Solutions to Summer 2025 Exam

1. Show that the following lines intersect, and find the equation of the plane that the contains the lines.

$$L_1: \begin{array}{l} x = 2 - t, \\ y = 1 + t, \\ z = 3t, \end{array} \qquad L_2: \begin{array}{l} x - y + 2z = 13, \\ 3x + y - z = -8. \end{array}$$

When we substitute the parametric equations for L_1 into $x - y + 2z = 13$, we get

$$(2-t) - (1+t) + 2(3t) = 13 \quad \Longrightarrow \quad 4t = 12 \quad \Longrightarrow \quad t = 3.$$

This gives the point $(-1, 4, 9)$. We check whether the point satisfies $3x+y-z = -8$. $3(-1)+4-9 = -8$. Thus, the point of intersection is $(-1, 4, 9)$.

A vector along L_1 is $\mathbf{v}_1 = (-1, 1, 3)$. A vector along L_2 is $\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = (-1, 7, 4)$. A vector normal to the required plane is $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 3 \\ -1 & 7 & 4 \end{vmatrix} = (-17, 1, -6)$. The equation of the required plane is $-17(x+1) + (y-4) - 6(z-9) = 0$.

2. Determine whether the following limit exists. Justify your answer.

$$\lim_{(x,y)\to(0,0))} \frac{x^6 - 2x^3y}{x^3y + y^2}$$

If we appoach $(0,0)$ along cubic curves $y = mx^3$, then

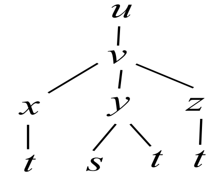
$$\lim_{(x,y)\to(0,0))} \frac{x^6 - 2x^3y}{x^3y + y^2} = \lim_{x\to 0} \frac{x^6 - 2x^6m}{mx^6 + m^2x^6} = \lim_{x\to 0} \frac{1-2m}{m+m^2} = \frac{1-2m}{m+m^2}.$$

Since this depends on m, the original limit does not exist.

3. Find the chain rule for $\left.\dfrac{\partial u}{\partial t}\right)_s$ when $u = f(v)$, $v = g(x, y, z)$, $x = h(t)$, $y = k(s, t)$, and $z = m(t)$. Indicate what variable(s) are being held constant in each partial derivative of your chain rule.

From the schematic,

$$\left.\frac{\partial u}{\partial t}\right)_s = \frac{du}{dv}\left.\frac{\partial v}{\partial x}\right)_{y,z}\frac{dx}{dt} + \frac{du}{dv}\left.\frac{\partial v}{\partial y}\right)_{x,z}\left.\frac{\partial y}{\partial t}\right)_s + \frac{du}{dv}\left.\frac{\partial v}{\partial z}\right)_{x,y}\frac{dz}{dt}$$



4. Find the equation of the tangent plane to the surface $x^3y^2z + xy = 2$ at the point $(1, -1, 3)$.

Since $\nabla(x^3y^2z + xy - 2) = (3x^2y^2z + y, 2x^3yz + x, x^3y^2)$, a normal to the surface at $(1, -1, 3)$ is $\mathbf{N} = (8, -5, 1)$. The equation of the tangent plane is $8(x-1) - 5(y+1) + (z-3) = 0$.

5. Does Euler's Theorem imply that

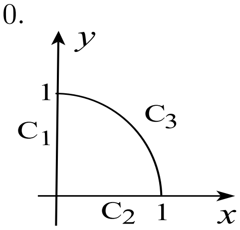
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 2f(x, .y, z)$$

when $f(x, y, z) = x^2 \cos\left(\frac{xy}{z}\right) + y^2 + yz$. Explain.

Since $f(x, y, z)$ is not homogeneous, Euler's Theorem does not apply.

