§3.5 Complex Logarithm Function

The real logarithm function $\ln x$ is defined as the inverse of the exponential function — $y = \ln x$ is the unique solution of the equation $x = e^y$. This works because e^x is a one-to-one function; if $x_1 \neq x_2$, then $e^{x_1} \neq e^{x_2}$. This is not the case for e^z; we have seen that e^z is $2\pi i$-periodic so that all complex numbers of the form $z + 2n\pi i$ are mapped by $w = e^z$ onto the same complex number as z. To define the logarithm function, $\log z$, as the inverse of e^z is clearly going to lead to difficulties, and these difficulties are much like those encountered when finding the inverse function of $\sin x$ in real-variable calculus. Let us proceed. We call w a logarithm of z, and write $w = \log z$, if $z = e^w$. To find w we let $w = u + vi$ be the Cartesian form for w and $z = re^{\theta i}$ be the exponential form for z. When we substitute these into $z = e^w$,

$$re^{\theta i} = e^{u+vi} = e^u e^{vi}.$$

According to conditions 1.20, equality of these complex numbers implies that

$$e^u = r \quad \text{or} \quad u = \ln r,$$

and

$$v = \theta = \arg z.$$

Thus, $w = \ln r + \theta i$, and a logarithm of a complex number z is

$$\log z = \ln |z| + (\arg z)i. \tag{3.21}$$

We use ln only for logarithms of real numbers; log denotes logarithms of complex numbers using base e (and no other base is used).

Because equation 3.21 yields logarithms of every nonzero complex number, we have defined the complex logarithm function. It is defined for all $z \neq 0$, and because $\arg z$ is determined only to a multiple of 2π, each nonzero complex number has an infinite number of logarithms. For example,

$$\log(1+i) = \ln\sqrt{2} + (\pi/4 + 2k\pi)i = (1/2)\ln 2 + (8k+1)\pi i/4.$$

Thus, to the complex number $1+i$, the logarithm function assigns an infinite number of values, $\log(1+i) = (1/2)\ln 2 + (8k+1)\pi i/4$. They all have the same real part, but their imaginary parts differ by multiples of 2π. In other words, the logarithm function is a multiple-valued function; to each complex number in its domain, it assigns an infinity of values.

Example 3.15 Express $\log(2-3i)$ in Cartesian form.

Solution Since $|2-3i| = \sqrt{13}$, and $\arg(2-3i) = 2k\pi - \text{Tan}^{-1}(3/2)$,

$$\log(2-3i) = \ln\sqrt{13} + [2k\pi - \text{Tan}^{-1}(3/2)]i = \frac{1}{2}\ln 13 + [2k\pi - \text{Tan}^{-1}(3/2)]i. \bullet$$

Some of the properties of the real logarithm function have counterparts in the complex logarithm. For example, if $z = x + yi = re^{\theta i}$, then

$$e^{\log z} = e^{\ln r + \theta i} = e^{\ln r} e^{\theta i} = re^{\theta i} = z, \tag{3.22a}$$

(as should be expected from the definition of $\log z$), and

$$\begin{aligned}\log(e^z) = \log(e^{x+yi}) &= \ln(e^x) + (y + 2k\pi)i \qquad (k \text{ an integer})\\ &= x + yi + 2k\pi i\\ &= z + 2k\pi i. \end{aligned} \tag{3.22b}$$

In real analysis the counterpart of this equation is $\log e^x = x$. The $z + 2k\pi$ on the right side of 3.22b is a reflection of the facts that $\log z$ is multiple-valued and the logarithm is the last operation on the left side of the equation.

If $z_1 = re^{\theta i}$ and $z_2 = Re^{\phi i}$, then

$$\begin{aligned}\log(z_1 z_2) = \log[rRe^{(\theta+\phi)i}] &= \ln(rR) + (\theta + \phi + 2p\pi)i \qquad (p \text{ an integer})\\ &= (\ln r + \theta i) + (\ln R + \phi i) + 2p\pi i.\end{aligned}$$

But $\log z_1 = \ln r + (\theta + 2n\pi)i$ and $\log z_2 = \ln R + (\phi + 2m\pi)i$. Hence,

$$\begin{aligned}\log(z_1 z_2) &= (\log z_1 - 2n\pi i) + (\log z_2 - 2m\pi i) + 2p\pi i\\ &= \log z_1 + \log z_2 + 2(p - n - m)\pi i\\ &= \log z_1 + \log z_2 + 2k\pi i.\end{aligned} \tag{3.23a}$$

Similarly,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 + 2k\pi i. \tag{3.23b}$$

The last two results must be approached with care. Because the logarithm function is multiple-valued, each equation must be interpreted as saying that given values for the logarithm terms, there is a value of k for which the equation holds. It is also possible to write these equations in the forms

$$\log(z_1 z_2) = \log z_1 + \log z_2, \tag{3.24a}$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2. \tag{3.24b}$$

We interpret them as saying that given values for two of the logarithm terms, there is a value of the third logarithm for which the equation is valid.

