Extremal Graph Theory

Can be viewed as the study of how graph constants ensure certain properties

Graph Constant	$\Longrightarrow$	Property
order		contains K_3
size		contains K_m
connectivity		contains G
min degree		contains a cycle
max degree		is r-colourable
$\chi(G)$		is k-partite
diameter		

For a first example,

Theorem (Mantel's Theorem (1905)).
For $n \geq 3$, if a graph G with n vertices has more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, then G contains a triangle (a subgraph isomorphic to K_3).

Note this is best possible, since the graph

$$K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$$

has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, and does not contain a triangle.

In general, for any $n \in \mathbb{Z}^+$, and any graph G, let

$$ex(n; G)$$

denote the largest integer e (if any exists) such that there exists a graph on n vertices with e edges not containing a subgraph isomorphic to G.

The number $ex(n; G)$ is called an **extremal number** (or sometimes a **Turan number**, for reasons we will see shortly).

A graph F on n vertices is called an **extremal graph for** G if F contains no copy of G, and $|E(F)| = ex(n; G)$.

Fix a graph G, and $n, e \in \mathbb{Z}^+$.

To show that $ex(n; G) \geq e$, it is necessary and sufficient to show there exists a graph F on n vertices with e edges such that F does not contain a copy of G.

To show that $ex(n; G) < e$, it is necessary and sufficient to show that EVERY graph with n vertices and e edges contains a copy of G (or equivalently that if a graph F on n vertices does not contain a copy of G, then $|E(F)| < e$).

For any $n, k \in \mathbb{Z}^+$, let $T(n, k)$ denote the k-partite graph with n vertices where the partite sets are as equal in size as possible.

For example, $T(12, 5) = \ldots$

Note that if $n < k$, then $T(n, k) = K_n$.

Let $t(n,k) = |E(T(n,k))|$.

Proposition.

$$t(n,k) = \binom{k}{2} + (n-k)(k-1) + t(n-k,k).$$

Theorem (Turán's Theorem (1941/1955)).
Let $n, k \in \mathbb{Z}^+$. Then $ex(n; K_{k+1}) = t(n,k)$ and $T(n,k)$ is the unique extremal K_{k+1}-free graph.

Proof...

The extremal graph for K_3 is bipartite. If we look at non-bipartite graphs, we can shave off many edges.

Theorem (Erdős-Gallai (1962)). For any non-bipartite graph G with n vertices, if

$$E(G) \geq ex(n-1; K_3) + 2 = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2,$$

then G contains a triangle.

Theorem (not sure who did this). Let V be a set with $|V| = 2n$, and let G be a graph with vertex set V. Then G contains a bipartite subgraph H in which each partite set has size n, and

$$|E(H)| \geq \frac{1}{4}|E(G)|.$$

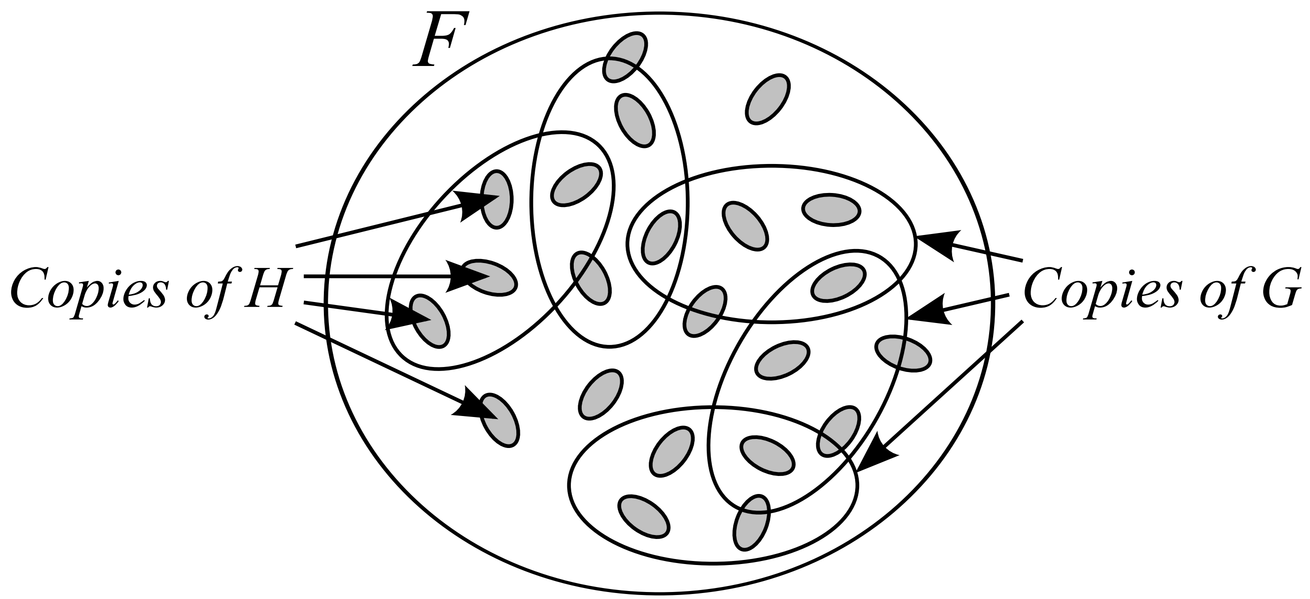
Generalizing to hypergraphs...

