Abstract

The seven following theorems, while seemingly unrelated, are equivalent (i.e., any one of them may be proved by assuming any other is true). These theorems relate to graph theory, set theory, flow theory, and even marriage: Menger's theorem (1929), König's theorem for matrices (1931), the König-Egerváry theorem (1931), Hall's marriage theorem (1935), the Birkhoff-Von Neumann theorem (1946), Dilworth's theorem (1950) and the Max Flow-Min Cut theorem (1962). I will attempt to explain each theorem, and give some indications why all are equivalent.

Graph Theory

A *graph* can be defined as a set of points, called *vertices* or *nodes*, and a set of 2-element sets of points, called *edges*.

We say a graph is *bipartite* if it's vertices can be partitioned into two disjoint sets such that all edges in the graph go from one set to the other. A *covering* of the edges in a graph is a set C of vertices such that each edge of the graph contains at least one vertex of C.

König's Theorem

Let G be a bipartite graph. The size of a maximum matching of G is equal to the size of a minimum covering of G.

Menger's Theorem

Let G be a graph, v and w vertices in G, and S be a subset of vertices. S is a *vw-separating set* if there is no path from v to w in $G \setminus S$. Two paths from v to w are *vertex-disjoint* if they only have v and w in common.

Menger's Theorem (1929)

The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices v and w is equal to the minimum number of vertices in a vw-separating set.

Flow Theory

A *directed graph* is a graph in which each edge has associated with it a direction. A *network* is a directed graph with two distinct vertices singled out as the *source* s and the *target* t, and with each edge assigned an integer called it's *capacity*.

Let S be a set of vertices, and S' it's complement. We define an *edge cut* [S, S'] as all the edges directed from S to S'. A *minimum edge cut* is one such that the sum of the capacities of the edges in [S, S'] is a minimum.

The easiest example to see would be a system of pipes, with water flowing from one source to one target where the capacities are the diameter of the pipes. You can also look at any kind of product distribution line as an example of a network.

Flow Theory cont., and the Max-Flow Min-Cut theorem

The *flow* on an edge uv is f(u,v), a positive integer such that f(u,v) $\leq$ the capacity of uv, and $f^{+}(v) = f^{-}(v)$ (the out flow from v equals the inflow to v). We will say a *flow* in the network N is a valid assignment of flows on all the edges. The value of a flow in N is $f^{+}(s) - f^{-}(s)$. A *maximum flow* is one whose value is maximum.

The Max-Flow Min-Cut theorem

A value of a maximum flow in a network N is equal to the value of a minimum cut of N.

(0,1)-Matrices and the König-Egerváry theorem

The *Term Rank* of a (0,1)-matrix is the largest number of 1s that can be chosen from the matrix such that no 2 selected 1s lie on the same line. A set S of rows and columns is a *cover* of a (0,1)-matrix if the matrix becomes a zero matrix after all the lines in S have been deleted.

König-Egerváry theorem (1931)

The term rank of a (0,1)-matrix is the cardinality of its smallest cover.

The Birkhoff-Von Neumann Theorem

A matrix D $= (d_{ij})$ with real non-negative entries is *Doubly Stochastic* if the sum of the entries in any row and any column equals 1. A *Permutation Matrix* is a doubly stochastic matrix with entries 0 and 1, that is, a matrix with exactly one 1 in each row and in each column. A matrix A is a *convex combination* of matrices $A_1, ..., A_s$ if there exist non-negative reals $\lambda_1, ..., \lambda_s$ such that $A = \sum_{i=1}^{s} \lambda_i A_i$ and $\sum_{i=1}^{s} \lambda_i = 1$.

Birkhoff-Von Neumann Theorem (1946)

Any doubly stochastic matrix can be written as a convex combination permutation matrices.

Hall's Theorem

Let X be a set of elements. Let S = $\{S_1, ..., S_n\}$ be a family of subsets of X. Then, a *Sequence of Distinct Representatives (SDR) for S* is a sequence $\{x_1, ..., x_n\}$ of pairwise distinct elements of X such that $x_i \in S_i, 1 \leq i \leq n$

Hall's Theorem

S has an SDR if and only if the union of any k members of S contains at least k elements.

Example:
X = {1, 2, 3, 4, 5}
S_1 = {1, 2, 3}, S_2 = {1, 4, 5}, S_3 = {3, 5}
SDR: {1, 4, 5}

The Marriage Theorem

This was the original motivation for Hall's Theorem:

Given a set of n men and a set of n women, let each man make a list of the women he is willing to marry. Then each man can be married to a woman on his list if, and only if, for every value of k ($1 \leq k \leq n$), the union of any k of the lists contain at least k names.

