CHAPTER 6 HERMITIAN, ORTHOGONAL, AND UNITARY OPERATORS

In Chapter 4, we saw advantages in using bases consisting of eigenvectors of linear operators in a number of applications. Chapter 5 illustrated the benefit of orthonormal bases. Unfortunately, eigenvectors of linear operators are not usually orthogonal, and vectors in an orthonormal basis are not likely to be eigenvectors of any pertinent linear operator. There are operators, however, for which eigenvectors are orthogonal, and hence it is possible to have a basis that is simultaneously orthonormal and consists of eigenvectors. This chapter introduces some of these operators.

§6.1 Hermitian Operators

When the basis for an n-dimensional real, inner product space is orthonormal, the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be calculated with formula 5.48. If $\mathbf{v}$ not only represents a vector, but also denotes its representation as a column matrix, we can write the inner product as the product of two matrices, one a row matrix and the other a column matrix,

$$(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T\mathbf{v}.$$

If A is an $n \times n$ real matrix, the inner product of $\mathbf{u}$ and the vector $A\mathbf{v}$ is

$$(\mathbf{u}, A\mathbf{v}) = \mathbf{u}^T(A\mathbf{v}) = (\mathbf{u}^TA)\mathbf{v} = (A^T\mathbf{u})^T\mathbf{v} = (A^T\mathbf{u}, \mathbf{v}). \tag{6.1}$$

This result,

$$(\mathbf{u}, A\mathbf{v}) = (A^T\mathbf{u}, \mathbf{v}), \tag{6.2}$$

allows us to move the matrix A from the second term to the first term in the inner product, but it must be replaced by its transpose A^T. A similar result can be derived for complex, inner product spaces. When A is a complex matrix, we can use equation 5.50 to write

$$(\mathbf{u}, A\mathbf{v}) = \overline{\mathbf{u}}^T(A\mathbf{v}) = (\overline{\mathbf{u}}^TA)\mathbf{v} = (A^T\overline{\mathbf{u}})^T\mathbf{v} = \left(\overline{\overline{A}^T\mathbf{u}}\right)^T\mathbf{v} = (\overline{A}^T\mathbf{u}, \mathbf{v}). \tag{6.3}$$

This is the complex counterpart of equation 6.2,

$$(\mathbf{u}, A\mathbf{v}) = (\overline{A}^T\mathbf{u}, \mathbf{v}), \tag{6.4}$$

but this time, we must also take complex conjugates of entries in A^T. We identify this matrix in the following definition.

Definition 6.1 The **Hermitian conjugate** of a complex matrix A is the transpose of its complex conjugate $\overline{A}^T$.

For example, the Hermitian conjugate of

$$\begin{pmatrix} 1+2i & 3 & 3-i \\ 2-3i & 2i & 1+6i \\ 2 & 4-2i & -6+7i \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 1-2i & 2+3i & 2 \\ 3 & -2i & 4+2i \\ 3+i & 1-6i & -6-7i \end{pmatrix}.$$

A real matrix is symmetric if it is equal to its transpose. The complex counterpart of a symmetric matrix is contained in the next definition.

Definition 6.2 A square complex matrix A is said to be **Hermitian** if it is equal to its Hermitian conjugate,

$$A = \overline{A}^T. \tag{6.5}$$

In a Hermitian matrix, entries on opposite sides of the diagonal are complex conjugates, and diagonal entries are real (see Exercise 7). A square real matrix is Hermitian if it is symmetric.

If L is a linear operator on a complex, inner product space V, we can associate a complex matrix A with L so that mapping a vector $\mathbf{v}$ can be written as matrix multiplication,

$$L(\mathbf{v}) = A\mathbf{v}.$$

We use Hermitian matrices to define *Hermitian operators*.

Definition 6.3 A linear operator H on a complex inner product space is said to be **Hermitian** if its matrix is Hermitian.

We use the letter H to represent a Hermitian operator to distinguish that it is Hermitian. The matrix A of a Hermitian operator satisfies equation 6.5, and for such an operator, equation 6.4 implies that

$$\big(\mathbf{u}, H(\mathbf{v})\big) = \big(H(\mathbf{u}), \mathbf{v}\big). \tag{6.6}$$

Hermitian operators are the first of our operators for which eigenvectors are orthogonal. This is proved in the following theorem.

