§8.5 Application of Laplace Transforms to Partial Differential Equations

In Sections 8.2 and 8.3, we illustrated the effective use of Laplace transforms in solving ordinary differential equations. The transform replaces a differential equation in $y(t)$ with an algebraic equation in its transform $\tilde{y}(s)$. It is then a matter of finding the inverse transform of $\tilde{y}(s)$ either by partial fractions and tables (Section 8.1) or by residues (Section 8.4). Laplace transforms also provide a potent technique for solving partial differential equations. When the transform is applied to the variable t in a partial differential equation for a function $y(x,t)$, the result is an ordinary differential equation for the transform $\tilde{y}(x,s)$. The ordinary differential equation is solved for $\tilde{y}(x,s)$ and the function is inverted to yield $y(x,t)$. We illustrate this procedure with five physical examples. The first two examples are on unbounded spatial intervals; inverse transforms are found in tables. The last three examples are on bounded spatial intervals; inverse transforms are calculated with residues.

Problems on Unbounded Intervals

Example 8.18 A very long cylindrical rod is placed along the positive x-axis with one end at $x = 0$ (Figure 8.22). The rod is so long that any effects due to its right end may be neglected. Its sides are covered with perfect insulation so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is 0°C throughout. Suddenly the left end of the rod has its temperature raised to U_0, and maintained at this temperature thereafter. The initial, boundary-value problem describing temperature $U(x,t)$ at points in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{8.19a}$$

$$U(0,t) = U_0, \quad t > 0, \tag{8.19b}$$

$$U(x,0) = 0, \quad x > 0, \tag{8.19c}$$

where k is a constant called the *thermal diffusivity* of the material in the rod. Use Laplace transforms on variable t to find $U(x,t)$.

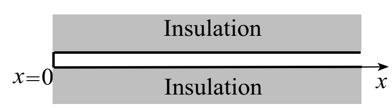

Figure 8.22

Solution When we apply the Laplace transform to the partial differential equation, and use property 8.10a,

$$s\tilde{U}(x,s) - U(x,0) = k\,\mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\}.$$

Since the integration with respect to t in the Laplace transform and the differentiation with respect to x are independent, we interchange the order of operations on the right,

$$s\tilde{U}(x,s) = k\frac{\partial^2 \tilde{U}}{\partial x^2},$$

where we have also used initial condition 8.19c. Because only derivatives with respect to x remain, we replace the partial derivative with an ordinary derivative,

$$s\tilde{U} = k\frac{d^2\tilde{U}}{dx^2}, \quad x > 0. \tag{8.20a}$$

When we take Laplace transforms of boundary condition 8.19b, we obtain

$$\tilde{U}(0,s) = \frac{U_0}{s}, \tag{8.20b}$$

a boundary condition to accompany differential equation 8.20a. A general solution of 8.20a is

$$\tilde{U}(x,s) = Ae^{\sqrt{s/k}x} + Be^{-\sqrt{s/k}x}.$$

Because $U(x,t)$ must remain bounded as x becomes infinite, so also must $\tilde{U}(x,s)$. We must therefore set $A = 0$, in which case 8.20b requires $B = U_0/s$. Thus,

$$\tilde{U}(x,s) = \frac{U_0}{s}e^{-\sqrt{s/k}x}. \tag{8.21}$$

The inverse transform of this function can be found in tables,

$$U(x,t) = U_0\,\mathcal{L}^{-1}\left\{\frac{e^{-\sqrt{s/k}x}}{s}\right\} = U_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right), \tag{8.22a}$$

where $\mathrm{erfc}\,(x)$ is the complementary error function

$$\mathrm{erfc}\,(x) = 1 - \mathrm{erf}\,(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du. \tag{8.22b}$$

Notice that for any $x > 0$ and any $t > 0$, temperature $U(x,t)$ is positive. This indicates that the abrupt change in temperature at the end $x = 0$ from 0°C to U_0 is felt instantaneously at every point in the rod. We have shown a plot of $U(x,t)$ for various fixed values of t in Figure 8.23.●

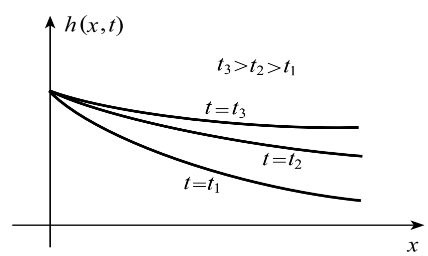

Figure 8.23

Example 8.19 A very long taut string is supported from below so that it lies motionless on the positive x-axis. At time $t = 0$, the support is removed and gravity is permitted to act on the string. If the end $x = 0$ is fixed at the origin, the initial, boundary-value problem describing displacements $y(x,t)$ of points in the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - g, \quad x > 0, \quad t > 0, \tag{8.23a}$$

$$y(0,t) = 0, \quad t > 0, \tag{8.23b}$$

$$y(x,0) = 0, \quad x > 0, \tag{8.23c}$$

$$y_t(x,0) = 0, \quad x > 0, \tag{8.23d}$$

where $g = 9.81$ and $c > 0$ is a constant depending on the material and tension of the string. Initial condition 8.23d expresses the fact that the initial velocity of the string is zero. Use Laplace transforms to solve this problem.