Multiple-valued functions cannot be analytic. To see why, consider the derivative of $\log z$,

$$\frac{d}{dz}\log z = \lim_{\Delta z \to 0} \frac{\log(z + \Delta z) - \log z}{\Delta z}.$$

This limit must exist, be unique, and be independent of the mode of approach of Δz to 0. But this is impossible if there is an infinity of possible choices for $\log z$ and for $\log(z + \Delta z)$ for each value of Δz. Thus, only single-valued functions can have derivatives. We therefore ask if it is possible to restrict the range of the logarithm function to obtain an analytic function; that is, can we make $\log z$ single-valued in such a way that it will have a derivative. The answer is yes, and there are many ways to do it. The most natural way to make $\log z$ single-valued is to restrict $\arg z$ in equation 3.21 to its principal value $\operatorname{Arg} z$. When this is done, we denote the resulting single-valued function by

$$\operatorname{Log} z = \ln|z| + (\operatorname{Arg} z)i, \quad z \neq 0. \tag{3.25}$$

Is $\operatorname{Log} z$ an analytic function? The answer is yes in a suitably restricted domain. To see this, we note that at any point on the negative real axis, the imaginary part of $\operatorname{Log} z$ is a discontinuous function. Theorem 2.4 implies therefore that $\operatorname{Log} z$ is discontinuous at points on the negative real axis, and $\operatorname{Log} z$ cannot be differentiable thereon. Suppose we consider the domain $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$. If we express $\operatorname{Log} z$ in the form

$$\operatorname{Log} z = \ln r + \theta i, \qquad -\pi < \theta < \pi,$$

partial derivatives of its real and imaginary parts are

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \qquad \frac{\partial v}{\partial \theta} = 1, \qquad \frac{\partial u}{\partial \theta} = 0, \qquad \frac{\partial v}{\partial r} = 0.$$

Since these derivatives satisfy Cauchy-Riemann equations 2.22 and are continuous in the domain $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$, it follows that $\operatorname{Log} z$ is an analytic function in this domain. According to formula 2.23a, the derivative of $\operatorname{Log} z$ is

$$e^{-\theta i}\left(\frac{1}{r}\right) = \frac{1}{re^{\theta i}};$$

that is,

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}. \tag{3.26}$$

Thus, $\operatorname{Log} z$ is analytic in the domain $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$. It is defined for all $z \neq 0$, but analytic only in the aforementioned domain. Points on the negative real axis and $z = 0$ are singularities of $\operatorname{Log} z$, but they are not isolated singularities (Figure 3.14).

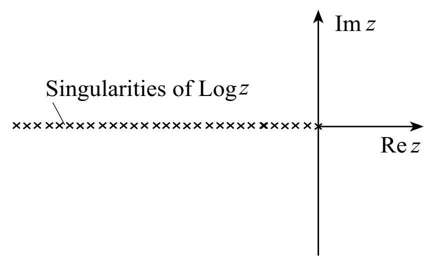


Figure 3.14

The restriction of $\arg z$ to $\operatorname{Arg} z$ in the definition of $\operatorname{Log} z$ produced a branch of the multiple-valued logarithm function called the **principal branch** of $\log z$. Other choices for $\arg z$ lead to different branches. For example, we could restrict $\arg z$ to $\arg_\phi z$. When this is done, we denote the resulting branch of the logarithm function by

$$\log_\phi z = \ln |z| + (\arg_\phi z)i. \tag{3.27}$$

These branches are analytic in any domain that does not contain $z = 0$, the branch point, or points on the branch cut, the half-line through $z = 0$ making an angle of ϕ radians with the positive real axis (Figure 3.15).

