136.323 Metric spaces, Assignment No.2

The assignment is due in class on Tuesday, November 2. Late assignments receive a mark of zero.
Prove all of your statements for each of the questions.

1. Let X be an inner product space and let ||.|| be a function on X defined by $||x|| = \langle x, x \rangle$, for all x in X. Prove that ||.|| defines a norm on X. [6]

2. Let d_∞ and d_1 denote the metrics on C[0,1] as defined in class and let

$$f_n(x) = \begin{cases} nx, & 0 \le x \le \frac{1}{n} \\ 1, & \frac{1}{n} < x \le 1 \end{cases}, \text{ for all n in } \mathbb{N}.$$

a) Prove that the sequence (f_n) converges pointwise on [0,1]. [4]

b) Check if the sequence (f_n) converges in (C[0,1], d_∞) by **using the definition** for convergence. [6]

c) Is the sequence (f_n) Cauchy in (C[0,1], d_∞) ? Explain. [2]

d) **Use the definition** of a Cauchy sequence to check if (f_n) is Cauchy in (C[0,1], d_1). [6]

e) Is (C[0,1], d_1) a closed subset of ($L^1[0,1], d_1$)? Explain. [2]

3. a) Let (X, d) be a metric space and let S be a connected subspace of X. Prove that if C is such that $S \subseteq C \subseteq cl(S)$, then C has to be connected. [6]

b) If $S \subseteq \mathbb{R}$ and S is connected, then S is an interval. (Use the definition of an interval on **page 65** in the textbook.) [5]

Total 37/35