Solution When we apply Laplace transforms to the partial differential equation, and use property 8.10b,

$$s^2\tilde{y}(x,s) - sy(x,0) - y_t(x,0) = c^2\mathcal{L}\left\{\frac{\partial^2 y}{\partial x^2}\right\} - \frac{g}{s}.$$

We now use initial conditions 8.23c,d, and interchange operations on the right,

$$s^2\tilde{y} = c^2\frac{d^2\tilde{y}}{dx^2} - \frac{g}{s},$$

or,

$$\frac{d^2\tilde{y}}{dx^2} - \frac{s^2}{c^2}\tilde{y} = \frac{g}{c^2 s}, \quad x > 0. \tag{8.24a}$$

This ordinary differential equation is subject to the transform of 8.23b,

$$\tilde{y}(0,s) = 0. \tag{8.24b}$$

A general solution of differential equation 8.24a is

$$\tilde{y}(x,s) = Ae^{sx/c} + Be^{-sx/c} - \frac{g}{s^3}.$$

For this function to remain bounded as $x \to \infty$, we must set $A = 0$, in which case condition 8.24b implies that $B = g/s^3$. Thus,

$$\tilde{y}(x,s) = -\frac{g}{s^3} + \frac{g}{s^3}e^{-sx/c}. \tag{8.25}$$

Property 8.4b gives

$$y(x,t) = -\frac{gt^2}{2} + \frac{g}{2}\left(t - \frac{x}{c}\right)^2 h\left(t - \frac{x}{c}\right) \tag{8.26}$$

where $h(t - x/c)$ is the Heaviside unit step function. What this says is that a point x in the string falls freely under gravity for $0 < t < x/c$, after which it falls with constant velocity $-gx/c$ [since for $t > x/c$, $y(x,t) = (g/2)(-2xt/c + x^2/c^2)$]. A picture of the string at any given time t_0 is shown in Figure 8.24. It is parabolic for $0 < x < ct_0$ and horizontal for $x > ct_0$. As t_0 increases, the parabolic portion lengthens and the horizontal section drops.●

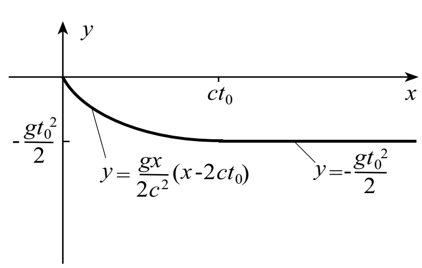

Figure 8.24

Problems on Bounded Intervals

The next three examples are on bounded intervals; we use residues to find inverse transforms.

Example 8.20 A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its sides are insulated so that no heat can enter or escape therethrough. At time $t = 0$, the left end ($x = 0$) has temperature 0°C, and the right end ($x = L$) has temperature L°C, and the temperature rises linearly between the ends. Suddenly at time $t = 0$, the temperature of the right end is reduced to 0°C, and both ends are held at temperature zero thereafter. The initial, boundary-value problem describing temperature $U(x, t)$ at points in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{8.27a}$$

$$U(0, t) = 0, \quad t > 0, \tag{8.27b}$$

$$U(L, t) = 0, \quad t > 0, \tag{8.27c}$$

$$U(x, 0) = x, \quad 0 < x < L, \tag{8.27d}$$

k again being the thermal diffusivity of the material in the rod. Use Laplace transforms to find $U(x, t)$.

Solution When we take Laplace transforms of 8.27a, and use property 8.10a,

$$s\tilde{U}(x, s) - x = k\frac{d^2\tilde{U}}{dx^2},$$

or,

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = -\frac{x}{k}, \quad 0 < x < L. \tag{8.28a}$$

This ordinary differential equation is subject to the transforms of 8.27b,c,

$$\tilde{U}(0, s) = 0, \tag{8.28b}$$

$$\tilde{U}(L, s) = 0. \tag{8.28c}$$

A general solution of differential equation 8.28a is

$$\tilde{U}(x, s) = A \cosh\sqrt{\frac{s}{k}}x + B \sinh\sqrt{\frac{s}{k}}x + \frac{x}{s}.$$

For Example 8.18 on an unbounded interval, we used exponentials in the solution of 8.20a. On bounded intervals, hyperbolic functions are preferable. Boundary conditions 8.28b,c require

$$0 = A, \qquad 0 = A \cosh\sqrt{\frac{s}{k}}L + B \sinh\sqrt{\frac{s}{k}}L + \frac{L}{s}.$$

From these,

$$\tilde{U}(x, s) = \frac{1}{s}\left(x - \frac{L \sinh\sqrt{s/k}\,x}{\sinh\sqrt{s/k}\,L}\right). \tag{8.29}$$

Although $\sqrt{s/k}$ denotes the principal square root function, $\tilde{U}(x, s)$ in 8.29 is a solution of problem 8.28 for any branch of the root function.

It remains now to find the inverse transform of $\tilde{U}(x, s)$. We do this by calculating residues of $e^{st}\tilde{U}(x, s)$ at its singularities. To discover the nature of the singularity at $s = 0$, we expand $\tilde{U}(x, s)$ in a Laurent series around $s = 0$. Provided we use the same branch for the root function in numerator and denominator of 8.29, we may write that

$$\begin{aligned}
\tilde{U}(x,s) &= \frac{1}{s}\left\{x - \frac{L[\sqrt{s/k}x + (\sqrt{s/k}x)^3/3! + \cdots]}{\sqrt{s/k}L + (\sqrt{s/k}L)^3/3! + \cdots}\right\} \\
&= \frac{1}{s}\left[x - \frac{x + sx^3/(6k) + \cdots}{1 + sL^2/(6k) + \cdots}\right] \\
&= \frac{1}{s}\left[\frac{sx(L^2 - x^2)}{6k} + \cdots\right] \\
&= \frac{x(L^2 - x^2)}{6k} + \text{ terms in } s,\ s^2,\ \cdots .
\end{aligned}$$

Since this is the Laurent series of $\tilde{U}(x, s)$ valid in some annulus around $s = 0$, it follows that $\tilde{U}(x, s)$ has a removable singularity at $s = 0$.

