

**Algebraic Properties of  $\mathbf{R}$**  On the set  $\mathbf{R}$  of real numbers there are two binary operations, denoted by  $+$  and  $\cdot$  and called **addition** and **multiplication**, respectively. These operations satisfy the following properties:

- (A1)  $a + b = b + a$  for all  $a, b$  in  $\mathbf{R}$  (*commutative property of addition*);
- (A2)  $(a + b) + c = a + (b + c)$  for all  $a, b, c$  in  $\mathbf{R}$  (*associative property of addition*);
- (A3) there exists an element  $0$  in  $\mathbf{R}$  such that  $0 + a = a$  and  $a + 0 = a$  for all  $a$  in  $\mathbf{R}$  (*existence of a zero element*);
- (A4) for each  $a$  in  $\mathbf{R}$  there exists an element  $-a$  in  $\mathbf{R}$  such that  $a + (-a) = 0$  and  $(-a) + a = 0$  (*existence of negative elements*);
- (M1)  $a \cdot b = b \cdot a$  for all  $a, b$  in  $\mathbf{R}$  (*commutative property of multiplication*);
- (M2)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c$  in  $\mathbf{R}$  (*associative property of multiplication*);
- (M3) there exists an element  $1$  in  $\mathbf{R}$  distinct from  $0$  such that  $1 \cdot a = a$  and  $a \cdot 1 = a$  for all  $a$  in  $\mathbf{R}$  (*existence of a unit element*);
- (M4) for each  $a \neq 0$  in  $\mathbf{R}$  there exists an element  $1/a$  in  $\mathbf{R}$  such that  $a \cdot (1/a) = 1$  and  $(1/a) \cdot a = 1$  (*existence of reciprocals*);
- (D)  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$  and  $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$  for all  $a, b, c$  in  $\mathbf{R}$  (*distributive property of multiplication over addition*).

**The Order Properties of  $\mathbf{R}$**  There is a nonempty subset  $P$  of  $\mathbf{R}$ , called the set of positive real numbers, that satisfies the following properties:

- (i) If  $a, b$  belong to  $P$ , then  $a + b$  belongs to  $P$ .
- (ii) If  $a, b$  belong to  $P$ , then  $ab$  belongs to  $P$ .
- (iii) If  $a$  belongs to  $\mathbf{R}$ , then exactly one of the following holds:

$$a \in P, \quad a = 0, \quad -a \in P.$$

(COMPLETENESS)

**The Supremum Property of  $\mathbf{R}$**  Every nonempty set of real numbers that has an upper bound has a supremum in  $\mathbf{R}$ .