Theorem. Let V be a set with $|V| = rn$, and let G be an r-uniform hypergraph with vertex set V. Then G contains a r-partite r-uniform subhypergraph H in which each partite set has size n, and

$$|E(H)| \geq \frac{r!}{r^r}|E(G)|.$$

Ramsey Theory

Can be viewed as the study of "structure" under partition (or colouring).



"Complete disorder is impossible" — T. S. Motzkin

"Of three ordinary people, two must have the same sex."
—D. J. Kleitman.

The concept of a partition, or a colouring, is crucial to Ramsey theory.

If S and C are sets with $|C| = r \geq 2$, any function $f : S \longrightarrow C$ is an r-*colouring* of S, and the elements of C are called *colours*. For each $i \in C$, $f^{-1}(i) \subseteq S$ is called a *colour class*, and any subset of a colour class is said to be *monochromatic*. For example, if $S = \{a, b, c\}$, $C = \{\text{red}, \text{blue}\}$, and $f(a) = f(c) =$ red, and $f(b) =$ blue, then the colour classes are $\{a, c\}$ and $\{b\}$.

The pigeonhole principle is a basic tool in Ramsey theory, and is itself now considered the simplest "Ramsey-type" theorem.

Theorem (Pigeonhole principle).
If at least $mr+1$ objects are partitioned into r (possibly empty) subsets, at least one subset contains $m+1$ elements.

The pigeonhole principle can be restated in various ways, *e.g.*:

(a) If S is a set with $|S| \geq mr+1$, and $S = S_1 \cup \cdots \cup S_r$ is a partition of S, then $\exists\, i \in [1,r]$ such that $|S_i| \geq m+1$.

(b) If $f : [1, mr+1] \longrightarrow [1,r]$, then $\exists\, i \in [1,r]$ such that $|f^{-1}(i)| \geq m+1$.

Theorem (Infinite pigeonhole principle).
$\forall$ finite colouring of an infinite set, there exists a colour class which is infinite.

Two theorems due to Ramsey generalize the finite and infinite versions of the pigeonhole principle to the colouring of k-sets, rather than just singletons.

Theorem (Ramsey's theorem for finite sets, 1930). $\forall\ m, k, r \in \mathbb{Z}^+$, $\exists$ a least $n = R_k(m;r) \in \mathbb{Z}^+$ such that $\forall$ n-set N, and $\forall$ r-colouring $f : [N]^k \longrightarrow [1, r]$ of the k-subsets of N, $\exists$ $M \in [N]^m$ such that $[M]^k$ is monochromatic.

Theorem (Ramsey's theorem for infinite sets, 1930). $\forall\ k, r \in \mathbb{Z}^+$, every infinite set X, and every $f : [X]^k \longrightarrow [1, r]$, $\exists$ an infinite set $Y \subseteq X$ such that $[Y]^k$ is monochromatic.

When $k = 1$, the two versions of Ramsey's theorem are exactly the two versions of the pigeonhole principle.

To show that $R_k(m;r) > x$, it suffices to exhibit an r-colouring of the k-tuples of an x-set containing no m-set M such that $[M]^k$ is monochromatic.

To show that $R_k(m;r) \leq x$, it suffices to prove that for every r-colouring of the k-tuples of an x-set, there is an m-set M such that $[M]^k$ is monochromatic.

It is a fairly quick proof (a good exercise) to see that if Ramsey's theorem is true for $r = 2$, then it is true for all r. So we will prove it in the case $r = 2$.

In order to prove Ramsey's theorem, we will introduce an off-diagonal version. Let $R_k(a, b)$ be the least integer n such that $\forall$ n-set N, and $\forall$ r-colouring $f : [N]^k \longrightarrow \{\text{red}, \text{blue}\}$ of the k-subsets of N, $\exists$ either $M_1 \in [N]^a$ such that $[M_1]^k$ is monochromatic red, or a $M_2 \in [N]^b$ such that $[M_2]^k$ is monochromatic blue.