To invert $\tilde{U}(x, s)$ with residues, the function cannot have a branch cut in the left half-plane $\operatorname{Im} s < 0$. Suppose we choose a branch of $\sqrt{s/k}$ in 8.29 with branch cut along the positive real axis. We continue to denote this branch by $\sqrt{s/k}$ notwithstanding the fact that it is no longer the principal branch. Singularities of $\tilde{U}(x, s)$ in the left half-plane then occur at the zeros of $\sinh\sqrt{s/k}L$; that is, when $\sqrt{s/k}L = n\pi i$, or, $s = -n^2\pi^2 k/L^2$, n a positive integer. Because the derivative of $\sinh\sqrt{s/k}L$ does not vanish at $s = -n^2\pi^2 k/L^2$, this function has zeros of order 1 at $s = -n^2\pi^2 k/L^2$. It follows that $\tilde{U}(x, s)$ has simple poles at these singularities. Residues of $e^{st}\tilde{U}(x, s)$ at these poles are

$$\begin{aligned}
\operatorname{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2 k}{L^2}\right] &= \lim_{s\to -n^2\pi^2 k/L^2}\left(s + \frac{n^2\pi^2 k}{L^2}\right)\frac{e^{st}}{s}\left(x - \frac{L\sinh\sqrt{s/k}x}{\sinh\sqrt{s/k}L}\right) \\
&= -\frac{e^{-n^2\pi^2 kt/L^2}}{-n^2\pi^2 k/L^2} L\sinh\frac{n\pi x i}{L}\lim_{s\to -n^2\pi^2 k/L^2}\frac{s + n^2\pi^2 k/L^2}{\sinh\sqrt{s/k}L}.
\end{aligned}$$

L'Hôpital's rule yields

$$\begin{aligned}
\operatorname{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2 k}{L^2}\right] &= \frac{iL^3}{n^2\pi^2 k}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}\lim_{s\to -n^2\pi^2 k/L^2}\frac{1}{\frac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}L} \\
&= \frac{2iL^2}{n^2\pi^2 k}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}\frac{1}{\frac{L}{n\pi k i}\cosh n\pi i} \\
&= \frac{2L}{n\pi}(-1)^{n+1}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.
\end{aligned}$$

We sum these residues to find the inverse transform of $\tilde{U}(x, s)$,

$$U(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.\bullet \tag{8.30}$$

Readers familiar with other methods for solving partial differential equations will be well aware that this solution can also be obtained by separation of variables.

Exponentials enhance convergence for large values of t. For instance, suppose the rod is 1/5 m in length and is made from stainless steel with thermal diffusivity $k = 3.87 \times 10^{-6}$ m^2/s. Consider finding the temperature at the midpoint $x = 1/10$ m of the rod at the four times $t = 2$, 5, 30, and 100 minutes. Series 8.30 gives

$$U(1/10, 120) = \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-0.1145861n^2}\sin\frac{n\pi}{2} \approx 0.100°\text{C};$$

$$U(1/10, 300) = \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-0.28646526n^2}\sin\frac{n\pi}{2} \approx 0.092°\text{C};$$

$$U(1/10, 1800) = \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-1.7187915n^2}\sin\frac{n\pi}{2} \approx 0.023°\text{C};$$

$$U(1/10, 6000) = \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-5.7293052n^2}\sin\frac{n\pi}{2} \approx 0.0004°\text{C}.$$

To obtain these approximations we used four nonzero terms in the first series, three in the second, and one in each of the third and fourth.

Laplace transforms also yield a representation for temperature in the rod that is particularly valuable when t is small. This representation is not available through separation of variables. We write transform 8.29 in the form

$$\begin{aligned}\tilde{U}(x,s) &= \frac{x}{s} - \frac{L\sinh\sqrt{s/k}x}{s\,\sinh\sqrt{s/k}L} = \frac{x}{s} - \frac{L}{s}\frac{e^{\sqrt{s/k}x} - e^{-\sqrt{s/k}x}}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}}\\ &= \frac{x}{s} - \frac{L}{s}\frac{e^{-\sqrt{s/k}L}(e^{\sqrt{s/k}x} - e^{-\sqrt{s/k}x})}{1 - e^{-2\sqrt{s/k}L}}.\end{aligned}$$

If we regard $1/(1-e^{-2\sqrt{s/k}L})$ as the sum of an infinite geometric series with common ratio $e^{-2\sqrt{s/k}L}$, then

$$\begin{aligned}\tilde{U}(x,s) &= \frac{x}{s} - \frac{L}{s}\left[e^{\sqrt{s/k}(x-L)} - e^{-\sqrt{s/k}(x+L)}\right]\sum_{n=0}^{\infty}e^{-2n\sqrt{s/k}L}\\ &= \frac{x}{s} - L\sum_{n=0}^{\infty}\left[\frac{e^{-\sqrt{s/k}[(2n+1)L-x]}}{s} - \frac{e^{-\sqrt{s/k}[(2n+1)L+x]}}{s}\right].\end{aligned}$$

