Mathematics of Eventown and Oddtown

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- The 32 citizens of each town love forming clubs.
- The mayors impose club-forming rules:
  - Every pair of clubs has an even number of common members.
  - Every club must have even/odd membership.
- **Question:** How many distinct clubs could possibly be made in each town?

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Every club must have an even number of people	For all i, $\|C_i\|$ is even.

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Under this situation, the number of possible distinct non-empty clubs is $2^{16} - 1 = 65535$. $\square$

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- The surprising result is the following:
- **Thm.** There is no way to form more than 32 clubs under the rules of Oddtown.

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- Pf on overhead.

Generalizations

Let X be a set, $n = |X|$.

Oddtown Thm. (Berlekamp 1969)

$\mathcal{F} \subseteq \mathcal{P}(X)$ s.t.

- $\forall A \in \mathcal{F}$, $|A|$ is odd, and
- $\forall A, B \in \mathcal{F}$, $A \neq B$, $|A \cap B|$ is even.

Then $|\mathcal{F}| \leq n$.

This bound is best possible.

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Eventown Thm. (Berlekamp 1969, Graver 1975)

$\mathcal{F} \subseteq \mathcal{P}(X)$ s.t. $\forall A, B \in \mathcal{F}$, $|A \cap B|$ is even. Then

$$|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor} + \begin{cases} 1 \text{ if } n \text{ is odd,} \\ 0 \text{ if } n \text{ is even.} \end{cases}$$

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Generalizations

Fisher's Inequality (Fisher 1940)

$\ell, k \in \mathbb{Z}^+$, $\mathcal{F} \subseteq \mathcal{P}(X)$ s.t. $\forall A, B \in \mathcal{F}$, $A \neq B$,

- $|A| = |B| = \ell$
- $|A \cap B| = k$.

Then $|\mathcal{F}| \leq n$.

(this is related to BIBDs.)

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Nonuniform Fisher's Inequality. (Majumdar 1953)

$k \in \mathbb{Z}^+$, $\mathcal{F} \subseteq \mathcal{P}(X)$ s.t.

$$\forall A, B \in \mathcal{F},\ A \neq B,\ |A \cap B| = k.$$

Then $|\mathcal{F}| \leq n$.

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Generalizations

Theorem. (Frankl and Wilson 1981)

$L \subseteq \mathbb{Z}^+ \cup \{0\}$, $\mathcal{F} \subseteq \mathcal{P}(X)$ s.t.

$$\forall A, B \in \mathcal{F},\ A \neq B,\ |A \cap B| \in L.$$

Then

$$|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}.$$

This bound is best possible.