MATH 2202
Final exam, 3 h

December 16, 2008

Instructions: Attempt all questions. The total value of all questions is 110/100. Values of individual questions are printed beside the statement of the question. If you need more space use the reverse side of the page, but indicate clearly that you are doing so. There are two blank pages at the end of the test for you to use as scrap paper.
Please fill in the information requested below.

Name (print)___________________________

Student No.___________________________

Signature______________________________

QUESTION #	*MARKS*
1	/10
2	/12
3	/7
4	/10
5	/12
6	/12
7	/14
8	/5
9	/10
10	/18
TOTAL	/100

Value

1. Let f: A $\rightarrow$ B and let H be a subset of the co-domain B.

a) Give the definitions of $f^{-1}(H)$ and f a surjection.

[2]

b) Prove that if f is surjective, then $f(f^{-1}(H)) = H$.

[6]

c) Give an example of f , A, B and H such that $H \not\subset f(f^{-1}(H))$.

[2]

2. Let $F(+,\bullet)$ be an ordered field with zero $\hat{0}$ and identity $\hat{1}$.
Prove, by using the field and order axioms, that for x, y, z in F:

a) $x \bullet y=\hat{0}$ if and only if $x=\hat{0}$ or $y=\hat{0}$. (You can use that $z\bullet \hat{0}=\hat{0}$, for every z in F.)
[6]

b) Let x^2 be defined as $x \bullet x$. Prove that $x^2+y^2=\hat{0}$ if and only if $x=\hat{0}$ and $y=\hat{0}$.
[6] (You can use part a) and the fact that if $x\neq \hat{0}$, then $x^2 \in \mathbb{P}$.)

3. Prove, by using the Principal of Mathematical Induction, that every non-empty finite subset of $\mathbb{R}$ has a greatest member (maximum). (Namely,
[7] if A = $\{a_1, a_2, \ldots, a_n\}$, then there exists a_i in A such that $a_i \geq a_j$, for j =1, 2, …n. You can use that if a $\geq$ b and b $\geq$c then a $\geq$ c.)

4. a) Define the infimum of a set A that is bounded from below.

[2]

b) State the Archimedean property.

[2]

c) Show, by using the **definition** of infimum and part b), that the infimum of the set A = $\left\{ \dfrac{3-2x}{x} : x \geq 1 \right\}$ is equal to –2.

[6]

5. a) State the definition of limit of a sequence.
[3]

b) Prove by using the definition that $\lim_{n\to\infty}\frac{3n}{2n+1}=\frac{3}{2}$.
[5]

Bonus Question: c) Prove that if $\lim_{n\to\infty}x_n=0$ and (y_n) is bounded, then $\lim_{n\to\infty}x_ny_n=0$.
[4]

6. a) State the Nested Intervals theorem.
[2]

b) Give an example of a sequence of nested, bounded, open intervals I_n
[2] such that $\bigcap_{n=1}^{\infty}I_n=\varnothing$.

c) Give an example of a sequence of nested, unbounded, closed intervals I_n
[2] such that $\bigcap_{n=1}^{\infty}I_n=\varnothing$. (Note: [a, ∞) is a closed interval.)

d) State the Bolzano-Weierstrass theorem.
[2]

e) Give an example of an unbounded sequence that has a bounded subsequence.
[2]

f) Give an example of a bounded sequence that does not converge.
[2]

7. (a) State the definition of $\lim_{x \to c} f(x) = L$ (where $f : A \to \mathbb{R}$ and c is a cluster point of A.)
[3]

b) Prove by using the **definition** that $\lim_{x \to -1} \frac{x+2}{x-3} = \frac{-1}{4}$.
[6]

Bonus question:

c) b) Let f: A$\to \mathbb{R}$ be nonnegative on A, and let f be continuous at c. Prove by using the **definition** of limit that then $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$,
[5]

8. Let f, g : $\mathbb{R} \to \mathbb{R}$ and let h = f+g and k = f–g. Use the Theorem on limits of combination of functions to prove that if h and k are continuous at c, then f and g are also continuous at c.
[5]

9. a) Define when is f uniformly continuous on A.
[3]

b) Prove by using the definition that $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on A=$[1, \infty)$.
[7]

10. Let f be continuous on (–2, 2] and such that $\lim_{x \to -2} f(x)$ does not exist. Answer the following questions and **explain your answer. State the Theorems, if you are using any.**

a) Does $\lim_{n \to \infty} f((2 - \frac{1}{n})^2 - 3)$ exist? If yes, is it equal to $\lim_{n \to \infty} f\left(\frac{\sin \frac{1}{n}}{\frac{1}{n}}\right)$?

[4]

b) If $f(-1) = -f(1)$ can the range of f equal to (3, ∞)?

[3]

c) Can the image of [–1, 2] under f be equal to [0, 3) ?

[2]

d) Is f uniformly continuous on (–2, 2)? Is it uniformly continuous on [–1, 2]?

[4]

e) If (x_n) is a Cauchy sequence in [–1, 2], must $(f(x_n))$ also be Cauchy?

[2]

f) Give an example of f as above and (x_n) in (–2, 2] such that (x_n) is a Cauchy sequence but $f(x_n)$ is not Cauchy.

[3]