

Values

- Let f be a function and let A and B be subsets of the domain of f .
 - Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$.
 - Give an example of f , A and B such that $f(A) \cap f(B) \not\subseteq f(A \cap B)$.
 - Must $f^{-1}(f(A))$ be always equal to A ? If yes, justify your answer; if no, give a counter-example.
- Define when is a function $f: A \rightarrow B$ an injection.
 - Suppose $f: A \rightarrow B$ is an injection. Prove by using mathematical induction that if A has n elements then $f(A)$ must have n elements too.
 - Let $f: \mathbf{Q} \rightarrow B$ be a surjection. Which of the three "sizes" can B have: finite, countable infinite, uncountable?
- Let $F(+, \bullet)$ be an ordered field with unit $\hat{1}$ and zero $\hat{0}$ (note that $\hat{1} \neq \hat{0}$).
 - Prove that $\hat{1} > \hat{0}$.
 - Prove that for every a in F , there exists b in F such that $a < b$.
 - Can an ordered field be finite?
- State the definition of a supremum of a bounded set.
 - State the Archimedean property.
 - Let $A = \{ 3 - \frac{1}{2n}; n \in \mathbf{N} \}$. Find $\inf A$ and $\sup A$. Prove the claim about the supremum of A by using (a) and (b).
- State the definition of the limit of a sequence.
 - Prove by using the definition that $\lim_{n \rightarrow \infty} \frac{2n+3}{3n-1} = \frac{2}{3}$.
- State the definition of $\lim_{x \rightarrow c} f(x) = L$ (where $f: A \rightarrow \mathbf{R}$ and c is a cluster point of A .)
 - Prove by using (a) that $\lim_{x \rightarrow 1} \sqrt{2x-1} = 1$
- State the sequential criteria for continuity of $f(x)$ at a point c (where $f: A \rightarrow \mathbf{R}$ and c is in A).
 - Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be continuous on \mathbf{R} and such that $f(r) = 3$ for all rational r . Prove by using (a) that $f(x) = 3$ for all x in \mathbf{R} .
- State Max-Min Theorem for continuous functions.
 - Give an example of a function f defined on the interval $(1, 2)$ but such that f does not attain its maximal value on $(1, 2)$.
 - Give an example of a function f bounded on the interval $[1, 2]$ but such that f does not attain its maximal value on $[1, 2]$.

(d) Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be continuous and bounded on \mathbf{R} by a constant M . Could there exist a sequence (x_n) such that $\lim_{n \rightarrow \infty} f(x_n) = M$ but still f does not attain its maximal value on \mathbf{R} ? Explain !

9. (a) State the Bolzano Intermediate Value Theorem.

(b) Let f be a continuous function on $[0,2]$ with $f(0) = f(2)$. Show that there is an x in $[0, 1]$ such that $f(x) = f(x+1)$. [Hint: look at $g(x) = f(x) - f(x+1)$.]

(c) Let f be continuous on $(a, b]$, $f^{-1}(\{2\}) = \emptyset$ and $f(\frac{a+b}{2}) = 3$. Can

$\lim_{n \rightarrow \infty} f(b - \frac{1}{n}) = 1$? Explain.

10.(a) State the definition of a function f being uniformly continuous on a set A .

(b) Let f be uniformly continuous on an interval (a,b) , Let (x_n) be a Cauchy sequence such that x_n is in (a,b) for all n . Prove that $(f(x_n))$ is also a Cauchy sequence.

(c) Give an example of a function f that is continuous on $(0,1)$, and an example of a Cauchy sequence (x_n) from the interval $(0,1)$ such that $(f(x_n))$ is not a Cauchy sequence.