136.323 Metric spaces, Assignment No.1

The assignment is due in class on Tuesday, September 28. Late assignments receive a mark of zero.

Prove all of your statements for each of the questions.

1. Let (X, d) be a metric space. Check if d_1, d_2 and d_3 are metrics on X:
   a) $d_1(x,y) = \min\{1, d(x,y)\}$, [3]
   b) $d_2(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$, [4]
   c) $d_3(x,y) = (d(x,y))^2$. [4]

2. Let (X, d) be a metric space.
   a) Show that for $r > 0$ and a in X, N(a, r) is always open. [4]
   b) For $S \subset X$, S is open if and only if S is a union of open balls. [3]
   c) For a, b in X with $d(a, b) > 2r$, $N(a, r) \cap N(b, r) = \varnothing$. [3]
   d) Is it possible that X has more than one point and the only open sets are X and $\varnothing$? [2]

3. Let $X = C([0, \frac{5}{4}])$ with d the "max' metric.
   a) Find d(f, g) for $f(x)=x^2$ and $g(x) = x$. [3]
   b) Let $f(x) < g(x), \forall x \in [0, \frac{5}{4}]$, and let S = $\{h \in X : f(x) < h(x) < g(x)\}$.
      Is S an open set? [4]
   c) Find S' and cl(S) for S = $\{f \in X : f(\frac{1}{2}) \in (4,5]\}$. [4]
   d) Prove that S = $\{f \in X : \int_0^{\frac{5}{4}} f(x)dx = 0\}$ is closed. [4]

4. Question No.3, page 88 from Kasriel. [6]

5. Let (X, d) be a metric space and let $S \subset X$. A point a in X is a **boundary point** for S if $\forall r > 0$, $N(a, r) \cap S \neq \varnothing$ and $N(a, r) \cap S^c \neq \varnothing$. Let bd(S) be the set of all boundary points of S.
   a) Prove that $cl(S) = S \cup bd(S)$, [3]
   b) S is open if and only if $bd(S) \cap S = \varnothing$. [3]
   c) Find the boundary points of $\mathbb{N}$ and $\mathbb{Q}$ as subsets of $\mathbb{R}$. [2]

Total 52/50