Note that:

$$R_k(a, b) \leq R_k(\max\{a, b\}; 2),$$
$$R_k(m; 2) = R_k(m, m).$$

An "arrow notation" (known as a "Ramsey arrow") is used to simplify statements that are similar to Ramsey's theorem. (introduced by Erdős and Rado in 1953).

For positive integers n, m, k and r, write

$$n \longrightarrow (m)_r^k$$

if $\forall$ n-set N, and $\forall$ $f : [N]^k \to [1, r]$, $\exists$ $M \in [N]^m$ such that $[M]^k$ is monochromatic.

Ramsey-type theorems can be hard to read and understand at first due to the number of alternating quantifiers. For example, there are a total of four quantification switches in Ramsey's theorem (stated below for colouring the integers):

$$\begin{aligned}&\forall m, k, r \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \text{ s.t. } \forall f : [1, n]^k \to [1, r],\\ &\qquad \exists i \in [1, r] \text{ and } S \in [1, n]^m \text{ s.t. } \forall S' \in [S]^k, f(S') = i.\end{aligned}$$

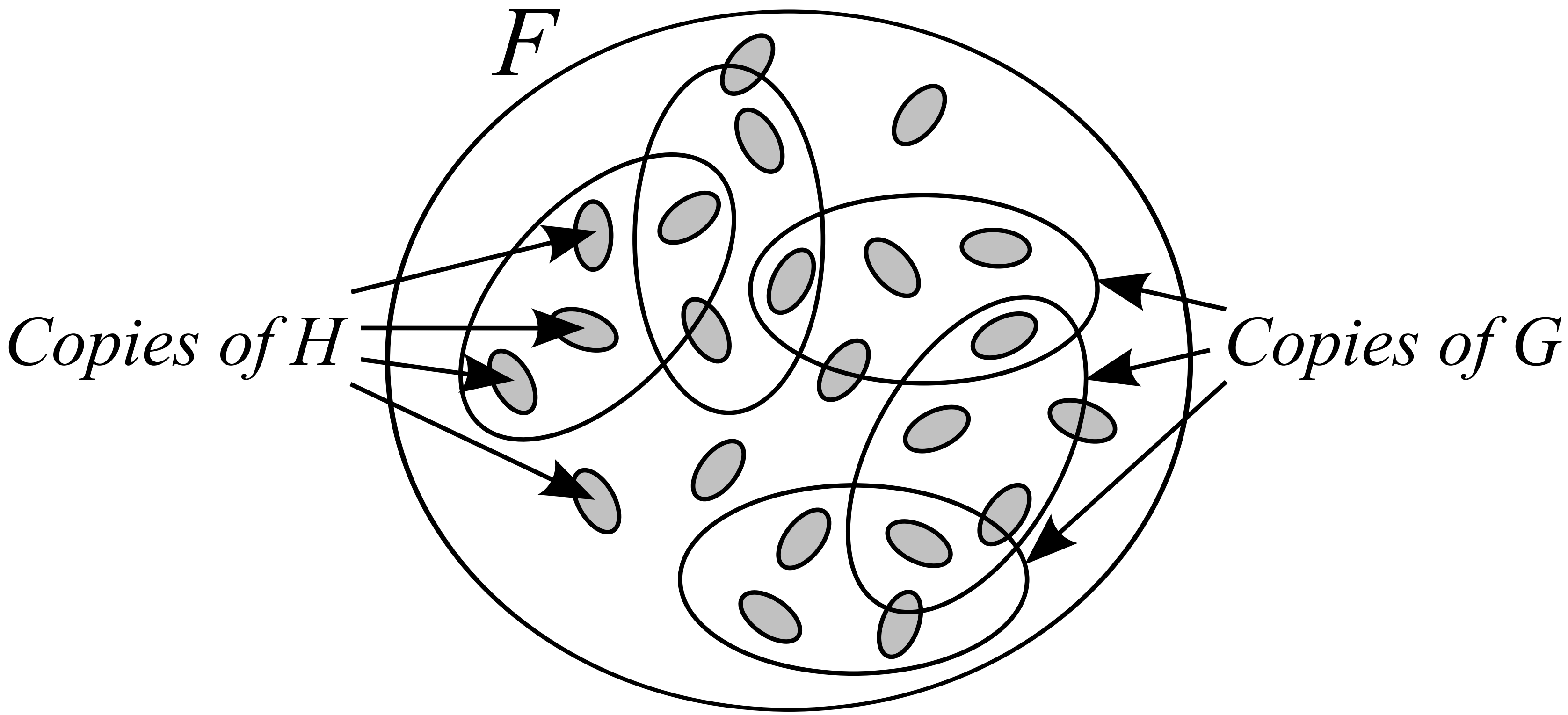
Using the arrow notation, Ramsey's theorem can be restated briefly:

Theorem (Ramsey's theorem restated).
$\forall\ m, k, r \in \mathbb{Z}^+$, $\exists$ a least integer $n = R_k(m; r)$ such that

$$n \longrightarrow (m)^k_r.$$

<u>Graph Ramsey Theory</u>

The arrow notation has been generalized to graphs as follows: let F be a "large" graph, G be a "medium sized" graph, and H a "small" graph.