Tables of Laplace transforms indicate that

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \text{erfc}\left(\frac{a}{2\sqrt{t}}\right)$$

where erfc (x) is the complementary error function in equation 8.22b. Hence $U(x,t)$ may be expressed as a series of complementary error functions,

$$\begin{aligned}U(x,t) &= x - L\sum_{n=0}^{\infty}\left\{\text{erfc}\left[\frac{(2n+1)L-x}{2\sqrt{kt}}\right] - \text{erfc}\left[\frac{(2n+1)L+x}{2\sqrt{kt}}\right]\right\}\\ &= x - L\sum_{n=0}^{\infty}\left\{\text{erf}\left[\frac{(2n+1)L+x}{2\sqrt{kt}}\right] - \text{erf}\left[\frac{(2n+1)L-x}{2\sqrt{kt}}\right]\right\},\end{aligned}$$

where we have used the fact that $\text{erfc}\,(x) = 1 - \text{erf}\,(x)$. This series converges rapidly for small values of t (as opposed to solution 8.30 which converges rapidly for large t). To see this consider temperature at the midpoint of the stainless steel rod at $t = 300$ s:

$$U(1/10, 300) = \frac{1}{10} - \frac{1}{5}\sum_{n=0}^{\infty}\left\{\text{erf}\left[\frac{(2n+1)/5 + 1/10}{2\sqrt{3.87\times 10^{-6}(300)}}\right] - \text{erf}\left[\frac{(2n+1)/5 - 1/10}{2\sqrt{3.87\times 10^{-6}(300)}}\right]\right\}.$$

For $n > 0$, all terms essentially vanish, and

$$U(1/10, 300) \approx \frac{1}{10} - \frac{1}{5}[\text{erf}\,(4.402) - \text{erf}\,(1.467)] = 0.092°\text{C}.$$

For $t = 1800$,

$$U(1/10, 1800) = \frac{1}{10} - \frac{1}{5}\sum_{n=0}^{\infty}\left\{\text{erf}\left[\frac{(2n+1)/5 + 1/10}{2\sqrt{3.87\times 10^{-6}(1800)}}\right] - \text{erf}\left[\frac{(2n+1)/5 - 1/10}{2\sqrt{3.87\times 10^{-6}(1800)}}\right]\right\}.$$

Once again, only the $n = 0$ term is required; it yields

$$U(1/10, 1800) \approx \frac{1}{10} - \frac{1}{5}[\text{erf}\,(0.1797) - \text{erf}\,(0.5991)] = 0.023°\text{C}.$$

Even for t as large as 6000, we need only the $n = 0$ and $n = 1$ terms to give

$$\begin{aligned} U(1/10, 6000) &= \frac{1}{10} - \frac{1}{5}\sum_{n=0}^{\infty}\left\{\text{erf}\left[\frac{(2n+1)/5 + 1/10}{2\sqrt{3.87\times 10^{-6}(6000)}}\right] - \text{erf}\left[\frac{(2n+1)/5 - 1/10}{2\sqrt{3.87\times 10^{-6}(6000)}}\right]\right\} \\ &\approx \frac{1}{10} - \frac{1}{5}[\text{erf}\,(0.9843) - \text{erf}\,(0.3281) + \text{erf}\,(2.297) - \text{erf}\,(1.641)] \\ &= 0.0004°\text{C}. \end{aligned}$$

Example 8.21 The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest along the axis and then at time $t = 0$, it is allowed to drop under its own weight. The initial, boundary-value problem describing displacements $y(x, t)$ of points in the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \tag{8.31a}$$

$$y(0, t) = 0, \quad t > 0, \tag{8.31b}$$

$$y(L, t) = 0, \quad t > 0, \tag{8.31c}$$

$$y(x, 0) = 0, \quad 0 < x < L, \tag{8.31d}$$

$$y_t(x, 0) = 0, \quad 0 < x < L, \tag{8.31e}$$

where $g = 9.81$ and $c > 0$ is a constant depending on the material and tension of the string. Initial condition 8.31e expresses the fact that the initial velocity of the string is zero. Use Laplace transforms to solve this problem.

Solution When we take Laplace transforms of 8.31a, and use initial conditions 8.31d,e,

$$s^2\tilde{y}(x, s) = c^2\frac{\partial^2\tilde{y}}{\partial x^2} - \frac{g}{s},$$

or,

$$\frac{d^2\tilde{y}}{dx^2} - \frac{s^2}{c^2}\tilde{y} = \frac{g}{c^2 s}, \quad 0 < x < L. \tag{8.32a}$$

This ordinary differential equation is subject to the transforms of 8.31b,c,

$$\tilde{y}(0,s) = 0, \tag{8.32b}$$

$$\tilde{y}(L,s) = 0. \tag{8.32c}$$

A general solution of 8.32a is

$$\tilde{y}(x,s) = A\cosh\frac{sx}{c} + B\sinh\frac{sx}{c} - \frac{g}{s^3}.$$

Boundary conditions 8.32b,c require

$$0 = A - \frac{g}{s^3}, \qquad 0 = A\cosh\frac{sL}{c} + B\sinh\frac{sL}{c} - \frac{g}{s^3}.$$

Thus,

$$\tilde{y}(x,s) = \frac{g}{s^3}\cosh\frac{sx}{c} - \frac{g}{s^3\sinh sL/c}\left(-1+\cosh\frac{sL}{c}\right)\sinh\frac{sx}{c} - \frac{g}{s^3}. \tag{8.33}$$