Theorem 6.1 Eigenvalues of a Hermitian operator on an inner product space are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof Let λ and $\mathbf{v}$ be an eigenpair for a Hermitian operator H. Not out of necessity, but to simplify calculations, suppose that $\mathbf{v}$ has length one. Then

$$\begin{aligned}
\lambda &= \lambda(\mathbf{v}, \mathbf{v}) && \text{(the length of } \mathbf{v} \text{ is one)}\\
&= (\mathbf{v}, \lambda\mathbf{v}) && \text{(property 5.19b)}\\
&= \big(\mathbf{v}, H(\mathbf{v})\big) && (\lambda \text{ is an eigenvalue for } H \text{ with eigenvector } \mathbf{v})\\
&= \big(H(\mathbf{v}), \mathbf{v}\big) && \text{(equation 6.6)}\\
&= (\lambda\mathbf{v}, \mathbf{v}) && (\lambda \text{ is an eigenvalue for } H \text{ with eigenvector } \mathbf{v})\\
&= \overline{\lambda}(\mathbf{v}, \mathbf{v}) && \text{(property 5.19a)}\\
&= \overline{\lambda} && \text{(the length of } \mathbf{v} \text{ is one).}
\end{aligned}$$

Hence, λ is real. Now suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to (real) distinct eigenvalues λ_1 and λ_2. Then

$$\begin{aligned}
\lambda_1(\mathbf{v}_1, \mathbf{v}_2) &= (\lambda_1\mathbf{v}_1, \mathbf{v}_2) && (\lambda_1 \text{ is real and property 5.19a})\\
&= \big(H(\mathbf{v}_1), \mathbf{v}_2\big) && (\lambda_1 \text{ is an eigenvalue for } H \text{ with eigenvector } \mathbf{v}_1)\\
&= \big(\mathbf{v}_1, H(\mathbf{v}_2)\big) && \text{(equation 6.6)}\\
&= (\mathbf{v}_1, \lambda_2\mathbf{v}_2) && (\lambda_2 \text{ is an eigenvalue for } H \text{ with eigenvector } \mathbf{v}_2)\\
&= \lambda_2(\mathbf{v}_1, \mathbf{v}_2). && \text{(property 5.19b)}
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, this implies that $(\mathbf{v}_1, \mathbf{v}_2) = 0$, and the eigenvectors are orthogonal.∎

It is important to note that different eigenvectors corresponding to the same eigenvalue are not usually orthogonal. Indeed, if $\mathbf{v}$ is an eigenvector of an operator, then so also is $3\,\mathbf{v}$, and these vectors are certainly not orthogonal. When an eigenvalue has two, or more, linearly independent eigenvectors, it is possible to have orthogonal eigenvectors corresponding to this eigenvalue. Here are some examples to illustrate the theorem and this remark.

Example 6.1 The symmetric matrix corresponding to a Hermitian operator on a real inner product space is

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Show that eigenvalues are real, and eigenvectors corresponding to different eigenvalues are orthogonal.

Solution Eigenvalues are given by

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{pmatrix} = -\lambda(\lambda-3)^2.$$

Thus, $\lambda = 0$ and $\lambda = 3$ are the eigenvalues. Eigenvectors corresponding to $\lambda = 0$ are $v_3(1, -1, 1)$. Eigenvectors corresponding to $\lambda = 3$ are $v_2(1, 1, 0) + v_3(-1, 0, 1)$. If we take the inner product of any eigenvector $\mathbf{v}_1 = v(1, -1, 1)$ corresponding to $\lambda = 0$, and any eigenvector $\mathbf{v}_2 = u(1, 1, 0) + w(-1, 0, 1)$ corresponding to $\lambda = 3$, we obtain

$$(\mathbf{v}_1, \mathbf{v}_2) = \big(v(1, -1, 1), u(1, 1, 0) + w(-1, 0, 1)\big) = vu(1 - 1 + 0) + vw(-1 + 0 + 1) = 0.$$

They are orthogonal. The inner product of the eigenvectors $(1, 1, 0)$ and $(-1, 0, 1)$ corresponding to $\lambda = 3$ is $\big((1, 1, 0), (-1, 0, 1)\big) = -1$. They are not orthogonal. Notice, however, the vector $(1, 1, 0) + 2(-1, 0, 1) = (-1, 1, 2)$ is an eigenvector corresponding to $\lambda = 3$ and it is orthogonal to the eigenvector $(1, 1, 0)$.•

Example 6.2 The Hermitian matrix corresponding to a Hermitian operator on a complex inner product space is

$$A = \begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}.$$

Find its eigenvalues and eigenvectors. Demonstrate orthogonality of eigenvectors corresponding to different eigenvalues.

