MATH 2202
Test 1

50 min October 28, 2009

Instructions: The total value of all questions is 39(+2). Values of individual questions are printed beside the statement of the question. If you need more space, use the reverse side of the page, but indicate clearly that you are doing so.
There are two blank pages at the end of the test for you to use as scrap paper.
Please fill in the information requested below.

Name (print)______________________________

Student No.______________________________

Signature______________________________

QUESTION#	MARK
1	/6
2	/8
3	/7
4	/10
5	/8
TOTAL	/39

Value

1. Let f: A $\rightarrow$ B and let G and H be subsets of B.

a) State the definition of $f^{-1}(G)$.

[1]

$f^{-1}(G) = \{x \in A : f(x) \in G\}$

b) Prove that if $G \subseteq H$, then $f^{-1}(G) \subseteq f^{-1}(H)$.

[3]

LET $G \subseteq H$ AND LET $x \in f^{-1}(G)$. (WE WANT $x \in f^{-1}(H)$)

THEN $f(x) \in G$ AND $G \subseteq H$, I.E. $f(x) \in H$.

BUT THEN $x \in f^{-1}(H)$ AND WE ARE DONE.

c) Give an example of f, domain A, codomain B, and sets G and H such that $G \subseteq H$, $f^{-1}(G) = f^{-1}(H)$, but $G \neq H$.

[2]

$f(x) = 1$, $A = B = \mathbb{R}$, $G = \{1, 7\}$, $H = \{1\}$

$f^{-1}(G) = \mathbb{R} = f^{-1}(H)$, BUT $G \not\subseteq H$.

2. a) Prove that for every a and b in $\mathbb{R}$, $-(a + b) = (-a) + (-b)$. State which field axioms or theorems you are using in each step.

[3]

$-(a+b) \overset{\text{THM.}}{=} (-1)(a+b) \overset{\text{F}}{=} (-1)a + (-1)b \overset{\text{THM}}{=} -a - b$

b) Prove that if $a > 0$, then $a^{-1} > 0$. State the order and field axioms, or the theorems that you are using in each step.

[5]

LET $a > 0$. PROOF BY CONTRADICTION: SUPPOSE $a^{-1} \leq 0$:

Case 1: If $a^{-1} = 0$, THEN $a \cdot a^{-1} = a \cdot 0 \overset{\text{THM.}}{=} 0$. BUT $a \cdot a^{-1} \overset{(*)}{=} 1$ AND $1 \neq 0$ (CONTRADICTION)

[2] 3. a) Define when is a function f: A $\to$B an injection and when is it a surjection.

f IS AN INJECTION IF $\forall x_1, x_2 \in A$, IF $x_1 \neq x_2$, THEN $f(x_1) \neq f(x_2)$
(OR: IF $f(x_1) = f(x_2)$, THEN $x_1 = x_2$).

f IS A SURJECTION IF $\forall b \in B$, $\exists a \in A$ SUCH THAT $b = f(a)$.

[5] b) Prove in details that the set A = { $3n^2 + \sqrt{2}$: n in $\mathbb{N}$ } is denumerable (i.e. infinite countable). (You have to find a bijection f between $\mathbb{N}$ and A and show that that f is a bijection.)

$f: \mathbb{N} \to A$ DEFINED BY $f(n) = 3n^2 + \sqrt{2}$

(i) f IS AN INJECTION: IF $f(n_1) = f(n_2)$, THEN $3n_1^2 + \sqrt{2} = 3n_2^2 + \sqrt{2}$, I.E.
$3n_1^2 = 3n_2^2$, I.E. $n_1^2 = n_2^2$, I.E. $|n_1| = |n_2|$. BUT $n_1 > 0$, $n_2 > 0$, AND SO $n_1 = n_2$.

(ii) f IS A SURJECTION: IF $a \in A$, THEN $\exists n_a \in \mathbb{N}$ SUCH THAT
$a = 3n_a^2 + \sqrt{2}$. BUT THEN $f(n_a) = a$.

[4] 4. a) Prove, by using the Principal of Mathematical Induction, that $\frac{1}{2^n} < \frac{1}{n}$, for every n in $\mathbb{N}$.

(i) $P(1)$: $\frac{1}{2} < 1$. TRUE

(ii) $P(k)$: $\frac{1}{2^k} \overset{(*)}{<} \frac{1}{k}$. CHECK IF $P(k)$ TRUE $\Rightarrow$ $P(k+1)$ IS TRUE.

$P(k+1)$: $\frac{1}{2^{k+1}} \overset{?}{<} \frac{1}{k+1}$. BUT $\frac{1}{2^{k+1}} = \frac{1}{2^k} \cdot \frac{1}{2} \overset{(*)}{<} \frac{1}{k} \cdot \frac{1}{2} = \frac{1}{2k}$

SINCE $k \geq 1$: $2k = k + k \geq k+1$, $\frac{1}{2k} \leq \frac{1}{k+1}$, AND SO
$\frac{1}{2^{k+1}} < \frac{1}{k+1}$, I.E. $P(k+1)$ IS TRUE.

THEN: $P(n)$ IS TRUE $\forall n \in \mathbb{N}$

c) Show that the infimum of the set A = $\{\frac{1}{2^n} : n \in \mathbb{N}\}$ is 0. (Hint: use a corollary of the Archimedean property and part a).)

[4]

$\inf A \overset{?}{=} 0$

(i) $0 < \frac{1}{2^n} = \frac{1}{2} \cdots \frac{1}{2}$, SINCE $\frac{1}{2} > 0$ (APPLY O2 $(n-1)$ MANY TIMES)

(ii) LET $0 < b$. By COR. TO ARCH. PROPERTY: $\exists n_b$ s.t. $0 < \frac{1}{n_b} < b$.

By PART a): $0 < \frac{1}{2^{n_b}} < \frac{1}{n_b} < b$, i.e. $\frac{1}{2^{n_b}} < b$; $\frac{1}{2^{n_b}} \in A$.

d) **BONUS QUESTION**: Find the set A = $\bigcap_{n=1}^{\infty} I_n$, with $I_n = [-1, \frac{1}{2^n}]$, by using c)

[2] and the conclusion in the proof of the Nested Intervals Theorem as done in class.

$\bigcap_{n=1}^{\infty} I_n = [\sup A, \inf B]$, $A = \{-1\}$, $\sup A = -1$

$B = \{\frac{1}{2^n} : n \in \mathbb{N}\}$, $\inf B = 0$ FROM c)

So: $A = [-1, 0]$.

5. a) State the definition of $\lim_{n\to\infty} x_n = x$.

[2]

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$: $|x_n - x| < \varepsilon$.

c) Prove that the sequence with $x_n = n$ diverges. (Hint: use proof by contradiction, part b) and the Archimedean Theorem.)

[4]

PROOF BY CONTRADICTION:

SUPPOSE $\lim_{n\to\infty} x_n = x$. BY b): (x_n) IS BOUNDED, I.E.

$\exists M > 0$ S.T. $|x_n| \leq M, \forall n \in \mathbb{N}$.

BUT $|x_n| = n \leq M$, $\forall n \in \mathbb{N}$ IS CONTRADICTION TO THE

ARCHIMEDEAN THM: $0 < M \Rightarrow \exists n_M \in \mathbb{N}$ S.T. $0 < M < n_M$.

SO $\lim_{n\to\infty} x_n$ DOES NOT EXIST, I.E. x_n DIVERGES.