Some Interesting Infinite Sequences of Natural Numbers

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Abstract

For any positive integer n, let $f_5(n)$ denote the integer obtained by reversing the digits of $n+5$. In 1959, M. W. Gorzkowski proved that if $n = 10^{2k+3} + 10^{k+1} + 1$ where $k = 0, 1, 2, ...$, then the sequence

$$n, f_5(n), f_5(f_5(n)), ...$$

is a purely periodic sequence with period $36 * 10^k$. In this talk, I will present a proof by W. Sierpinski (based on Gorzkowski's idea), as well as a slightly stronger theorem for which the same proof works (which may be contained in the original literature).

Notation

Throughout this seminar, n will always be a natural number ($n \in \mathbb{Z} \cup \{0\}$), and $s \in \mathbb{Z}$. We have already defined $f_5(n)$, but in general, let $f_s(n)$ denote the integer obtained by reversing the digits of $n + s$, and let $f_s^r(n)$ denote $\underbrace{(f_s \circ f_s \circ \ldots f_s)}_{\text{r times}}(n)$.

In this seminar, we will be interested in sequences of the form:

$$S_s(n) = (n, f_s(n), f_s^2(n), \ldots)$$

How Do These Sequences Behave?

Example: for n=0, s=4: $0, 4, 8, 21, 52, 65, 96, \underline{???}$

Example: for n=0, s=5: $0, 5, 1, 6, 11, 61, 66, \underline{???}$

Example: for n=1, s=6: $1, 7, 31, 73, 97, 301, 703, 907, \underline{???}$

Are these sequences monotonic, or eventually monotonic? Are they convergent? Are they periodic? If so, how long is the period? If not, are they bounded at all? These sequences have been studied extensively by W. Sierpinski and others during the mid 20th century.

Interesting Observations

- $f_s(n)$ is never one to one. Example:

$$2 = f_5(15) = f_5(195) = f_5(1995) = f_5(199...95)$$

- $f_s(n)$ is never onto: Because we write 01 as 1, $f_s(n) = 10$ has no solutions.

Question 1: Convergent, Periodic, or Unbounded?

While the sequence $S_4(0) = (0, 4, 8, 21, 52, 65, 96, ...)$ looks monotonic increasing, it's not. In fact, it's periodic with period 55.

0, 4, 8, 21, 52, 65, 96, **1**, 5, 9, 31, 53, 75, 97, 101, 501, 505, 905, 909, 319, 323, 723, 727, 137, 141, 541, 545, 945, 949, 359, 363, 763, 767, 177, 181, 581, 585, 985, 989, 399, 304, 803, 708, 217, 122, 621, 526, 35, 93, 79, 38, 24, 82, 6 8, 27, 13, 71, 57, 16, 2, 6, **1**

Here are some examples of sequences: one that converges and one that is unbounded:

- $S_9(30)$: $30, 93, 201, 12, 12, 12, ...$

- $S_{10}(0)$: 0, 1, 11, 12, 22, 23, 33, 34, 44, 45, 55, 56, 66, 67, 77, 78, 88, 89, 99, 901, 119, 921, 139, 941, 159, 961, 179, 981, 199, 902, 219, 922, 239, 942, 259, 962, 279, 982, 299, 903, 319, 923, 339, 943, 359, 963, 379, 983, 399, 904, 419, 924, 439, 944, 459, 964, 479, 984, 499, 905, 519, 925, 539, 945, 559, 965, 579, 985, 599, 906, 619, 926, 639, 946, 659, 966, 679, 986, 699, 907, 719, 927, 739, 947, 759, 967, 779, 987, 799, 908, 819, 928, 839, 948, 859, 968, 879, 988, 899, 909, 919, 929, 939, 949, 959, 969, 979, 989, 999, 9001,...

In General:

Theorem 1 *For s = 3, 7, 9, or 11, and for any* $n < 100$, $S_s(n)$ *is periodic.*

Schinzel (1959) proved this (essentially) by exhaustion. He then used this in the following theorem:

Theorem 2 *For s = 3, 7, 9, or 11 and for any n, there occurs in the sequence* $S_s(n)$ *a number less than 100.*

From this it follows that the set of all periods of all such sequences ($s \in \{3, 7, 9, 11\}$) is finite.