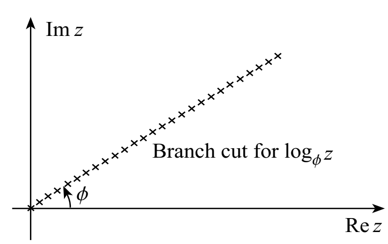


Figure 3.15

In real variable work, the notation $\log_a x$ indicates that a is the base of the logarithm function. This is not the case in complex function theory. Only e is used as the base for logarithms, and ϕ in $\log_\phi z$ indicates a particular branch of the $\log z$ function.

Example 3.16 Express $\mathrm{Log}(-1+\sqrt{3}i)$, $\log_{\pi/4}(-1+\sqrt{3}i)$, and $\log_{-3\pi/2}(-1+\sqrt{3}i)$ in Cartesian form.

Solution The complex number $-1+\sqrt{3}i$ is shown in Figure 3.16. We see that

$$\mathrm{Log}(-1+\sqrt{3}i) = \ln 2 + (2\pi/3)i,$$
$$\log_{\pi/4}(-1+\sqrt{3}i) = \ln 2 + (2\pi/3)i,$$
$$\log_{-3\pi/2}(-1+\sqrt{3}i) = \ln 2 + (-4\pi/3)i. \bullet$$

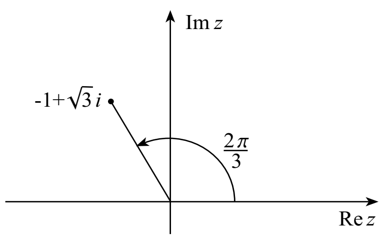


Figure 3.16

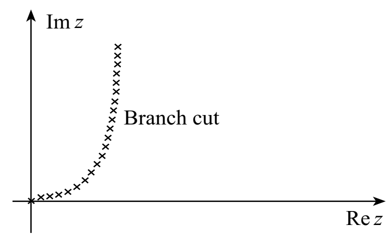


Figure 3.17

Branch cuts for the logarithm function need not be straight lines. We could stipulate that the branch cut of a branch of $\log z$ be the parabolic curve in Figure 3.17. We simply agree that at each point on this curve, arguments of z will be specified in a certain way, perhaps as $0 < \arg z < \pi/2$, that arguments will increase to the left of the curve, and that they will jump by 2π across the curve.

To understand branches of a multiple-valued function f, it is sometimes helpful to visualize ranges of the branches in the $w = f(z)$ plane. The three branches $\mathrm{Log}\, z$, $\log_0 z$, and $\log_{2\pi} z$ of $\log z$ are illustrated in Figures 3.18–3.20. Each branch maps the z-plane (less $z = 0$) onto a horizontal strip of width 2π.

