\frac{1}{3})(6) = -64$.

(b) $\det(A)\det(B)\det(A^T) = \det(A)\det(B)\det(A) = \frac{2}{3}$.

(c) $\det(A)\det(B)\det(A^{-1}) = \det(A)\det(B)(\det(A))^{-1} = \det(B) = 6$.

(d) $A(\mathrm{adj}(A)) = \det(A)I = -\frac{1}{3}I_5 = \begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} \end{bmatrix}$

7. (a) By expansion down third column, $-(1)\begin{vmatrix} 2 & 4 & 1 \\ 0 & 8 & 2 \\ 0 & 3 & 5 \end{vmatrix} = -(2)\begin{vmatrix} 8 & 2 \\ 3 & 5 \end{vmatrix} = -68$.

(b) Row 2 is a multiple of row 1, so the determinant is zero.

(c) This is a diagonal matrix, so the determinant is $(-20)(2)(-5)(10)(2)(-1) = -4000$.

8. (a) $a = C_{12} = -\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} = 3$, $b = C_{13} = +\begin{vmatrix} -3 & \\ 3 & 3 \end{vmatrix} = -24$

(b) $\mathrm{adj}(A) = \mathrm{cof}(A)^T = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 2 & 0 \\ -24 & -3 & 13 \end{bmatrix}$, so

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{13}\begin{bmatrix} 5 & -1 & 0 \\ 3 & 2 & 0 \\ -24 & -3 & 13 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 \\ \frac{3}{13} & \frac{2}{13} & 0 \\ -\frac{24}{13} & -\frac{3}{13} & 1 \end{bmatrix}.$$