1. Let f be a function and let A and B be subsets of the domain of f.

(a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(b) Give an example of f, A and B such that $f(A) \cap f(B) \not\subseteq f(A \cap B)$.

(c) Must $f^{-1}(f(A))$ be always equal to A ? If yes, justify your answer; if no, give a counter-example.

2. (a) Define when is a function $f: A \to B$ an injection.

(b) Suppose $f: A \to B$ is an injection. Prove by using mathematical induction that if A has n elements then $f(A)$ must have n elements too.

(c) Let $f: \mathbf{Q} \to B$ be a surjection. Which of the three "sizes" can B have: finite, countable infinite, uncountable ?

3. Let $F(+,\bullet)$ be an ordered field with unit $\hat{1}$ and zero $\hat{0}$ (note that $\hat{1} \neq \hat{0}$).

(a) Prove that $\hat{1} > \hat{0}$.

(b) Prove that for every a in F, there exists b in F such that a < b.

(c) Can an ordered field be finite? Explain.

4. (a) State the definition of a supremum of a bounded set.

(b) State the Archimedean property.

(c) Let $A = \{ 3 - \frac{1}{2n}; n \text{ in } \mathbf{N}\}$. Find $\inf A$ and $\sup A$. Prove the claim about the supremum of A by using (a) and (b).

5. (a) State the definition of the limit of a sequence.

(b) Prove by using the definition that $\lim_{n \to \infty} \frac{2n+3}{3n-1} = \frac{2}{3}$.

6. (a) State the definition of $\lim_{x \to c} f(x) = L$ (where $f: A \to \mathbf{R}$ and c is a cluster point of A.)

(b) Prove by using (a) that $\lim_{x \to 1} \sqrt{2x-1} = 1$.

7. (a) State the sequential criteria for continuity of $f(x)$ at a point c (where $f: A \to \mathbf{R}$ and c is in A).

(b) Let $f: \mathbf{R} \to \mathbf{R}$ be continuous on $\mathbf{R}$ and such that $f(r) = 3$ for all rational r. Prove by using (a) that $f(x) = 3$ for all x in $\mathbf{R}$.

8. (a) State Max-Min Theorem for continuous functions.

(b) Give an example of a function f defined on the interval (1,2) but such that f does not attain its maximal value on (1, 2).

(c) Give an example of a function f bounded on the interval [1,2] but such that f does not attain its maximal value on [1, 2].

(d) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and bounded on $\mathbf{R}$ by a constant M. Could there exist a sequence (x_n) such that $\lim_{n\to\infty} f(x_n) = M$ but still f does not attain its maximal value on $\mathbf{R}$? Explain !

9. (a) State the Bolzano Intermediate Value Theorem.

(b) Let f be a continuous function on [0,2] with $f(0) = f(2)$. Show that there is an x in [0,1] such that $f(x) = f(x+1)$. [Hint: look at $g(x) = f(x) - f(x+1)$.]

(c) Let f be continuous on (a, b], $f^{-1}(\{2\}) = \varnothing$ and $f(\frac{a+b}{2}) = 3$. Can $\lim_{n\to\infty} f(b - \frac{1}{n}) = 1$? Explain.

10.(a) State the definition of a function f being uniformly continuous on a set A.

(b) Let f be uniformly continuous on an interval (a, b), Let (x_n) be a Cauchy sequence such that x_n is in (a, b) for all n. Prove that $(f(x_n))$ is also a Cauchy sequence.

(c) Give an example of a function f that is continuous on (0,1), and an example of a Cauchy sequence (x_n) from the interval (0,1) such that $(f(x_n))$ is not a Cauchy sequence.