Solution Eigenvalues are given by

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 0 & i \\ 0 & 1-\lambda & 0 \\ -i & 0 & 2-\lambda \end{pmatrix} = -(\lambda-1)^2(\lambda-3).$$

Thus, $\lambda = 1$ and $\lambda = 3$ are eigenvalues. Eigenvectors corresponding to $\lambda = 1$ are $v_2(0, 1, 0) + v_3(-i, 0, 1)$. Eigenvectors corresponding to $\lambda = 3$ are $v_3(i, 0, 1)$. If we take the inner product of any eigenvector $\mathbf{v} = v_2(0, 1, 0) + v_3(-i, 0, 1)$ corresponding to $\lambda = 1$, and any eigenvector $\mathbf{w} = w_3(i, 0, 1)$ corresponding to $\lambda = 3$, we obtain

$$(\mathbf{v}, \mathbf{w}) = \big(v_2(0, 1, 0) + v_3(-i, 0, 1), w_3(i, 0, 1)\big) = v_2w_3(0) + v_3w_3(-1 + 1) = 0.$$

They are orthogonal.•

Eigenvectors corresponding to eigenvalues with algebraic multiplicity larger than one need not be orthogonal, but from nonorthogonal eigenvectors, the Gram-Schmidt process can be used to construct orthogonal eigenvectors corresponding to the same eigenvalue. Here is an example.

Example 6.3 The symmetric (Hermitian) matrix of an operator on a 3-dimensional space is

$$A = \begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}.$$

Show that A has two eigenvalues one with (algebraic) multiplicity 2. Find orthonormal eigenvectors corresponding to the eigenvalue with algebraic multiplicity 2.

Solution Eigenvalues are given by

$$0 = \det \begin{pmatrix} 5-\lambda & -2 & 4 \\ -2 & 8-\lambda & 2 \\ 4 & 2 & 5-\lambda \end{pmatrix} = -\lambda(\lambda-9)^2.$$

Eigenvalues are $\lambda = 0$ and $\lambda = 9$. Eigenvectors corresponding to $\lambda = 9$ satisfy

$$\mathbf{0} = (A - 9I)\mathbf{v} = \begin{pmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The reduced row echelon form for the augmented matrix of this system is

$$\left(\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

When we convert to equations,

$$v_1 + \frac{v_2}{2} - v_3 = 0.$$

Eigenvectors are therefore

$$\mathbf{v} = \begin{pmatrix} -v_2/2 + v_3 \\ v_2 \\ v_3 \end{pmatrix} = \frac{v_2}{2} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Linearly independent eigenvectors corresponding to $\lambda = 9$ are $\mathbf{v}_1 = (1, -2, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$. We can construct an eigenvector corresponding to $\lambda = 9$ orthogonal to $\mathbf{v}_1$ using Gram-Schmidt,

$$\mathbf{v}_3 = (1, 0, 1) - \frac{\big((1, 0, 1), (1, -2, 0)\big)}{5}(1, -2, 0) = (1, 0, 1) - \frac{1}{5}(1, -2, 0) = \left(\frac{4}{5}, \frac{2}{5}, 1\right).$$

Orthonormal eigenvectors corresponding to $\lambda = 9$ are

$$\hat{\mathbf{v}}_1 = \frac{(1, -2, 0)}{\sqrt{5}} \qquad \text{and} \qquad \hat{\mathbf{v}}_3 = \frac{(4, 2, 5)}{3\sqrt{5}}.\bullet$$

We now know that eigenvectors corresponding to different eigenvalues of a Hermitian operator are orthogonal, and should an eigenvalue with algebraic multiplicity greater than 1 have more than one linearly independent eigenvector, then orthogonal eigenvectors can be constructed from them. The following theorem shows that the geometric multiplicity of every eigenvalue of a Hermitian operator on a finite-dimensional vector space must be equal to its algebraic multiplicity. In other words, the number of linearly independent eigenvectors of a Hermitian operator on a finite-dimensional space must be equal to the dimension of the space.

Theorem 6.2 A Hermitian operator on an n-dimensional vector space has n linearly independent eigenvectors.