J. Browkin in 1959 provided a strengthening of Theorem 2:

Theorem 3 *Let* $n \in \mathbb{N}$, $s \in \mathbb{Z}^+$ *such that* $gcd(s, 10) = 1$. *Then* $S_s(n)$ *is eventually periodic.*

Theorem 4 *For* $s \in \mathbb{Z}^+$ *such that* $gcd(s, 10) = 10$, *there exists* $n \in \mathbb{N}$ *such that* $S_s(n)$ *is not periodic.*

Q2: If $S_s(n)$ is periodic, how long is the period?

Theorem 5 *If $n = 10^{2k+3} + 10^{k+1} + 1$ where $k = 0, 1, 2, ...,$ then the sequence*

$$n, f_5(n), f_5^2(n), ...$$

is a purely periodic sequence with period $36 * 10^k$.

Proof?

Theorem 6 *If $s \neq 1$ and $s|10$, then the length of the period for a suitable n can be arbitrarily large.*

Question 3: How Many Different Cycles Are There?

We have already seen the following: Let $s = 3$, 7, 9, or 11, and let n be a positive integer. Then the set of all periods of $S_s(n)$ is finite. But how many are there? What about the other values for s? What happens in general when $gcd(s, 10) \neq 1$ and $\neq 10$?

My Idea:

Define an equivalence relation on the positive integers as follows: $a \equiv b$ iff $S_s(a)$ and $S_s(b)$ are eventually the same sequence (i.e., $S_s^r(a) = S_s^t(b)$ for some $r, t \in \mathbb{Z}^+$). I then wrote a java program to calculate the equivalence classes of $S_s(n)$ for some range of n, their sizes, and the sizes of the cycles therein.

Example: $S_3(n)$, $n \in \{0, ..., 1000000\}$

```
Number of classes found:  10
Class 1=<9>, size=9, Cycle Size=6
Class 2=<1>, size=1000013, Cycle Size=3
Class 3=<8>, size=8, Cycle Size=6
Class 4=<12>, size=12, Cycle Size=6
Class 5=<13>, size=6, Cycle Size=6
Class 6=<15>, size=6, Cycle Size=3
Class 7=<16>, size=6, Cycle Size=6
Class 8=<19>, size=7, Cycle Size=6
Class 9=<23>, size=17, Cycle Size=6
Class 10=<26>, size=6, Cycle Size=3
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Example: $S_4(n)$, $n \in \{0, ..., 1000000\}$

```
Number of classes found:  496
Class 1=<1>, size=1500071, Cycle Size=54
Class 2=<1011>, size=3917, Cycle Size=90
Class 3=<1013>, size=93, Cycle Size=90
Class 4=<1015>, size=97, Cycle Size=90
...
Class 22=<1051>, size=99, Cycle Size=90
Class 23=<1053>, size=93, Cycle Size=90
Class 24=<100011>, size=349907, Cycle Size=1890
Class 25=<100013>, size=1893, Cycle Size=1890
Class 26=<100015>, size=1897, Cycle Size=1890
...
Class 44=<100051>, size=1899, Cycle Size=1890
Class 45=<100053>, size=1893, Cycle Size=1890
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Class 46=<100101>, size=572, Cycle Size=450
Class 47=<100103>, size=453, Cycle Size=450
Class 48=<100105>, size=457, Cycle Size=450
...
Class 269=<100547>, size=451, Cycle Size=450
Class 270=<100549>, size=230, Cycle Size=225
Class 271=<100551>, size=459, Cycle Size=450
Class 272=<100555>, size=460, Cycle Size=450
Class 273=<100559>, size=456, Cycle Size=450
Class 274=<100563>, size=462, Cycle Size=450
Class 275=<100567>, size=458, Cycle Size=450
Class 276=<100571>, size=464, Cycle Size=450
Class 277=<100575>, size=460, Cycle Size=450
Class 278=<100579>, size=456, Cycle Size=450
Class 279=<100583>, size=462, Cycle Size=450
Class 280=<100587>, size=458, Cycle Size=450
...
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