

Prove the following statements by using Mathematical Induction:

1. $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all n in \mathbb{N} .

2. If $r \neq 1$, then $1 + r + r^2 + \dots + r^n = \frac{1-r^{n+1}}{1-r}$, for all n in \mathbb{N} .

3. $5^{2n} - 1$ is divisible by 6, for all n in \mathbb{N} .

4. $n < 2^n$, for all n in \mathbb{N} .

5. $2^n < n!$, for all $n \geq 4$.

6. $a - b$ is a factor of $a^n - b^n$, for all n in \mathbb{N} .

7. Let $a_1 = 3$, and let $a_{n+1} = 2a_n + 1$, for $n \geq 1$. Find the formula for a_n in terms of n .

8.

$$\begin{aligned} 1 &= 1 \\ 1-4 &= -(1+2) \\ 1-4+9 &= 1+2+3 \\ 1-4+9-16 &= -(1+2+3+4) \\ &\dots\dots\dots \end{aligned}$$

Guess the general formula suggested and prove it by using Mathematical Induction.

9. Prove that for every $x \geq 0$, and every n in \mathbb{N} , $e^x \geq 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$.

10. Let $\binom{n}{m} = \frac{n!}{(n-m)!m!}$, for n and m non-negative integers, and let a and b be in \mathbb{R} .

Prove that $\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m}$, for $1 \leq m \leq n$, and use it to prove that

$$(a + b)^n = \sum_{m=0}^n \binom{n}{m} a^{n-m} b^m, \text{ for all } n \text{ in } \mathbb{N}.$$