Proof We can prove this by showing that the $n \times n$ matrix A of the Hermitian operator is diagonalizable. We do so by mathematical induction. If the space has dimension 1, then A is a 1×1 matrix that is automatically diagonal. Assume that the $k \times k$ matrix A of a Hermitian operator on a k-dimensional space is diagonalizable. Let H be a Hermitian operator on a $(k+1)$-dimensional space. The operator must have at least one eigenvalue, call it λ_1, and let $\hat{\mathbf{v}}_1$ be a corresponding unit eigenvector. Expand $\hat{\mathbf{v}}_1$ to an orthonormal basis $\{\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_{k+1}\}$ for the space. The matrix of a Hermitian operator is equal to its Hermitian conjugate, and therefore if B is the matrix of H with respect to the orthonormal basis, then it is also equal to its Hermitian conjugate. In addition, because the basis is orthonormal, equation 5.46 indicates that entries in the first column are

$$v_{i1} = \big(\hat{\mathbf{v}}_i, H(\hat{\mathbf{v}}_1)\big) = \big(\hat{\mathbf{v}}_i, \lambda_1 \hat{\mathbf{v}}_j\big) = \lambda_1\big(\hat{\mathbf{v}}_i, \hat{\mathbf{v}}_j\big) = 0.$$

Since B is Hermitian, $v_{1i} = \overline{v_{i1}}$, and matrix B must therefore be of the form

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & C \end{pmatrix},$$

where C is a $k \times k$ Hermitian matrix. By the induction hypothesis, C is diagonalizable, with therefore k linearly independent eigenvectors $\mathbf{b}_2$, $\mathbf{b_3}$, ..., $\mathbf{b}_{k+1}$. Thus, B is diagonalizable, and $\mathbf{v}_1$, $(0, \mathbf{b}_2)$, $(0, \mathbf{b_3})$, ..., $(0, \mathbf{b}_{k+1})$ must be $k+1$ linearly independent eigenvectors of H. This verifies the result for all n.∎

Two immediate results of this theorem are the following corollaries.

Corollary 6.2.1 Algebraic and geometric multiplicities of eigenvalues of Hermitian operators are equal.

Corollary 6.2.2 Matrices associated with Hermitian operators are always diagonalizable.

EXERCISES 6.1

In Exercises 1–6 determine whether the matrix is Hermitian.

1. $\begin{pmatrix} 3 & 2-5i \\ 2+5i & -4 \end{pmatrix}$ **2.** $\begin{pmatrix} 3 & 6+5i \\ 6+5i & 1 \end{pmatrix}$ **3.** $\begin{pmatrix} i & 2-3i \\ 2+3i & 41 \end{pmatrix}$

4. $\begin{pmatrix} -2 & 1-2i & i \\ 1+2i & 16 & 4+3i \\ -i & 4-3i & \pi \end{pmatrix}$ **5.** $\begin{pmatrix} 1+i & 2 & -2+5i \\ 2 & -2 & 3i \\ -2-5i & -3i & 4 \end{pmatrix}$ **6.** $\begin{pmatrix} 1 & 2i & 1-3i \\ -2i & 4 & 6-5i \\ 1+3i & 6+5i & 42 \end{pmatrix}$

7. Verify that diagonal entries in a Hermitian matrix must be real.

In Exercises 8–12 find eigenvalues and eigenvectors of the Hermitian operator with given matrix. Verify that eigenvectors corresponding to distinct eigenvalues are orthogonal.

8. $\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$ **9.** $\begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}$ **10.** $\begin{pmatrix} 2 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 2 \end{pmatrix}$

11. $\begin{pmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 0 \end{pmatrix}$ **12.** $\begin{pmatrix} 2\sqrt{2} & -i & i \\ i & 2\sqrt{2} & 0 \\ -i & 0 & 2\sqrt{2} \end{pmatrix}$

