MATH 2202, Assignment No. 3
November 9, 2009

The assignment is due Monday, November 16, in class. Late assignments receive a mark zero.

1. a) Using the **definition of the limit** of a sequence prove that $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$. [5]
   b) Prove that if $\lim\limits_{n\to\infty} x_n = 0$ and the sequence (y_n) is bounded, then $\lim\limits_{n\to\infty} x_n y_n = 0$. [5]
   c) Give examples of sequences (x_n) and (y_n) such that $\lim\limits_{n\to\infty} x_n = x \neq 0$, (y_n) is bounded, but $\lim\limits_{n\to\infty} x_n y_n$ does not exist. [2]

2. Determine if (x_n) converges or diverges: (**Explain** why by showing all of your work and using theorems done in class.)
   a) $x_n = \frac{\sin 2n}{n} + \frac{n^2}{1-n^2}$. [5]
   b) $x_n = (-1)^n \sqrt{n}(\sqrt{n+1} - \sqrt{n})$. [5]
   c) $x_n = \frac{2^n}{n}$. [5]

   (**Do not** use theorems on limits not done in class, which you might know from other courses, such as for example "L'Hospital's rule".)

3. a) Let (x_n) be a bounded sequence and let $s = \sup\{ x_n : n \text{ in } \mathbb{N} \}$. Show that if s is not in $\{ x_n : n \text{ in } \mathbb{N} \}$, then there is a subsequence of (x_n) that converges to s. [6]
   b) Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim\limits_{k\to\infty} \frac{5}{x_{n_k}} = 0$. [6]

4. a) Show, by using the **definition of a Cauchy sequence**, that if (x_n) and (y_n) are Cauchy sequences, then (x_n+y_n) and $(x_n y_n)$ are also Cauchy sequences. [6]
   b) Let (x_n) be a Cauchy sequence such that x_n is an integer, for all n in $\mathbb{N}$. Show that (x_n) is ultimately constant, i.e. $\exists N \in \mathbb{N}$ such that $x_n = x_N, \forall n \geq N$. [3]

5. Let (x_n) and (y_n) be sequences of positive numbers and suppose that $\lim(x_n / y_n) = 0$.
   a) Show that if $\exists N \in \mathbb{N}, \exists c > 0$, such that $x_n \geq c, \forall n \geq N$, then $\lim (y_n) = \infty$. [4]
   b) Use a) to show that if $\lim(x_n) = \infty$, then $\lim (y_n) = \infty$. [3]

Total [55/54]