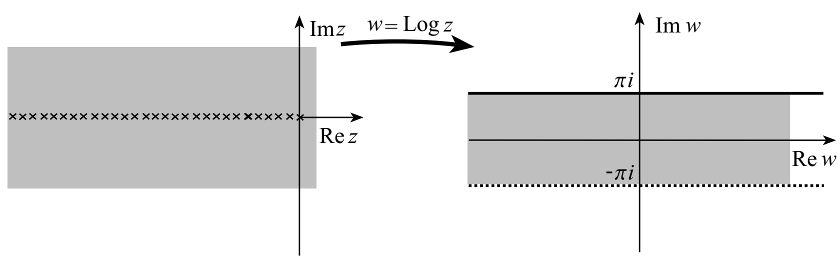


Figure 3.18

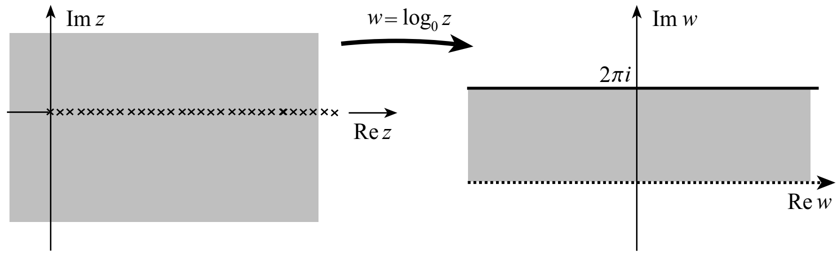


Figure 3.19

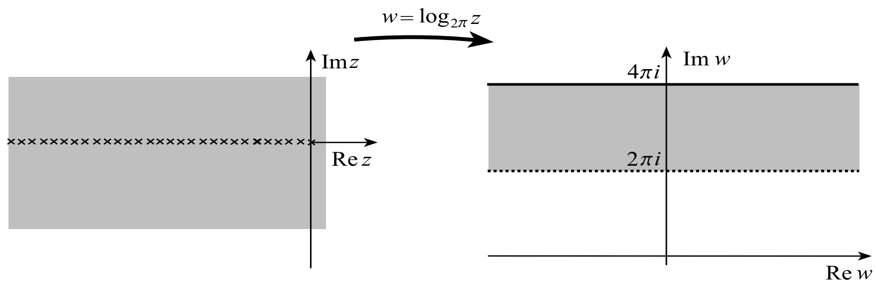


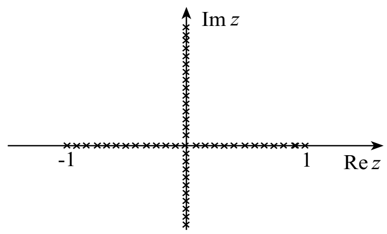
Figure 3.20

Defining branches of $\log z$ is reminiscent of the derivation of the inverse sine function in real-variable work. The situations are very similar. Because $\sin x$ is 2π-periodic, defining $y = \sin^{-1} x$ when $x = \sin y$ results in many values for $\sin^{-1} x$. Principal values $-\pi/2 \leq \text{Sin}^{-1} x \leq \pi/2$ are chosen in order to create a single-valued function. The difference in the two situations is that we almost always use principal values of $\text{Sin}^{-1} x$; other choices are possible, but they are seldom used. In order that $\log z$ be single-valued, we restrict $\arg z$ to an interval of length 2π. This leads to branches of $\log z$, the principal branch $\text{Log}\, z$, but other branches as well. Because we have taken the effort to discuss other branches of $\log z$, and even given some of them special notations ($\log_\phi z$), the implication is that they are important. This is indeed true. Different problems require different branches of $\log z$.

Quite often in applications, we encounter logarithm functions where arguments are not just z; they are functions $f(z)$ of z. For example, branch points of the function $\text{Log}[f(z)]$ are at the zeros of $f(z)$, and the branch cuts are where $f(z)$ is real and negative. An example follows.

Example 3.17 Find branch points and branch cuts for the function $\text{Log}(z^2 - 1)$.

Solution Branch points of the function occur at the zeros of $z^2 - 1$, namely $z = \pm 1$. Branch cuts occur where $z^2 - 1$ is real and negative. If we set $z = x + yi$, then $z^2 - 1 = (x^2 - y^2 - 1) + 2xyi$. This is real and negative if $x^2 - y^2 - 1 < 0$ and $2xy = 0$. The second gives $x = 0$ or $y = 0$. When $x = 0$, the inequality requires $-y^2 - 1 < 0$, which is valid for all y. When $y = 0$, the inequality requires $x^2 - 1 < 0$, that part of the real axis between $x = \pm 1$. Branch cuts are therefore the real axis between ± 1, including ± 1, and the imaginary axis (Figure 3.21).●



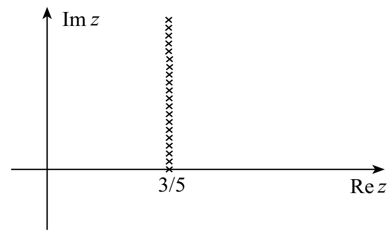


Figure 3.21 **Figure 3.22**

Branch points of the function $\log_\phi[f(z)]$ are once again at the zeros of $f(z)$. Branch cuts are where one of the arguments of $f(z)$ is equal to ϕ. Here is an example.

Example 3.18 Find branch points and branch cuts for the function $\log_{3\pi/2}(3-5z)$.

Solution Branch points occur at the zeros of $3-5z$, there being only one $z=3/5$. Branch cuts occur where one of the arguments of $3-5z$ is equal to $3\pi/2$. If we set $z=x+yi$, then $3-5z=(3-5x)-5yi$. If θ is an argument of this complex number, then

$$\cos\theta = \frac{3-5x}{\sqrt{(3-5x)^2+25y^2}}, \qquad \sin\theta = \frac{-5y}{\sqrt{(3-5x)^2+25y^2}}.$$

For θ to be equal to $3\pi/2$, the first of these requires $x=3/5$, and then the second implies that

$$-1 = \frac{-5y}{\sqrt{25y^2}} \quad \Longrightarrow \quad 5|y| = 5y \quad \Longrightarrow \quad y>0.$$

The branch cut is therefore the vertical half-line above $z=3/5$, including $z=3/5$, (Figure 3.22).•

EXERCISES 3.5

In Exercises 1–4 express the complex number(s) in Cartesian form.

