Value

1. Let $f: A \to B$ and let X and Y be subsets of A.

[3] a) Prove that if $X \subseteq Y$, then $f(X) \subseteq f(Y)$.

LET $y \in f(X)$. THEN $\exists x \in X$ SUCH THAT $f(x) = y$. BUT $X \subseteq Y$ AND SO $y \in Y$. THUS $y = f(x)$, FOR $x \in Y$ AND SO $y \in f(Y)$.

[3] b) Show that the converse of the statement in a) is false, i.e. give an example of f, A, B, X and Y (subsets of A) such that $f(X) \subseteq f(Y)$, but X is not a subset of Y.

$f(x) = 1$, $A = B = \mathbb{R}$, $X = \mathbb{R}$, $Y = [0, 1]$, $Y \subset X$

AND $f(X) = f(Y) = \{1\}$.

[5] c) Show that if f is an injection, then $f(X) \subseteq f(Y)$ implies that $X \subseteq Y$. (State first

LET f BE AN INJECTION AND LET $f(X) \subseteq f(Y)$.

WE WANT TO SHOW THAT $X \subseteq Y$.

LET $x \in X$. THEN $f(x) \in f(X) \subseteq f(Y)$, i.e $f(x) \in f(Y)$.

SO: $\exists y \in Y$ SUCH THAT $f(y) = f(x)$. BUT f IS AN INJECTION AND SO $x = y$, i.e. $x \in Y$.

(DEFINITION: f INJECTION $\Leftrightarrow$ $f(x) = f(y) \Rightarrow x = y$.)

2. Prove the following by using the Field and Order axioms.

[5] a) Show that if $a > 0$, then $a^{-1} > 0$ too. (You can use Theorems proven in class such as $0x=0$, $-x = (-1)x$ and $1 > 0$.)

LET $a > 0$. THEN a^{-1} EXISTS BY $*$.

CASE 1. $a^{-1} = 0 \Rightarrow a \cdot a^{-1} = a \cdot 0 = 0$. BUT $a \cdot a^{-1} = 1 \neq 0$.

CASE 2. $a^{-1} < 0$. THEN $-a^{-1} > 0$ AND $(-a^{-1})a \overset{*}{=} ((-1)a^{-1})a =$
$\overset{*}{=} (-1)(a^{-1}a) \overset{*}{=} (-1) \cdot 1 \overset{*}{=} -1 < 0$. CONTRAD. TO O3:
$a > 0$, $-a^{-1} > 0 \Rightarrow (-a^{-1})a > 0$.

THUS $a^{-1} > 0$.

[3] b) $1^{-1} = 1$. (You can use the Theorem on uniqueness of the inverse.)

$1 \cdot 1^{-1} = 1$ BY $*$. $1 \cdot 1 = 1$ BY $*$ $\Rightarrow 1^{-1} = 1$

[3] 3.. a) Define the infimum of a set A that is bounded from below.

$\inf A = i$ iff

1. $i \le a, \ \forall a \in A$

2. IF $b > i$, $\exists a \in A$ SUCH THAT $b > a$.

[7] b) Let $A = \{ x : |x-1| < x \}$. Find the interval form of A and the infimum and the supremum of A if they exist. Use a) and show all of your work.

IF $x-1 \ge 0$ i.e. $x \ge 1$, THEN $x-1 < x$, $-1 < 0$. So $(x \ge 1)$; $[1, \infty)$ UNION
OR
IF $x-1 < 0$, i.e. $x < 1$, THEN $1-x < x$, $1 < 2x$, $x > \frac{1}{2}$. So: $(\frac{1}{2}, 1)$

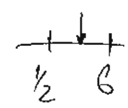
Thus: $A = (\frac{1}{2}, \infty) = \{x : x \in \mathbb{R}, \ x > \frac{1}{2}\}$

sup A DOES NOT EXIST, SINCE A is NOT BDD: $(\forall M > 0 \ ; \ M+1 \in A)$...

Claim: $\inf A = \frac{1}{2}$. PROOF: (i) $x \in A \Rightarrow x > \frac{1}{2} \Rightarrow \frac{1}{2}$ is LOWER BOUND.