A "copy" of H in G means a subgraph (either weak or induced depending on context) of G isomorphic to H. If $\forall$ r-colouring of the copies of H in F, there is a copy G' of G in F such that every copy of H in G' is the same colour, then write

$$F \longrightarrow (G)^H_r.$$

The field of "Graph Ramsey theory" is essentially the study of aspects of various graph Ramsey arrows. Since the copies of K_k inside K_n are in one-to-one correspondence with the k-subsets of an n-set, Ramsey's theorem can be restated in terms of graphs.

Theorem (Ramsey's theorem—graph theoretic).
$\forall\ m, k, r \in \mathbb{Z}^+$, $\exists$ a least integer $n = R_k(m; r)$ such that

$$K_n \longrightarrow (K_m)_r^{K_k}.$$

Hilbert's Cube Lemma

Hilbert's Cube lemma is one of the earliest examples of a Ramsey-type result, predating Ramsey himself, dating from 1892.

Fix $x_0 \in \mathbb{Z}^+ \cup \{0\}$, $d \in \mathbb{Z}^+$, and let $x_1, \ldots, x_d \in \mathbb{Z}^+$. The family of integers

$$H(x_0, x_1, \ldots, x_d) = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [1, d] \right\}$$

is called an affine cube of dimension d (or an affine d-cube). For example,

$$H(0,1,1,1) = \{0,1,2,3\} \quad \text{and} \quad H(5,2,3) = \{5,7,8,10\}.$$

Theorem (Hilbert, 1892).
$\forall\ r, d \in \mathbb{Z}^+$, $\exists\ n = h(d; r)$ such that $\forall\ r$-colouring $f : [1, n] \to [1, r]$, $\exists$ a monochromatic affine d-cube.

An **arithmetic progression of length k with difference b**, written AP_k, is a sequence of numbers of the form $\{a, a+b, a+2b, \cdots, a+(k-1)b\}$. For example,

$$3, 8, 13, 18$$

is an AP_4 with difference 5.

Theorem (van der Waerden, 1927).
Fix $k, r \in \mathbb{Z}^+$. Then $\exists\ n = W(k,r)$ such that for every r-colouring $f : [1,n] \to [1,r]$, $\exists$ a monochromatic AP_k (arithmetic progression of length k). *i.e.*, $\exists\ a, d \in \mathbb{Z}^+$, $c \in [1,r]$ such that $\{a + id : i \in [1,k]\} \subseteq [1,n]$, and for all $i \in [1,k]$, $f(a+id) = c$.

The bounds on $W(k,r)$ produced by the original proof to van der Waerden's theorem are very poor, and still today little is known.

Related to van der Waerden's theorem...

Theorem (Schur's Lemma, 1916).
Fix $r \in \mathbb{Z}^+$. Then $\exists$ a least integer $n = S(r)$ such that for every r-colouring $f : [1, n] \to \{1, 2, \ldots, r\}$, there exist $x, y \in [1, n]$ such that $f(x) = f(y) = f(x + y)$.

Proof...

Fix $r \in \mathbb{Z}^+$, and let $n = R_2(3; r) - 1$. Let $f : [1, n] \longrightarrow [1, r]$ be an r-colouring of $[1, n]$. Define an r-colouring f',

$$f' : [1, n + 1]^2 \longrightarrow [1, r],$$

where for $a, b \in [1, n + 1], a < b$, $f'(\{a, b\}) = f(b - a)$. Then by the choice of n, $\exists$ a triangle, monochromatic under f'. *i.e.*, $\exists$ $u, v, w \in [1, n + 1], u < v < w$ such that

$$f'(\{u, v\}) = f'(\{u, w\}) = f'(\{v, w\}).$$

By the definition of f', it then follows that

$$f(v - u) = f(w - u) = f(w - v).$$

Let $x = v - u$, $y = w - v$. Then $f(x) = f(y) = f(x + y)$. $\square$

Corollary.
For all $r \in \mathbb{Z}^+$, $S(r) \leq R_2(3; r) - 1$.