1. $\log i$

2. $\mathrm{Log}\,(2-6i)$

3. $\log_{\pi/2}(1+i)$

4. $\log_{-3}(-2+3i)$

In Exercises 5–8 find all solutions of the equation.

5. $\mathrm{Log}\,z=\pi i/2$

6. $\log_1 z = 2+3i$

7. $z^4=i$

8. $e^{z+2}=4$

Use the complex logarithm function to find all solutions of the equations in Exercises 9–22.

9. Exercise 15 in Section 3.2

10. Exercise 16 in Section 3.2

11. Exercise 17 in Section 3.2

12. Exercise 18 in Section 3.2

13. Exercise 19 in Section 3.2

14. Exercise 20 in Section 3.2

15. Exercise 21 in Section 3.2

16. Exercise 22 in Section 3.2

17. Exercise 13 in Section 3.3

18. Exercise 14 in Section 3.3

19. Exercise 15 in Section 3.3

20. Exercise 9 in Section 3.4

21. Exercise 10 in Section 3.4

22. Exercise 11 in Section 3.4

23. You may have noticed that we have not stated the complex analogue of $\ln(x^a) = a \ln x$. The reason for this is that in general

$$\log(z^a) \neq a \log z.$$

Illustrate this with $z = 1 + i$ and $a = 4$.

24. Is $\log_\phi(z_1 z_2) = \log_\phi z_1 + \log_\phi z_2$?

25. Which branches of $\log z$ have zeros? What are the zeros?

26. Find $f'(3+i)$ if $f(z) = \mathrm{Log}\,(2z + 3 - i)$.

27. Find $f''(2i)$ if $f(z) = \log_{3\pi/8}(-z + i)$.

28. Is $\mathrm{Log}\, e^z = z$?

The function Log $[f(z)]$ has branch points where $f(z) = 0$ and branch cuts where $f(z)$ is real and negative. In Exercises 29–32 identify branch points and branch cuts for the function.

29. $\mathrm{Log}\,(z + i)$

30. $\mathrm{Log}\,(z - 1)$

31. $\mathrm{Log}\,(3 - 2z)$

32. $\mathrm{Log}\,(3z - 2 + 4i)$

33. (a) What are the branch points for the function $f(z) = \mathrm{Log}\,(z^2 - 4)$?
(b) Show that the imaginary axis and that part of the real axis between the branch points are branch cuts.

34. Find branch points and branch cuts for $f(z) = \mathrm{Log}\,(z^2 + 1)$.

The function $\log_\phi[f(z)]$ has branch points where $f(z) = 0$ and branch cuts where one of the arguments of $f(z)$ is equal to ϕ. In Exercises 35–38 identify branch points and branch cuts for the function.

35. $\log_0(z - 2i)$

36. $\log_{\pi/2}(3 - z)$

37. $\log_{-\pi/2}(4 - 2z)$

38. $\log_2(2z + 1)$

39. (a) What are the branch points for the function $f(z) = \log_0(z^2 + 1)$?
(b) Show that the real axis and the line segment joining the branch points are branch cuts.

40. Find branch points and branch cuts for $f(z) = \log_{3\pi/2}(1 - z^2)$.

41. In this exercise we show that logarithm functions cannot always be combined in what might be thought to be natural ways.
(a) Show that in general

$$\log_\phi(z^2 - 1) \neq \log_\phi(z + 1) + \log_\phi(z - 1).$$

Illustrate with $\phi = -\pi$ and $z = -1 + i$. Why is this not a contradiction of property 3.24a?
(b) Show that in general

$$\log_\phi\left(\frac{z + 1}{z - 1}\right) \neq \log_\phi(z + 1) - \log_\phi(z - 1).$$

Illustrate with $\phi = 0$ and $z = -1 + i$. Why is this not a contradiction of property 3.24b?

42. Compare branch cuts for $f(z) = \operatorname{Log}(z^2 - 1)$ and $g(z) = \operatorname{Log}(z+1) + \operatorname{Log}(z-1)$.

43. Compare branch cuts for $f(z) = \operatorname{Log}(4 - z^2)$ and $g(z) = \operatorname{Log}(2+z) + \operatorname{Log}(2-z)$.