Similarly, we can apply this theorem to the Assignment Problem:

Given a set of n employees, fill out a list of the jobs each of them would be able to preform. Then, we can give each person a job suited to their abilities if, and only if, for every value of k ($1 \leq k \leq n$), the union of any k of the lists contains at least k jobs.

Partially Ordered Sets and Dilworth's Theorem

We define a *Partially Ordered Set*, or a *Poset*, as a set P with a partial ordering $\leq$ defined on it's elements. I.e, for any two elements x and y of P, either x $\leq$ y, y $\leq$ x (x and y are comparable), or x || y (x and y are incomparable). A *Chain* is any pairwise comparable subset of P, and an *Antichain* is any pairwise incomparable subset of P.

Dilworth's Theorem

Suppose that the largest antichain in the poset P has size r. Then P can be partitioned into r disjoint chains.

The König-Egerváry theorem $\Leftrightarrow$ König's Theorem

Let the bipartite graph G have bipartitions $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_m\}$. Construct the m*n adjacency matrix A with A_{ij} =1 if, and only if, there is an edge joining x_i to y_j. Now, the term rank of A is the cardinality of a maximum matching in G, and the size of a smallest cover of A is the size of a smallest covering of the edges of G. So, we have that the König-Egerváry Theorem $\Rightarrow$ König's Theorem. Conversely, since any (0,1)-matrix can be interpreted as the adjacency matrix of some bipartite graph, König's Theorem $\Rightarrow$ the König-Egerváry Theorem.

Hall's Theorem $\Rightarrow$ The König-Egerváry Theorem

Sketch: Let B be an m*n (0,1)-matrix, p = term rank of B, q = cardinality of smallest cover.

Claim: $p \leq q$. Removing r rows and s columns ($r + s = q$) removes all the ones from the matrix. Thus, there are at most $r + s$ ones in different rows and columns. Thus $p \leq q$.

Hall's Theorem $\Rightarrow$ The König-Egerváry Theorem (cont.)

Claim: p $\geq$ q. Permute the rows and columns of B (as figure). Let $A_i = \{j : j > s$ and $B_{ij} = 1\}$. Note $|A_i| > 0$ for all i. Also, the union of any k of the A_i's has at least k elements. If not, we would be able to replace k rows of the smallest cover with less than k columns. Thus, by Hall's theorem, we have r ones in the top right, each in it's own row and column.

Similarly, we get s ones in the bottom left, each in it's own row and column. So, p $\geq$ r + s = q. So, p = q.

Dilworth's Theorem $\Rightarrow$ Hall's Theorem

Sketch: Let $S_1, S_2, ..., S_n$ be the subsets of $\{x_1, ..., x_m\}$

Assume that the union of any k sets has at least k elements. Let $X = \{S_1, S_2, ..., S_n, x_1, x_2, ..., x_m\}$ and define a partial order on X such that $x_i < S_j$ iff $x_i \in A_j$, and there are no other comparable elements.

The xi's form an antichain of length m, and it can be shown this is the largest antichain. So, by Dilworth's theorem, we can partition X into m chains, n of which have 2 elements, and the rest have 1. The 2 element chains define an SDR.

Konig's Theorem $\Rightarrow$ Hall's Theorem

Sketch: Let $S = \{S_1, S_2, ..., S_n\}$ be a set of subsets of $X = \{x_1, x_2, ..., x_m\}$ such that the union of any k members of S contains at least k elements. Construct a bipartite graph G with bipartitions X and Y $= \{1, ..., n\}$ in which $\{x_i, j\}$ is an edge if, and only if, $x_i \in S_j$.

Let $K \subseteq Y$. Define N(K) to be the set of vertices in X such that each vertex in N(K) is joined to at least 1 vertex in K. Equivalently, $N(K) = \bigcup_{i \in K} S_i$. By assumption, $|N(K)| \geq |K|$ (1). Now, we need to show that there is a matching of Y into X.

Konig's Theorem $\Rightarrow$ Hall's Theorem (cont.)

By König's Theorem, a complete matching from Y into X exists if and only if the cardinality of every covering of the edges in this graph is at least n.

Let C be a covering of G. Let $C_y = C \bigcap Y$. Then, $N(X \setminus C_y)$ is a subset of $C \setminus C_y$ (2). Thus,

$$\begin{aligned} |C| &= |C_y| + |C \setminus C_y| \\ &\geq |C_y| + |N(X \setminus C_y)| \text{ (by (2))} \\ &\geq |C_y| + |X \setminus C_y| \text{ (by (1))} = \text{m} \end{aligned}$$

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