MATH 1300 Mid-Term Solutions for February 22, 2007

1. (a) Augmented matrix

$$= \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 4 \\ 1 & 2 & 2 & -1 & 8 \end{array}\right] \xrightarrow{R_2=R_2-R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 4 \\ 0 & 0 & 2 & -6 & 4 \end{array}\right] \xrightarrow{R_2=R_2\times\frac{1}{2}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 4 \\ 0 & 0 & 1 & -3 & 2 \end{array}\right]$$

Then $x_1 + 2x_2 + 5x_4 = 4$ and $x_3 - 3x_4 = 2$. Set $x_2 = s$ and $x_4 = t$ (arbitrary). Then the general solution is

$$x_1 = 4 - 2s - 5t, \quad x_2 = s, \quad x_3 = 2 + 3t, \quad x_4 = t, \quad s, t \in \mathbb{R}.$$

(b) Set $s = -2$ and $t = 3$ to get

$$x_1 = -7, \quad x_2 = -2, \quad x_3 = 11, \quad x_4 = 3.$$

2. (a) **Does not exist**, because AB will be a 2×3 matrix, and C is 3×2, so the **sum** cannot be formed.

(b) **Does not exist**, because A has 2 columns and C has 3 rows, so the **product** cannot be formed.

(c) $BC = \begin{bmatrix} -7 & -3 \\ -5 & 4 \end{bmatrix}$, so $BC + A = \begin{bmatrix} -6 & -1 \\ -9 & 10 \end{bmatrix}$.

3.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3=R_3-2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3=R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_3\times\frac{1}{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow[R_1=R_1+R_3]{R_2=R_2-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] = [I|A^{-1}].$$

So $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

4. First, reduce A to I:

$$\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \xrightarrow{R_2 = R_2 \times \frac{1}{2}} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 + 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then apply those EROs to I:

$$I \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1, \quad I \xrightarrow{R_2 = R_2 \times \frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = E_2, \quad I \xrightarrow{R_1 = R_1 + 3R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = E_3.$$

Find the inverses: $E_1^{-1} = E_1$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

Finally, $A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

5. (a) $\det A = -(2) \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + 0 - (1) \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -2(0-3) - (1-6) = 11.$

(b) $\det(AB^2) = \det A (\det B)^2 = 11 \cdot 100 = 1100.$

(c) $\det(A^{-1}) = (\det A)^{-1}$ and $\det A^T = \det A$, so
$\det(A^{-1}(2B)A^T) = \det(A^{-1}) \det(2B) \det(A^t) = \det(2B) = 2^3 \det B = 80.$

6. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 6 & 5 \\ -7 & 2 \end{bmatrix}$, and $A_2 = \begin{bmatrix} 2 & 6 \\ 3 & -7 \end{bmatrix}$. Then by Cramer's rule,

$$x = \frac{\det A_1}{\det A} = \frac{47}{-11} = -\frac{47}{11}, \quad y = \frac{\det A_2}{\det A} = \frac{-32}{-11} = \frac{32}{11}.$$

7. (a) The system has no solutions if and only if p is zero and q is not zero.

(b) The system has a unique solution if p is not zero. (q can be any value.)

(c) The system has infinitely many solutions if p and q are both zero.

(d) If $p = q = 0$, the matrix becomes $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$, so setting $x_3 = t$ gives the general solution

$$x_1 = -2 + t, \quad x_2 = -3, \quad x_3 = t, \quad t \in \mathbb{R}.$$