We invert this transform function by finding residues of $e^{st}\tilde{y}(x,s)$ at its singularities. To determine the type of singularity at $s = 0$, we expand $\tilde{y}(x,s)$ in a Laurent series around $s = 0$:

$$\begin{aligned}\tilde{y}(x,s) &= -\frac{g}{s^3}\left[1 - \left(1 + \frac{s^2x^2}{2c^2} + \frac{s^4x^4}{24c^4} + \cdots\right) + \left(\frac{s^2L^2}{2c^2} + \frac{s^4L^4}{24c^4} + \cdots\right)\left(\frac{\dfrac{sx}{c} + \dfrac{s^3x^3}{6c^3} + \cdots}{\dfrac{sL}{c} + \dfrac{s^3L^3}{6c^3} + \cdots}\right)\right]\\ &= -\frac{g}{s^3}\left[-\left(\frac{s^2x^2}{2c^2} + \frac{s^4x^4}{24c^4} + \cdots\right) + \left(\frac{s^2L^2}{2c^2} + \frac{s^4L^4}{24c^4} + \cdots\right)\left(\frac{x}{L} + \frac{s^2x(x-L)}{6Lc^2} + \cdots\right)\right]\\ &= -\frac{gx(L-x)}{2c^2s} + \cdots\end{aligned}$$

This shows that $\tilde{y}(x,s)$ has a simple pole at $s = 0$, and from the product

$$\begin{aligned}e^{st}\tilde{y}(x,s) &= \left(1 + st + \frac{s^2t^2}{2!} + \cdots\right)\left[-\frac{gx(L-x)}{2c^2s} + \cdots\right]\\ &= -\frac{gx(L-x)}{2c^2s} + \cdots,\end{aligned}$$

the residue of $e^{st}\tilde{y}(x,s)$ at $s = 0$ is $-gx(L-x)/(2c^2)$.

The remaining singularities of $\tilde{y}(x,s)$ occur at the zeros of $\sinh(sL/c)$; that is, when $sL/c = n\pi i$, or, $s = n\pi ci/L$, n an integer. Because the derivative of $\sinh(sL/c)$ does not vanish at $s = n\pi ci/L$, this function has zeros of order one at $s = n\pi ci/L$. When n is even, $s = n\pi ci/L$ is a simple zero of $-1 + \cosh(sL/c)$, and therefore these are removable singularities of $\tilde{y}(x,s)$. When n is odd, $s = n\pi ci/L$ is not a zero of $-1 + \cosh(sL/c)$, and these are therefore simple poles of $\tilde{y}(x,s)$. Residues of $e^{st}\tilde{y}(x,s)$ at these poles are

$$\begin{aligned}
\text{Res}\left[e^{st}\tilde{y}(x,s),\frac{n\pi ci}{L}\right] &= \lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)\frac{-ge^{st}}{s^3}\left[1-\cosh\frac{sx}{c}+\left(-1+\cosh\frac{sL}{c}\right)\frac{\sinh(sx/c)}{\sinh(sL/c)}\right]\\
&\quad -\frac{g}{(n\pi ci/L)^3}e^{n\pi cti/L}(-1+\cosh n\pi i)\sinh\frac{n\pi xi}{L}\lim_{s\to n\pi ci/L}\frac{s-n\pi ci/L}{\sinh(sL/c)}\\
&= -\frac{gL^3 i}{n^3\pi^3c^3}e^{n\pi cti/L}(-1+\cos n\pi)i\sin\frac{n\pi x}{L}\lim_{s\to n\pi ci/L}\frac{1}{(L/c)\cosh(sL/c)}\\
&= -\frac{gL^3}{n^3\pi^3c^3}e^{n\pi cti/L}[1+(-1)^{n+1}]\sin\frac{n\pi x}{L}\frac{1}{(L/c)\cosh n\pi i}\\
&= \frac{gL^2}{n^3\pi^3c^2}e^{n\pi cti/L}[1+(-1)^{n+1}]\sin\frac{n\pi x}{L}.
\end{aligned}$$

Since n is odd, we may write that residues of $e^{st}\tilde{y}(x,s)$ at the poles $s=(2n-1)\pi ci/L$ are

$$\text{Res}\left[e^{st}\tilde{y}(x,s),\frac{(2n-1)\pi ci}{L}\right]\frac{2gL^2}{(2n-1)^3\pi^3c^2}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}.$$

Displacements of points in the string are given by

$$y(x,t) = -\frac{gx(L-x)}{2c^2}+\sum_{n=-\infty}^{\infty}\frac{2gL^2}{(2n-1)^3\pi^3c^2}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}.$$

We separate the summation into two parts, one over positive n and the other over nonpositive n, and in the latter we set $m=1-n$,

$$\begin{aligned}
y(x,t) &= -\frac{gx(L-x)}{2c^2}+\frac{2gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}\\
&\quad +\frac{2gL^2}{\pi^3c^2}\sum_{n=-\infty}^{0}\frac{1}{(2n-1)^3}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}\\
&= -\frac{gx(L-x)}{2c^2}+\frac{2gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}\\
&\quad +\frac{2gL^2}{\pi^3c^2}\sum_{m=1}^{\infty}\frac{1}{[2(1-m)-1]^3}e^{[2(1-m)-1]\pi cti/L}\sin\frac{[2(1-m)-1]\pi x}{L}.
\end{aligned}$$

If we now replace m by n in the second summation, and combine it with the first,

$$\begin{aligned}
y(x,t) &= -\frac{gx(L-x)}{2c^2}+\frac{2gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}\\
&\quad +\frac{2gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)\pi cti/L}\sin\frac{(2n-1)\pi x}{L}\\
&= -\frac{gx(L-x)}{2c^2}+\frac{2gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi cti/L}+e^{-(2n-1)\pi cti/L}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}\\
&= -\frac{gx(L-x)}{2c^2}+\frac{4gL^2}{\pi^3c^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}. \quad (8.34)
\end{aligned}$$

The first term is the static position that the string would occupy were it slowly lowered under the force of gravity. The series represents oscillations about this position due to the fact that the string was dropped from a horizontal position.

