Abstract

In Euclidean geometry we have the following: Given a line L and a point P not on it, there is exactly one line through P that is parallel to L. It was discovered that assuming this is false produces the equally valid Hyperbolic Geometry, where there are in fact infinitely many lines through P that are parallel to L. This presentation is an introduction to Hyperbolic Geometry, and on modelling it in the Euclidean plane.

Outline

- Introduction to Hyperbolic Geometry
  - Interesting Results
- Metrics and Geodesics
- Hyperbolic Isometries
- Modelling Hyperbolic Geometry
- Program Demonstration
- Tessellations (Time Permitting)

Euclidean Geometry

Euclidean geometry rests upon the following axioms:

- Each pair of points can be joined together by one and only one straight-line segment.
- Any straight-line segment can be indefinitely extended in either direction.
- There is exactly one circle of any given radius with any given center.
- All right angles are congruent to one another, and finally,
- Given a line L and a point P not on it, there is exactly one line through P that is does not intersect L.

The fifth is known as the Parallel Postulate. It has been shown that this fifth axiom is independent of the others. So what if it wasn't there?

Absolute Geometry

The first four axioms by themselves are sufficient to prove many classical theorems true in geometry. Some of these are:

- If two lines intersect, they intersect at only one point.
- On the line L from A to B, there is exactly one point equidistant from A and B, and this point is between A and B.
- The Side-Angle-Side, Angle-Side-Angle, and Side-Side-Side rules for triangles.
- If one angle of a triangle is not acute then the other two angles of the triangle are acute.
- The sum of the lengths of each two sides of a triangle is greater than the length of the third side (Triangle Inequality).

Absolute Geometry continued

It is important to note that some theorems require the parallel postulate to be true, and some do not. This is essentially where Hyperbolic Geometry comes from. When the parallel postulate is true, the five axioms give us the usual Euclidean geometry. When false, they produce two different non-Euclidean geometries: Elliptic (or Projective) and Hyperbolic.

Since Absolute Geometry is based only on the first four axioms, all facts true in Absolute geometry are true in the Euclidean, Hyperbolic, and Elliptic geometries.

The Hyperbolic Axiom

In Hyperbolic Geometry, the Parallel Postulate is replaced with the following axiom:

If point P is not on a line T, then there are at least 2 lines through P that do not intersect T.

Using this axiom, we can prove many theorems that, in Euclidean geometry, are absolute nonsense.

The Elliptic Axiom

If point P is not on a line T, then there are no lines through P that do not intersect T. (i.e. all lines intersect)

Extension of the Hyperbolic Axiom

In fact, it can be shown that when you assume that there are at least 2 lines through P that do not intersect T, it can be proven that in fact there are infinitely many lines through P that do not intersect T.

We can also separate this class of lines into two groups: Hyper Parallel and Disjointly Parallel lines. Hyper parallel lines are those parallel lines that intersect only at infinity–the distance between the lines goes to zero as you move towards infinity. Disjointly parallel lines are all others.

For a list of interesting unintuitive theorems true in Hyperbolic Geometry, see [Kelly]

Metrics and Geodesics

A Geodesic is defined as the curve of shortest length between two points. In Euclidean geometry (the Cartesian Model), Geodesics are of the form $y = mx + b$, the classical definition of a straight line. In order to talk about the definition, we need a well defined distance function: a metric.

Metric

Formally, a Metric is a function d, defined on a set, satisfying:

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

It is a distance function, describing the distance between any two points in a set. In Euclidean Geometry, we have the following metric:

$$d(A, B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$$

Models of Hyperbolic Geometry

It is one thing to develop Theorems in this geometry, but it's another to perform constructions. A Euclidean model in which we could use a compass and straight edge construction would be quite helpful. Over the years, numerous models have been developed, but I will present the most popular ones here.

Poincare (interior to the) Disk Model
This is the space $\{(x, y)|x^2 + y^2 < 1\}$ (all the points within the unit circle) together with the metric

$$ds^2 = 4\left(\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}\right)$$

In this model, the geodesics are the circular arcs orthogonal to the unit circle, combined with the diameters of the circle (which can be viewed as circular arcs with infinite radius). (Note that this seems to be the most popular model for Hyperbolic Geometry–see the artwork of Escher)

Upper Half Plane Model

This is the space $\{(x, y)|y > 0\}$ (the upper half plane) together with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

In this model, the geodesics are the upper half portions of Euclidean circles of the form $(x-b)^2+y^2=r^2$ for some b and some r. We also allow r to be infinite, and get Euclidean vertical lines x = a for any a.

Klein-Beltrami

This is the space $\{(x,y)|x^2+y^2<1\}$ (all the points within the unit circle) together with the metric

$$ds^2 = \frac{dx^2+dy^2}{1-x^2-y^2} + \frac{xdx^2+ydy^2}{(1-x^2-y^2)^2}$$

In this model, the geodesics are the segments of Euclidean straight lines that lie inside the unit circle.

Logarithmic Half Plane Model

This is the space R^2 (the full plane) together with the metric

$$\frac{dx^2 + e^{2y}dy^2}{e^y}$$

This model is directly related to the Half Plane model. In order to help understand Hyperbolic Geometry, we extend the Half Plane model to the full plane by taking the natural logarithm of each point in the upper half plane. This produces geodesics of the form

$$ln\sqrt{r^2 - (x - x_0)^2}$$

where x_0 and r are the center and radius of a related half circle in the upper half plane model.

Quote

[The Upper Half Plane Model] is the hyperbolic plane, as much as anything can be, but we call it a model of the hyperbolic plane because any surface isometric to [The Upper Half Plane Model] is equally entitled to the name.

John Stillwell

Isometries

An Isometry is a distance preserving mapping.

In the Euclidean plane, there are four kinds of isometries - reflection, translation, rotation and glide reflection. These are the classic "movements" of the Euclidean plane. Similarly, there are four kinds of isometries in hyperbolic space: circle inversion (hyperbolic reflection), hyperbolic isometry (hyperbolic translations), the parabolic isometry (rotations at infinity) and the elliptic isometry (hyperbolic rotation).

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