Answers

1. Hermitian **2.** Not Hermitian **3.** Not Hermitian **4.** Hermitian **5.** Not Hermitian
6. Hermitian **8.** $\lambda = 0$ with $(-1-i, 1)$; $\lambda = 3$ with $(1+i, 2)$
9. $\lambda = 0$ with $(-2, -1, 2)$; $\lambda = 9$ with $(1, -2, 0)$ and $(0, 2, 1)$
10. $\lambda = 3$ with $(i, 0, 1)$; $\lambda = 1$ with $(0, 1, 0)$ and $(-i, 0, 1)$
11. $\lambda = -1$ with $(-1, 1+2i, 1)$; $\lambda = -2$ with $(1+3i, -2-i, 5)$; $\lambda = 6$ with $(1-21i, 6-9i, 13)$
12. $\lambda = \sqrt{2}$ with $(\sqrt{2}i, 1, -1)$; $\lambda = 2\sqrt{2}$ with $(0, 1, 1)$; $\lambda = 3\sqrt{2}$ with $(\sqrt{2}i, -1, 1)$

§6.2 Orthogonal and Unitary Operators

In this section we study linear operators on real inner product spaces that are called *orthogonal* operators, and their complex counterparts called *unitary* operators.

Orthogonal Operators

Definition 6.4 A linear operator R on a real inner product space is said to be **orthogonal** if it preserves norms of vectors; that is, for every vector $\mathbf{v}$ in the space,

$$\|R(\mathbf{v})\| = \|\mathbf{v}\|. \tag{6.7}$$

According to equation 5.32, the inner product of the space can be expressed in terms of norms, and therefore an orthogonal operator also preserves inner products,

$$\big(R(\mathbf{u}), R(\mathbf{v})\big) = (\mathbf{u}, \mathbf{v}). \tag{6.8}$$

The following theorem characterizes orthogonal operators in terms of their matrices.

Theorem 6.3 If the matrix associated with an orthogonal operator R on an n-dimensional real inner product space, relative to an orthonormal basis of the space, is A, then:

(1) The columns of A are orthonormal vectors.

(2) $A^TA = I$, $AA^T = I \quad \Longleftrightarrow \quad A^{-1} = A^T$

(3) The rows of A are orthonormal vectors.

(4) The determinant of A is ± 1.

Proof (1) Since the columns of A are images of the orthonormal basis vectors, and R preserves inner products and norms, the columns of A must be orthonormal.

(2) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in the space, then

$$(\mathbf{u}, \mathbf{v}) = (A\mathbf{u}, A\mathbf{v}) = (A^TA\mathbf{u}, \mathbf{v}), \qquad \text{(see equation 6.2)}.$$

But for this to be valid for all $\mathbf{u}$ and $\mathbf{v}$, we must have $A^TA = I$, the appropriately sized identity matrix (see Exercise 21 in Section 5.3). This means that the inverse of A is its transpose.

(3) Since the matrix $AA^T = I$ gives the inner product of rows of A, the rows are orthonormal vectors.

(4) This follows by taking determinants of $A^TA = I$, and using the fact that the determinant of A^T is the same as that of A.∎

A matrix whose columns are orthonormal vectors is called an **orthogonal matrix**. As a result, matrices associated with orthogonal operators are orthonormal. We might have expected to call a matrix orthogonal if its columns are orthogonal vectors, and call it orthonormal if its columns are orthonormal vectors. Unfortunately, this is not the common terminology. There is no name for a matrix whose columns are just orthogonal; the columns must be orthonormal for the matrix to be termed orthogonal.

Perhaps the easiest orthogonal operators to visualize are rotations in $\mathcal{G}^2$ and $\mathcal{G}^3$. They preserve lengths of vectors, inner products, and and angles between vectors. For instance, suppose that R is the linear operator on $\mathcal{G}^3$ that rotates vectors through angle θ around the z-axis, counterclockwise as viewed far up the axis. According to Example 2.15 in Section 2.2, the matrix of the operator, relative to the natural basis of $\mathcal{G}^3$ is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Columns are clearly orthonormal vectors.

Reflections of vectors in planes through the origin in $\mathcal{G}^3$ are also orthogonal operators. In Exercise 9 of Section 2.2, it was shown that the matrix for the linear operator that reflects vectors in the plane $Ax + By + Cz = 0$, where A and B are positive constants, is

$$\frac{1}{A^2 + B^2 + C^2}\begin{pmatrix} B^2 + C^2 - A^2 & -2AB & -2AC \\ -2AB & A^2 + C^2 - B^2 & -2BC \\ -2AC & -2BC & A^2 + B^2 - C^2 \end{pmatrix}.$$

Once again, columns are orthonormal vectors.

Example 6.4 Show that every orthogonal operator on $\mathcal{G}^2$ is either a rotation or a reflection.