The technique of separation of variables and eigenfunction expansions leads to the following solution of problem 8.31

$$y(x,t) = -\frac{2gL^2}{\pi^3 c^2}\sum_{n=1}^{\infty}\frac{[1+(-1)^{n+1}]}{n^3}\left(1-\cos\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}.$$

We can see the advantage of Laplace transforms. They have rendered part of the series solution in closed form, namely the term $gx(L-x)/(2c^2)$ in solution 8.34.•

Examples 8.18 and 8.21 contained specific nonhomogeneities and/or initial conditions. Although Laplace transforms can handle nonhomogeneities and initial conditions with arbitrary functions, they do not do so particularly efficiently. Our final example is an illustration of this.

Example 8.22 Solve the following heat conduction problem in a rod of length L when the initial temperature distribution is an unspecified function $f(x)$,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{8.35a}$$

$$U(0,t) = 0, \quad t > 0, \tag{8.35b}$$

$$U(L,t) = 0, \quad t > 0, \tag{8.35c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{8.35d}$$

Solution When we take Laplace transforms of both sides of PDE 8.35a,

$$s\tilde{U} - f(x) = k\frac{d^2\tilde{U}}{dx^2},$$

or,

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = -\frac{f(x)}{k}, \quad 0 < x < L,$$

subject to transforms of 8.35b,c,

$$\tilde{U}(0,s) = 0, \qquad \tilde{U}(L,s) = 0.$$

Variation of parameters leads to the following general solution of the differential equation

$$\tilde{U}(x,s) = A\cosh\sqrt{\frac{s}{k}}x + B\sinh\sqrt{\frac{s}{k}}x - \frac{1}{\sqrt{ks}}\int_0^x f(u)\sinh\sqrt{\frac{s}{k}}(x-u)\,du.$$

The boundary conditions require

$$0 = A, \qquad 0 = A\cosh\sqrt{\frac{s}{k}}L + B\sinh\sqrt{\frac{s}{k}}L - \frac{1}{\sqrt{ks}}\int_0^L f(u)\sinh\sqrt{\frac{s}{k}}(L-u)\,du.$$

Thus,

$$\tilde{U}(x,s) = \frac{\sinh\sqrt{\frac{s}{k}}x}{\sqrt{ks}\sinh\sqrt{\frac{s}{k}}L}\int_0^L f(u)\sinh\sqrt{\frac{s}{k}}(L-u)\,du - \frac{1}{\sqrt{ks}}\int_0^x f(u)\sinh\sqrt{\frac{s}{k}}(x-u)\,du$$

$$= \frac{1}{\sqrt{k}}\int_0^L f(u)\tilde{p}(x,u,s)\,du - \frac{1}{\sqrt{ks}}\int_0^x f(u)\sinh\sqrt{\frac{s}{k}}(x-u)\,du$$

where

$$\tilde{p}(x,u,s) = \frac{\sinh\sqrt{\frac{s}{k}}x\,\sinh\sqrt{\frac{s}{k}}(L-u)}{\sqrt{s}\,\sinh\sqrt{\frac{s}{k}}L}.$$

The second integral in $\tilde{U}(x,s)$ has a removable singularity at $s=0$, and therefore this term may be ignored in taking inverse transforms by residues. The function $\tilde{p}(x,u,s)$ has singularities when $\sqrt{s/k}L = n\pi i$, or, $s = -n^2\pi^2k/L^2$. Since

$$\tilde{p}(x,u,s) = \frac{\left[\frac{\sqrt{s}x}{\sqrt{k}} + \frac{1}{6}\left(\frac{\sqrt{s}x}{\sqrt{k}}\right)^3 + \cdots\right]\left[\frac{\sqrt{s}(L-u)}{\sqrt{k}} + \frac{1}{6}\left(\frac{\sqrt{s}(L-u)}{\sqrt{k}}\right)^3 + \cdots\right]}{\sqrt{s}\left[\frac{\sqrt{s}L}{\sqrt{k}} + \frac{1}{6}\left(\frac{\sqrt{s}L}{\sqrt{k}}\right)^3 + \cdots\right]} = \frac{x(L-u)}{L\sqrt{k}} + \cdots,$$

it follows that $\tilde{p}(x,u,s)$ has a removable singularity at $s=0$. The singularities $s = -n^2\pi^2k/L^2$ are poles of order one, and residues of $e^{st}\tilde{p}(x,u,s)$ at these poles are

$$\lim_{s\to -n^2\pi^2k/L^2}\left(s + \frac{n^2\pi^2k}{L^2}\right)e^{st}\frac{\sinh\sqrt{\frac{s}{k}}x\,\sinh\sqrt{\frac{s}{k}}(L-u)}{\sqrt{s}\,\sinh\sqrt{\frac{s}{k}}L}$$

