Graph Reconstruction

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Abstract

The concept of a graph is one of the most basic and readily understood mathematical concepts, and the **Reconstruction Conjecture** is one of the most engaging problems under the domain of Graph Theory. The conjecture proposes that every graph with at least three vertices can be uniquely reconstructed given the multiset of subgraphs produced by deleting each vertex of the original graph one by one. This conjecture has been proven true for several infinite classes of graphs, but the general case remains unsolved. In this talk, I will outline the problem, and introduce some of the most well known results.

Graph Theory

- $V(G)$ = the set of vertices of G.
- $E(G)$ = the set of edges of G.

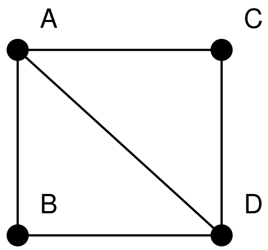
Definitions

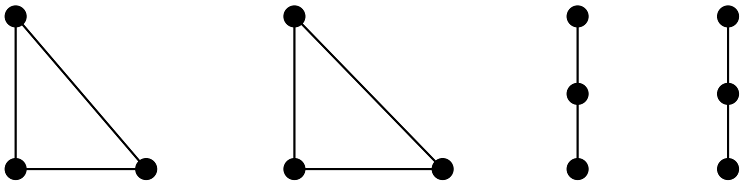
- For $v \in V(G)$, the **degree of** v, $deg(v)$, is the number of edges connected to v.
- A graph is **regular** if all it's vertices have the same degree.
- K_n denotes the **complete** graph on n vertices (every possible edge).
- A **card** of G is an <u>unlabelled</u> graph formed by deleting 1 vertex and all edges attached to it.
- The **deck** of G, $\mathscr{D}(G)$, is the collection of all G's cards. Note this is in general a multiset.

Example

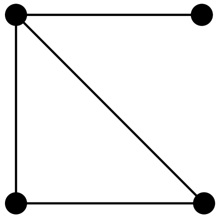
- A graph is **reconstructible** if, given it's deck, we can uniquely determine what the original graph was.
- Equivalently, G is not reconstructible if and only if there is an H such that $\mathscr{D}(G) = \mathscr{D}(H)$.

Example





- This graph is reconstructible.
- The last three cards are also in the deck of



- (these can be checked by exhaustion).

The Reconstruction Conjecture

- **Reconstruction Conjecture:**
  Every graph with at least 3 vertices is reconstructible.

- We need at least three vertices:

- Note: the reconstruction conjecture is false if and only if there exist two graphs with the same deck.

Reconstructible Properties

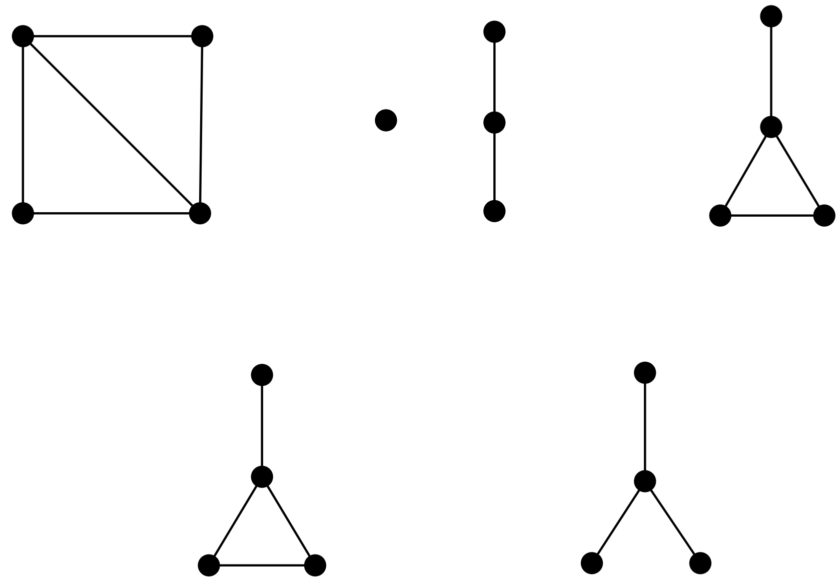
- Some properties of G are easily determined from $\mathscr{D}(G)$:
  - $|V(G)|$ = # of cards in $\mathscr{D}(G)$.
  - $|E(G)|$: Each edge in G is missing from exactly two cards. Thus each edge is in exactly $n-2$ cards, and we have

$$|E(G)| = \frac{1}{n-2} \sum_{v \in V(G)} |E(G-v)|.$$

Further...

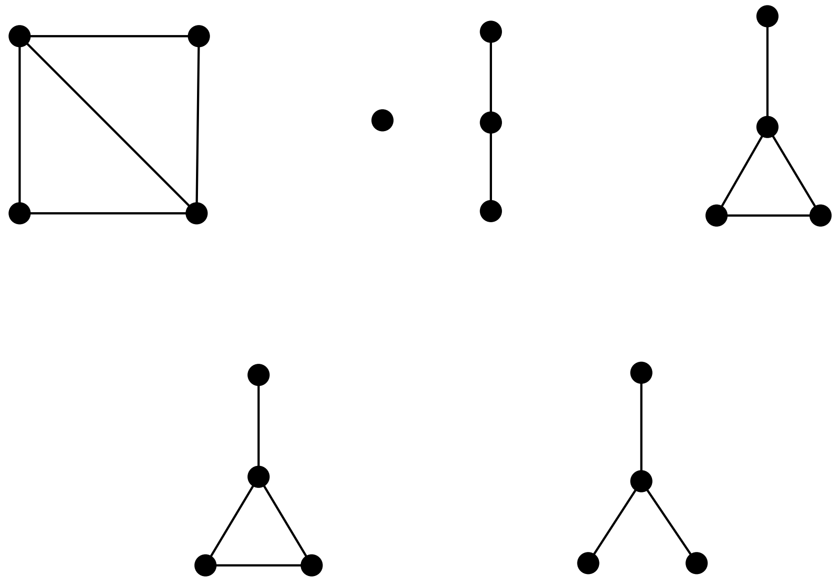
- Since $|E(G)|$ can be determined from $\mathscr{D}(G)$, we can also determine the degree of v for all $v \in V(G)$.
- $\implies$ We can determine the whole *degree sequence* of G.