Solution If A is the 2×2 matrix of an orthogonal operator R on $\mathcal{G}^2$, then its columns are orthonormal vectors. Without loss in generality, we can take the first column to be the vector $(\cos\theta, \sin\theta)^T$. There are then only two possible choices for the second column,

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

The first matrix represents a counterclockwise rotation of vectors through angle θ (see Example 2.15 in Section 2.2). The second matrix represents a reflection of vectors in the line making angle $\theta/2$ with the positive x-axis (see Exercise 12 in Section 2.2).

Unitary Operators

Unitary operators are the analogs in complex vector spaces of orthogonal operators in real spaces.

Definition 6.5 A linear operator U on a complex inner product space is said to be **unitary** if it preserves norms of vectors.

Unitary operators also preserve inner products of vectors. Matrices associated with unitary operators are characterized in the following theorem.

Theorem 6.4 If the matrix associated with an unitary operator U on an n-dimensional complex inner product space, relative to an orthonormal basis of the space, is A, then:

(1) The columns of A are orthonormal vectors.

(2) $\overline{A^T}A = I$, $A\overline{A^T} = I \quad \Longleftrightarrow \quad A^{-1} = \overline{A^T}$

(3) The rows of A are orthonormal vectors.

Proof (1) Since the columns of A are images of the orthonormal basis vectors, and U preserves inner products and norms, the columns of A must be orthonormal.

(2) If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in the space, then

$$(\mathbf{u}, \mathbf{v}) = (A\mathbf{u}, A\mathbf{v}) = (\overline{A}^T A\mathbf{u}, \mathbf{v}), \qquad \text{(see equation 6.4).}$$

But for this to be valid for all $\mathbf{u}$ and $\mathbf{v}$, we must have $\overline{A}^T A = I$, the appropriately sized identity matrix. This means that the inverse of A is its Hermitian conjugate.

(3) Since the matrix $AA^T = I$ gives the inner product of rows of A, the rows are orthonormal vectors.∎

Eigenvalues and Eigenvectors of Orthogonal and Unitary Operators

Like Hermitian operators, eigenvectors of unitary and orthogonal operators are orthogonal. We prove the result for unitary operators and state the result for orthogonal operators as a corollary.

Theorem 6.5 Eigenvalues of a unitary operator U on an n-dimensional complex vector space have magnitude one, and eigenvectors corresponding to distinct eigenvalues are orthogonal. Furthermore, the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, so that the operator has n linearly independent eigenvectors.

Proof If $\mathbf{v}$ is an eigenvector of U corresponding eigenvalue λ, then

$$\begin{aligned}\|\mathbf{v}\| &= \|U(\mathbf{v})\| && \text{(unitary operators preserve lengths)}\\ &= \|\lambda\mathbf{v}\| && \text{($\mathbf{v}$ is an eigenvector)}\\ &= |\lambda|\|\mathbf{v}\|.\end{aligned}$$

This implies that $|\lambda| = 1$, so that eigenvalues have magnitude one. Now suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. Then

$$\begin{aligned}(\mathbf{v}_1, \mathbf{v}_2) &= \big(U(\mathbf{v}_1), U(\mathbf{v}_2)\big) && \text{(unitary operators preserve inner products)}\\ &= (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2) && \text{($\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors)}\\ &= \overline{\lambda_1}\lambda_2(\mathbf{v}_1, \mathbf{v}_2) && \text{(see properties 5.19).}\end{aligned}$$

Since $|\lambda_1| = 1$, it follows that $\overline{\lambda_1} = 1/\lambda_1$, and $\overline{\lambda_1}\lambda_2 = \dfrac{\lambda_2}{\lambda_1} \neq 1$. It follows therefore that $(\mathbf{v}_1, \mathbf{v}_2) = 0$, and the eigenvectors are orthogonal. Verification that U has n linearly independent eigenvectors is similar to that in Theorem 6.2.■

Corollary 6.5.1 Eigenvalues of an orthogonal operator R on an n-dimensional real vector space may be real, or complex (in complex conjugate pairs), but, in either case, eigenvalues have magnitude one. Eigenvectors (which may be real or in complex conjugate pairs) corresponding to distinct eigenvalues are orthogonal. Furthermore, the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, so that the operator has n linearly independent eigenvectors.

EXERCISES 6.2

In Exercises 1–3 verify that the matrix is orthogonal. Find eigenvalues and orthonormal eigenvectors for the associated operator.