(ii) IF $b > \frac{1}{2}$, THEN $\exists r \in \mathbb{Q}$ s.t. $b > r > \frac{1}{2}$. BUT THEN $r \in A$ AND SO b is NOT AN UPPER BOUND.

(or take $x = \frac{b + \frac{1}{2}}{2} \in A$; since $\frac{1}{2} < \frac{b+\frac{1}{2}}{2} < b$)



[4] **Bonus question:** **c)** Define a supremum of a set A that is bounded above and prove that if A and B are nonempty subsets of $\mathbb{R}$, B is bounded and $A \subseteq B$, then A is also bounded and $\sup A \le \sup B$.

A is bdd above by sup B, since $\forall a \in A$, $a \in B$ and so $a \le \sup B$.

Let $\sup A = a$; $\sup B = b$.

Suppose $b < a$. Since $a = \sup A$ AND $b < a$, $\exists x \in A$ such that $b < x$. But $A \subseteq B$ AND SO $x \in B$.

CONTRADICTION: $b < x$ and $x \le b$, $\forall x \in B$.

[2] 4. a) State the Nested Intervals Theorem.

LET $I_n = [a_n, b_n]$, $a_n \leq b_n$ AND $I_{n+1} \subseteq I_n$, $\forall n \in \mathbb{N}$.

THEN $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

[2] b) State the Archimedean Property.

$\forall x \in \mathbb{R}$, $\exists\, n_x \in \mathbb{N}$ SUCH THAT $x < n_x$.

[6] c) Let $I_n = (0, \frac{1}{n}]$. Prove that $\bigcap_{n=1}^{\infty} I_n = \varnothing$, by using b). (Hint:Use proof by contradiction.)

SUPPOSE $x \in \bigcap_{n=1}^{\infty} I_n$. THEN $x \in I_n = (0, \frac{1}{n}]$, $\forall n \in \mathbb{N}$. i.e.

$0 < x < \frac{1}{n}$, $\forall n \in \mathbb{N}$. BY ARCH. PROPERTY, FOR $\frac{1}{x}$, $\exists\, n_x \in \mathbb{N}$

s.t. $\frac{1}{x} < n_x$, i.e. $\frac{1}{n_x} < x$. CONTRADICTION: $x \notin I_{n_x}$.

THUS $\bigcap_{n=1}^{\infty} I_n = \phi$

[2] d) Does c) contradict a) ? Explain why.

NO. $(0, \frac{1}{n}]$ IS NOT CLOSED.

5. Let (x_n) be a sequence of real numbers.

a) State the definition of $\lim_{n\to\infty} x_n = x$.

[2]

$\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq N$ we have $|x_n - x| < \varepsilon$.

b) Prove, by using the definition of limit, that for any b in $\mathbb{R}$, $\lim_{n\to\infty} \frac{b}{n} = 0$.

[5]

LET $\varepsilon > 0$ (WANT $|\frac{b}{n} - 0| = \frac{|b|}{n} \overset{?}{<} \varepsilon$.)

(NOTE: $b = 0 \Rightarrow x_n = \frac{0}{n} = 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} 0 = 0$)

WHENEVER $n > \frac{|b|}{\varepsilon}$ $(\varepsilon > 0)$; $\frac{|b|}{n} < \varepsilon$. So: TAKE $N(\varepsilon) = \left[\frac{|b|}{\varepsilon}\right] + 1$.

DONE.

c) Show that $\lim_{n\to\infty} \frac{2\cos n^3}{n} = 0$, by using b). (You can also use any theorem on limits proven in class, and the properties of the cosx function. State all theorems that you are using.)

[4]

$-1 \leq \cos n^3 \leq 1$, $-2 \leq 2\cos n^3 \leq 2$,

$$-\frac{2}{n} \leq \frac{2\cos n^3}{n} \leq \frac{2}{n}$$

Squ. Thm. plus $\lim \frac{-2}{n} = 0 = \lim \frac{2}{n}$

$$\Rightarrow \lim \frac{2\cos n^3}{n} = 0.$$

(Sq. Thm: LET $x_n \leq y_n \leq z_n$, $\forall n \in \mathbb{N}$ AND $\lim x_n = l = \lim z_n$. Then $\lim_{n\to\infty} y_n$ exists and $\lim y_n = l$.)