Reconstruction Example



- $|V(G)| = 5$.
- $|E(G)| = 6$.
- Thus the degrees of the missing vertices in order are 1,4,2,2,3.

Reconstruction Example



- Note the degree of the vertex deleted in the second graph is 4.
- Thus it must have been connected to every other vertex:

Recognizable Properties

- Let G have some property. This property is called **recognizable** if whenever we have a graph H with the same deck as G, H must also have the same property.
- Since we can determine the degree sequence uniquely given a deck, regular graphs are recognizable.

Class of reconstructible graphs

Theorem(Kelly, 1957) Regular graphs are reconstructible.

Proof.

- Let G be regular, and construct $\mathscr{D}(G)$.
- By previous slide, the regularity of G is recognizable.
- Let d be the degree of each vertex.
- Add a vertex to any card.
- Connect new vertex to each of the d vertices in the card with degree $d-1$, and G is reconstructed uniquely.

Kelly's Lemma

- $s(F, G)$ = # of subgraphs of G isomorphic to F.
- **Kelly's Theorem:** For all F with less vertices than G,

$$s(F,G) = \frac{1}{(|V(G)| - |V(F)|)} \sum_{v \in V(G)} s(F, G - v).$$

Proof of Kelly's Theorem:

Double count ordered pairs of the form

$$(H, G - v)$$

where $H \cong F$, and $H \subset G - v$.

A Nice Consequence of Kelly's Lemma

- Let $F = K_2$ (edges). Then $s(K_2, G) = |E(G)|$, and...

Even more

- However, Kelly's Lemma gives us even more: $s(F, G)$ **for any** F.
- Thus if there are two graphs with the same deck, they must have the same:
  - number of vertices.
  - number of edges.
  - number of subgraphs of EVERY type (same $s(F, G)$ for all F with $|V(F)| < |V(G)|$).

Thoughts on the Conjecture

- So maybe one would think then that all graphs are reconstructible, and the conjecture is true?
- The conjecture has been confirmed after all for all graphs with less than 11 vertices (12.3 million graphs).
- However, there are good reasons to believe it false...

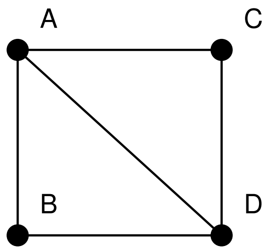
False Reconstruction Conjectures

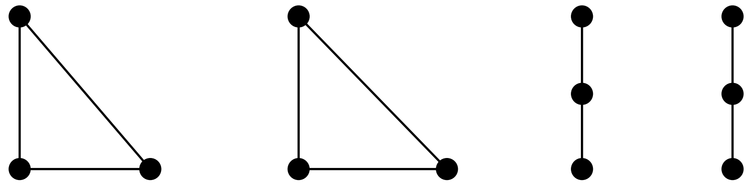
- Digraphs are not reconstructible in general.
  - There are arbitrarily large counterexamples. (Stockmeyer, 1977/1981)
- Hypergraphs are not, in general, reconstructible.
  - There is an infinite family of counterexamples. (Kocay, 1987)
- Infinite graphs are not, in general, reconstructible.

Reconstruction Numbers

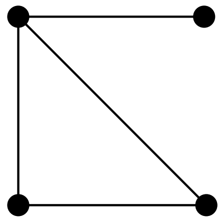
- Related Topic: **Graph Reconstruction Numbers.**
- The **existential reconstruction number** of G, $\exists rn(G)$, is the minimum number of cards in $\mathscr{D}(G)$ required to reconstruct G uniquely.
- The **universal reconstruction number** of G, $\forall rn(G)$, is the minimum number n such that all multisubsets of $\mathscr{D}(G)$ of size n uniquely reconstruct G.
- Almost all graphs have $\exists rn(G) = 3$. (Bollobas, 1990).

Example





- The last three cards in this deck are as well in the deck of



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- However, the first three graphs DO determine the graph G uniquely.
- Thus, for this G, $\exists rn(G) = 3$, $\forall rn(G) = 4$.

Reconstruction Numbers

- Found for all graphs on at most 10 vertices:

$\exists rn$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
3	4	8	34	150	1044	12,334	274,666	12,005,156
4		3		4		8		6
5				2		2	2	4
6						2		
7								2

- Very hard to get any further: there are 1,018,997,864 graphs on 11 vertices!

Other Related Problems

- **The Legitimate Deck Problem:** Given a multiset of graphs $\mathscr{G}$, does there exist a graph H such that $\mathscr{D}(H) = \mathscr{G}$?
- **The problem of k-reconstruction:** Instead of deleting 1 vertex to form each card, delete k vertices. Then determine $f(k)$, if it exists, such that all graphs with greater than $f(k)$ vertices are k-reconstructible.
  - The Graph Reconstruction Conjecture says $f(1) = 2$.

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