44. Compare branch cuts for $f(z) = \log_{-\pi/2}(z^2 - 1)$ and $g(z) = \log_{-\pi/2}(z+1) + \log_{-\pi/2}(z-1)$.

45. Compare branch cuts for $f(z) = \log_{-\pi/2}(4 - z^2)$ and $g(z) = \log_{-\pi/2}(2+z) + \log_{-\pi/2}(2-z)$.

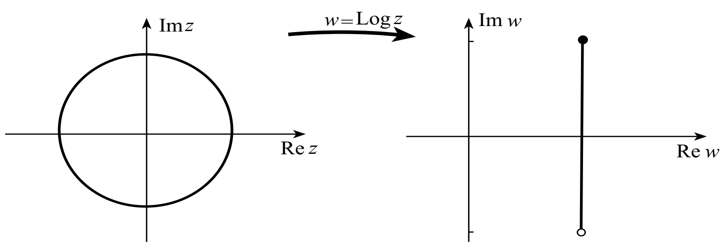
46. (a) Prove that for $b > a > 1$, the function $\operatorname{Log}\left(\dfrac{z-a}{z-b}\right)$ is analytic except for points on the $\operatorname{Re} z$ axis for which $a \le \operatorname{Re} z \le b$.

(b) Verify that for $|z| > b$, $\quad \operatorname{Log}\left(\dfrac{z-a}{z-b}\right) = \operatorname{Log}\left(1 - \dfrac{a}{z}\right) - \operatorname{Log}\left(1 - \dfrac{b}{z}\right)$.

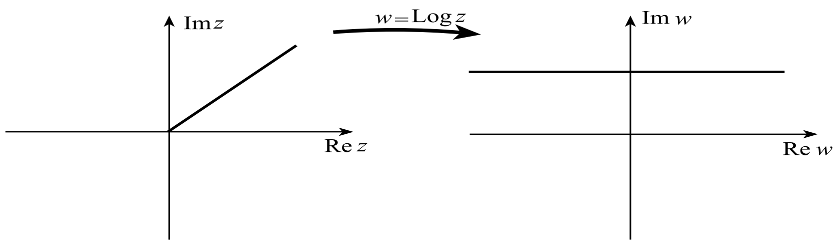
Mapping Exercises

47. Show that $w = \operatorname{Log} z$, regarded as a mapping from the z-plane to the w-plane, performs the mappings indicated in the figures below.

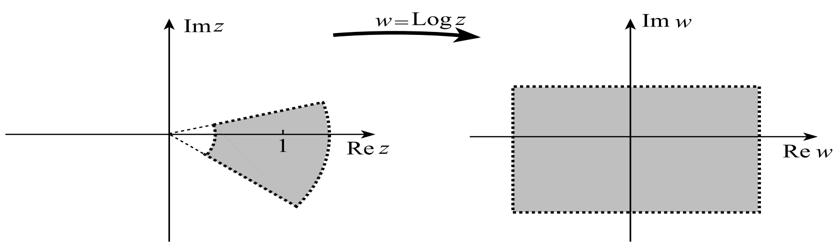
(a)



(b)



(c)



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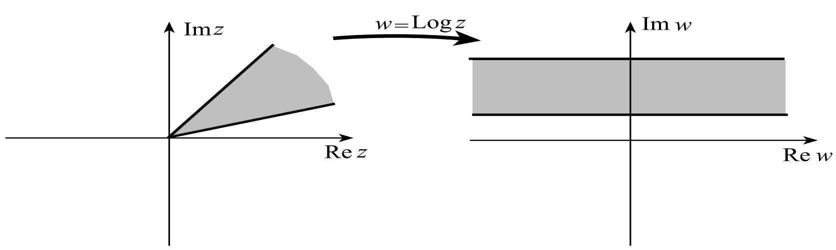
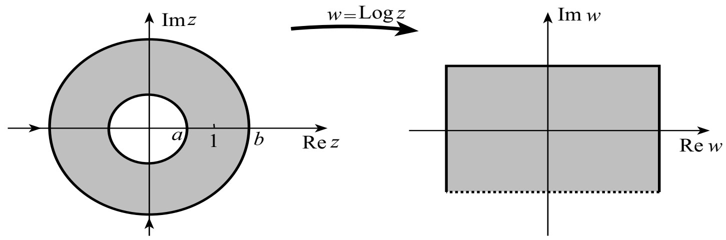


(e)



48. Show that the transformation $w = \operatorname{Log}\left[\coth\left(\frac{\pi z}{2a}\right)\right]$ maps the semi-infinite strip $-a \leq \operatorname{Im} z \leq a$, $\operatorname{Re} z \geq 0$ to the infinite strip $-\pi/2 \leq \operatorname{Im} w \leq \pi/2$.