$$= \frac{e^{-n^2\pi^2kt/L^2}}{n\pi\sqrt{k}i/L}\sinh\frac{n\pi xi}{L}\sinh\frac{n\pi i(L-u)}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\frac{L}{2\sqrt{ks}}\cosh\sqrt{\frac{s}{k}}L}$$

$$= \frac{-2}{n\pi i}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi(L-u)}{L}\frac{1}{\frac{L}{n\pi\sqrt{k}i}\cosh n\pi i}$$

$$= \frac{-2\sqrt{k}}{L}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}(-1)^{n+1}\sin\frac{n\pi u}{L}\frac{1}{(-1)^n}$$

$$= \frac{2\sqrt{k}}{L}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}.$$

Residues of e^{st} times the first integral in $\tilde{U}(x,s)$ can now be calculated by interchanging limits and integration in

$$\lim_{s\to -\frac{n^2\pi^2k}{L^2}}\left(s + \frac{n^2\pi^2k}{L^2}\right)\frac{e^{st}}{\sqrt{k}}\int_0^L f(u)\tilde{p}(x,u,s)\,du$$

to obtain

$$\frac{1}{\sqrt{k}}\int_0^L\left[\lim_{s\to -\frac{n^2\pi^2k}{L^2}}\left(s + \frac{n^2\pi^2k}{L^2}\right)e^{st}f(u)\tilde{p}(x,u,s)\right]du$$

$$= \frac{1}{\sqrt{k}}\int_0^L f(u)\left[\lim_{s\to -\frac{n^2\pi^2k}{L^2}}\left(s + \frac{n^2\pi^2k}{L^2}\right)e^{st}\tilde{p}(x,u,s)\right]du$$

$$= \frac{1}{\sqrt{k}}\int_0^L f(u)\left[\frac{2\sqrt{k}}{L}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}\right]du$$

$$= \frac{2}{L}e^{-n^2\pi^2kt/L^2}\left[\int_0^L f(u)\sin\frac{n\pi u}{L}\,du\right]\sin\frac{n\pi x}{L}.$$

The inverse transform of $\tilde{U}(x,s)$ is therefore

$$U(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(u)\sin\frac{n\pi u}{L}du\right]e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.\bullet$$

Readers who are familiar with the method of separation of variables for solving PDEs will attest to the fact that separation of variables is more efficient than Laplace transforms in obtaining the above solution. Similar complications arise when nonhomogeneities involve arbitrary functions. In general, Laplace transforms are less appealing as an alternative to separation of variables when nonhomogeneities and initial conditions contain unspecified functions. As a result exercises on bounded intervals will be confined to problems containing specific nonhomogeneities and initial conditions.

EXERCISES 8.5

In these exercises use Laplace transforms to solve the initial, boundary-value problem.

1. A very long cylindrical rod is placed along the positive x-axis with one end at $x = 0$. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is 0° C throughout. For $t > 0$, heat is added at a constant rate to the left end. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$U_x(0,t) = C, \quad t > 0,$$
$$U(x,0) = 0, \quad x > 0,$$

where $k > 0$ and $C < 0$ are constants.

2. A very long cylindrical rod is placed along the positive x-axis with one end at $x = 0$. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is a constant $\overline{U}$ throughout. For $t > 0$, the left end has temperature U_0. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0,$$
$$U(0,t) = U_0, \quad t > 0,$$
$$U(x,0) = \overline{U}, \quad x > 0,$$

where $k > 0$ is a constant.

3. Use convolutions to express the solution to Exercise 1 in integral form when the boundary condition at $x = 0$ is $U_x(0,t) = f(t),\ t > 0$.

4. Use convolutions to express the solution to Exercise 2 in integral form when the boundary condition at $x = 0$ is $U(0,t) = f(t),\ t > 0$.

5. A very long string lies motionless along the positive x-axis. If the left end $(x = 0)$ is subjected to vertical motion described by $f(t)$ for $t > 0$, subsequent displacements $y(x,t)$ of the string are described by the initial, boundary-value problem

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0, \quad t > 0, \\
y(0,t) &= f(t), \quad t > 0, \\
y(x,0) &= 0, \quad x > 0, \\
y_t(x,0) &= 0, \quad x > 0,
\end{aligned}
$$

where $c > 0$ is a constant.

6. A very long string lies motionless along the positive x-axis. At time $t = 0$, the support is removed and gravity is permitted to act on the string. If the left end ($x = 0$) is subjected to periodic vertical motion described by $\sin \omega t$ for $t > 0$, subsequent displacements $y(x,t)$ of the string are described by the initial, boundary-value problem

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad x > 0, \quad t > 0, \\
y(0,t) &= \sin \omega t, \quad t > 0, \\
y(x,0) &= 0, \quad x > 0, \\
y_t(x,0) &= 0, \quad x > 0,
\end{aligned}
$$

where $g = 9.81$ and $c > 0$ is a constant.

7. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is given by $f(x) = \sin(m\pi x/L)$, $0 \le x \le L$, where $m > 0$ is an integer. For $t > 0$, both ends of the rod are held at temperature 0° C. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= k \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\
U(0,t) &= 0, \quad t > 0, \\
U(L,t) &= 0, \quad t > 0, \\
U(x,0) &= \sin \frac{m\pi x}{L}, \quad 0 < x < L,
\end{aligned}
$$

where $k > 0$ is a constant.

8. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is given by $f(x) = x$, $0 \le x \le L$. For $t > 0$, both ends of the rod are insulated. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= k \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\
U_x(0,t) &= 0, \quad t > 0, \\
U_x(L,t) &= 0, \quad t > 0, \\
U(x,0) &= x, \quad 0 < x < L,
\end{aligned}
$$

where $k > 0$ is a constant.

9. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is 0° C throughout. For $t > 0$, its left end ($x = 0$) is kept at 0° C, and its right end ($x = L$) is kept at a constant U_L° C. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ U(0,t) &= 0, \quad t > 0, \\ U(L,t) &= U_L, \quad t > 0, \\ U(x,0) &= 0, \quad 0 < x < L, \end{aligned}$$

where $k > 0$ is a constant.

10. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is a constant U_0° C. For $t > 0$, its end $x = 0$ is insulated, heat is added to the end $x = L$ at a constant rate. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ U_x(0,t) &= 0, \quad t > 0, \\ U_x(L,t) &= C, \quad t > 0, \\ U(x,0) &= U_0, \quad 0 < x < L, \end{aligned}$$

where $k > 0$ and $C > 0$ are constants.

11. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is a constant 100° C. For $t > 0$, its end $x = 0$ is kept at temperature 0°, and end $x = L$ has temperature $100e^{-t}$. The initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ U(0,t) &= 0, \quad t > 0, \\ U(L,t) &= 100e^{-t}, \quad t > 0, \\ U(x,0) &= 100, \quad 0 < x < L, \end{aligned}$$

where $k > 0$ is a constant. Assume that $k \neq L^2/(n^2\pi^2)$ for any integer n.

12. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is 0° C, and for $t > 0$ the ends of the rod continue to be held at 0° C. When heat generation at each point of the rod is described by the function $e^{-\alpha t}$, where α is a positive constant, the initial, boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + e^{-\alpha t}, \quad 0 < x < L, \quad t > 0, \\ U(0,t) &= 0, \quad t > 0, \\ U(L,t) &= 0, \quad t > 0, \\ U(x,0) &= 0, \quad 0 < x < L, \end{aligned}$$

where $k > 0$ is a constant. Assume that $\alpha \neq n^2\pi^2/L^2$ for any integer n.

13. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If it is given an initial displacement at time $t = 0$ of $f(x) = kx(L - x)$, where $k > 0$ is a constant, and no initial velocity, the initial boundary-value problem for displacements $y(x, t)$ of points in the string is

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ y(0, t) &= 0, \quad t > 0, \\ y(L, t) &= 0, \quad t > 0, \\ y(x, 0) &= f(x), \quad 0 < x < L, \\ y_t(x, 0) &= 0, \quad x > 0, \end{aligned}$$

where $c > 0$ is a constant.

14. Repeat Exercise 13 if the initial displacement is zero and $f(x)$ is the initial velocity of the string.

15. A taut string initially at rest along the x-axis has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If gravity is taken into account, the initial boundary-value problem for displacements $y(x, t)$ of points in the string is

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \\ y(0, t) &= 0, \quad t > 0, \\ y(L, t) &= 0, \quad t > 0, \\ y(x, 0) &= 0, \quad 0 < x < L, \\ y_t(x, 0) &= 0, \quad 0 < x < L, \end{aligned}$$

where $g = 9.81$.

16. A taut string initially at rest along the x-axis has its ends at $x = 0$ and $x = L$ fixed on the axis. For $t \geq 0$, it is subjected to a force per unit x-length $F = F_0 \sin \omega t$, where F_0 is a constant, as is $\omega \neq n\pi/L$ for any positive integer n. The initial boundary-value problem for displacements $y(x, t)$ of points in the string is

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F_0}{\rho} \sin \omega t, \quad 0 < x < L, \quad t > 0, \\ y(0, t) &= 0, \quad t > 0, \\ y(L, t) &= 0, \quad t > 0, \\ y(x, 0) &= 0, \quad 0 < x < L, \\ y_t(x, 0) &= 0, \quad 0 < x < L, \end{aligned}$$

where $c > 0$ and $\rho > 0$ are constants.

17. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is $0°$ C, and for $t > 0$, the ends $x = 0$ and $x = L$ of the rod are held at temperature 100° C and 0° C respectively. The initial boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned}\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\ U(0,t) &= 100, \quad t > 0,\\ U(L,t) &= 0, \quad t > 0,\\ U(x,0) &= 0, \quad 0 < x < L,\end{aligned}$$

where $k > 0$ is a constant. Find two solutions, one in terms of error functions, and the other in terms of time exponentials.

18. A cylindrical rod of length L has its ends at $x = 0$ and $x = L$ on the x-axis. Its curved sides are perfectly insulated so that no heat can enter or escape therethrough. At time $t = 0$, the temperature of the rod is 0° C, and for $t > 0$, its left end $x = 0$ continues to be kept at temperature 0° C. If heat is added to the end $x = L$ at a constant rate, the initial boundary-value problem for temperature $U(x,t)$ in the rod is

$$\begin{aligned}\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\ U(0,t) &= 0, \quad t > 0,\\ U_x(L,t) &= C, \quad t > 0,\\ U(x,0) &= 0, \quad 0 < x < L,\end{aligned}$$

where $k > 0$ and $C > 0$ are constants. Find two solutions, one in terms of error functions, and the other in terms of time exponentials.