MATH 2202, Assignment No. 3
November 3, 2008

The assignment is due Monday, November 10, in class. Late assignments receive a mark zero.

1. a) Using the **definition of the limit** of a sequence prove that $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$. [5]

   b) Prove that if $\lim_{n\to\infty} x_n = 0$ and the sequence (y_n) is bounded, then $\lim_{n\to\infty} x_n y_n = 0$. [5]

   c) Give examples of sequences (x_n) and (y_n) such that $\lim_{n\to\infty} x_n = x \neq 0$, (y_n) is bounded , but $\lim_{n\to\infty} x_n y_n$ does not exist. [2]

2. Determine if (x_n) converges or diverges: (**Explain** why by showing all of your work.)

   a) $x_n = \frac{\sin 2n}{n} + \frac{n^2}{1-n^2}$. [4]

   b) $x_n = (-1)^n \sqrt{n}(\sqrt{n+1} - \sqrt{n})$. [5]

   c) $x_n = \frac{2^n}{n}$. [6]

   (**Do not** use any theorems that you know on limits of functions, such as for example L'Hospital's rule.) [6]

3. a) Let (x_n) be a bounded sequence and let $s = \sup\{ x_n : n \text{ in } \mathbb{N} \}$. Show that if s is not in $\{ x_n : n \text{ in } \mathbb{N} \}$, then there is a subsequence of (x_n) that converges to s. [6]

   b) Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} \frac{5}{x_{n_k}} = 0$. [5]

   c) Show that if (x_n) is monotonic increasing and bounded above, then (x_n) converges, by using the Bolzano-Weierstrass Theorem.[5]

4. a) Show, by using the **definition of a Cauchy sequence**, that if (x_n) and (y_n) are Cauchy sequences, then (x_n+y_n) and $(x_n y_n)$ are also Cauchy sequences. [7]

   b) Let (x_n) be a Cauchy sequence such that x_n is an integer, for all n in $\mathbb{N}$. Show that (x_n) is ultimately constant. [6]

Total [56/54]