1. $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ **2.** $\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$ **3.** $\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$

In Exercises 4–6 verify that the matrix is orthogonal. Find eigenvalues and verify that they have magnitude equal to one. Find a normalized eigenvector corresponding to the real eigenvalue.

4. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ **5.** $\begin{pmatrix} 1/3 & 2\sqrt{2}/3 & 0 \\ 2/3 & -\sqrt{2}/6 & \sqrt{2}/2 \\ -2/3 & \sqrt{2}/6 & \sqrt{2}/2 \end{pmatrix}$ **6.** $\begin{pmatrix} \sqrt{3}/2 & -\sqrt{3}/4 & 1/4 \\ 1/2 & 3/4 & -\sqrt{3}/4 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}$

7. Verify that the following matrix is orthogonal. Find eigenvalues and orthonormal eigenvectors for the associated operator.

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

In Exercises 8–9 verify that the matrix is unitary. Find eigenvalues and orthonormal eigenvectors for the associated operator.

8. $\begin{pmatrix} 2i/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2i/\sqrt{5} \end{pmatrix}$

9. $\begin{pmatrix} (1+i)/2 & (1+i)/2 \\ (-1+i)/2 & (1-i)/2 \end{pmatrix}$

10. Verify that the following matrix is unitary.

$$\begin{pmatrix} 1/\sqrt{6} & (1+i)/2 & 1/\sqrt{3} \\ 2i/\sqrt{6} & 0 & -i/\sqrt{3} \\ 1/\sqrt{6} & -(1+i)/2 & 1/\sqrt{3} \end{pmatrix}$$

11. Prove that the product of two orthogonal matrices is orthogonal. Is this true for unitary matrices?

12. Prove that the inverse of an orthogonal matrix is also orthogonal.

Answers

1. $\lambda = (1+i)/\sqrt{2}$, $\mathbf{v} = (1, i)/\sqrt{2}$; $\lambda = (1-i)/\sqrt{2}$, $\mathbf{v} = (1, -i)/\sqrt{2}$
2. $\lambda = (1+2i)/\sqrt{5}$, $\mathbf{v} = (1, i)/\sqrt{2}$; $\lambda = (1-2i)/\sqrt{5}$, $\mathbf{v} = (1, -i)/\sqrt{2}$
3. $\lambda = (2+i)/\sqrt{5}$, $\mathbf{v} = (2, 1+i)/\sqrt{6}$; $\lambda = -(1+2i)/\sqrt{5}$, $\mathbf{v} = (-1+i, 2)/\sqrt{6}$
4. $\lambda = 1$, $\lambda = (-1 \pm \sqrt{3}i)/2$, $\mathbf{v} = (1, 1, 1)/\sqrt{3}$
5. $\lambda = -1$, $\lambda = \dfrac{1}{6}\left(4 + \sqrt{2} \pm i\sqrt{18 - 8\sqrt{2}}\right)$, $\mathbf{v} = (2+\sqrt{2}, -2-2\sqrt{2}, 2)/\sqrt{22 - 4\sqrt{2}}$
6. $\lambda = 1$, $\lambda = \dfrac{1}{8}\left(4\sqrt{3} - 1 \pm i\sqrt{15 + 8\sqrt{3}}\right)$, $\mathbf{v} = (1, 2-\sqrt{3}, 1)/\sqrt{9 - 4\sqrt{3}}$
7. $\lambda = -1$, $\mathbf{v} = (0, 1, -1, 0)/\sqrt{2}$, $\mathbf{v} = (0, 1, 1, -2)/\sqrt{6}$; $\lambda = (1 \pm \sqrt{3}i)/2$, $\mathbf{v} = (\pm\sqrt{3}i, 1, 1, 1)$
8. $\lambda = (1+2i)/\sqrt{5}$, $\mathbf{v} = (1, 1)/\sqrt{2}$; $\lambda = (-1+2i)/\sqrt{5}$, $\mathbf{v} = (1, -1)/\sqrt{2}$
9. $\lambda = (1+\sqrt{3}i)/2$, $(1+i, (\sqrt{3}-1)i)/\sqrt{6 - 2\sqrt{3}}$; $\lambda = (1-\sqrt{3}i)/2$, $\big(1+i, -(1+\sqrt{3})i\big)/\sqrt{6 + 2\sqrt{3}}$
11. Yes