6. Find the maximum value of the function $f(x, y) = x(1 - x - y)$ on the closed region bounded by the curves

$$x = \sqrt{1-y}, \quad x = 0, \quad y = 0.$$



For critical points interior to the region, we solve
$0 = f_x = 1 - 2x - y, \quad 0 = f_y = -x.$
The only solution is $(0, 1)$ (which happens to be on the edge), at which $f(0, 1) = \boxed{0}$. Along C_1, $x = 0$, in which case $f(0, y) = \boxed{0}$. Along C_2, $y = 0$, in which case

$$f(x, 0) = g(x) = x - x^2, \quad 0 \le x \le 1.$$

For critical values, $0 = g'(x) = 1 - 2x \quad \Longrightarrow \quad x = 1/2$. We evaluate

$$g(0) = \boxed{0}, \quad g(1/2) = \boxed{1/4}, \quad g(1) = \boxed{0}.$$

Along C_3, $y = 1 - x^2$, in which case

$$f(x, 1 - x^2) = h(x) = x - x^2 - x(1 - x^2) = x^3 - x^2, \quad 0 \le x \le 1.$$

For critical values, $0 = h'(x) = 3x^2 - 2x \quad \Longrightarrow \quad x = 0, 2/3$. We evaluate

$$h(0) = \boxed{0}, \quad h(2/3) = \boxed{-4/27}, \quad h(1) = \boxed{0}.$$

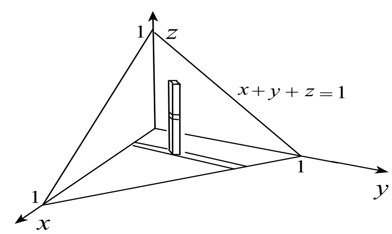
The maximum value is 1/4.

7. Evaluate the triple integral

$$\iiint_V (x+y)\,dV$$

where V is bounded by

$$x+y+z=1, \quad x=0, \quad y=0, \quad z=0.$$



The value is

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y) dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (x+y)(1-x-y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} [(x+y)-(x+y)^2]\,dy\,dx$$

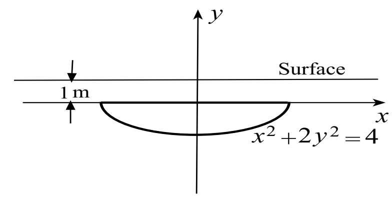
$$= \int_0^1 \left\{ \frac{1}{2}(x+y)^2 - \frac{1}{3}(x+y)^3 \right\}_0^{1-x} dx$$

$$= \int_0^1 \left[\frac{1}{2} - \frac{1}{3} - \frac{x^2}{2} + \frac{x^3}{3} \right] dx$$

$$= \left\{ \frac{x}{6} - \frac{x^3}{6} + \frac{x^4}{12} \right\}_0^1 = \frac{1}{12}$$

8. Do part (a) or part (b), but not both.

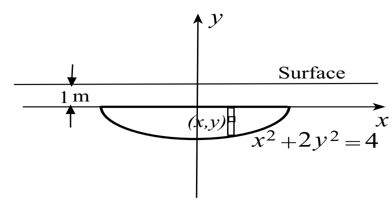
(a) The figure to the right shows a semi-elliptic plate submerged vertically in water with its flat edge 1 metre below the surface. Set up, **but do NOT evaluate**, a double iterated integral to find the force due to water pressure on each vertical side of the plate. Identify numerical values for any physical constants.



(b) Set up. **but do NOT evaluate**, a double iterated integral to find the volume of the solid of revolution when the area bounded by the curves $x = -\sqrt{y}$ and $y = x^3 + 12$ is rotated around the line $y = x - 10$. Simplify your integrand as much as possible.

(a)

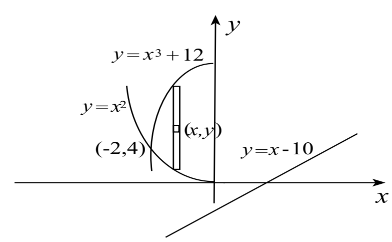
$$F = 2\int_0^2 \int_{-\sqrt{2-x^2/2}}^{0} 1000(9.81)(1-y)\,dy\,dx \quad \text{N}$$



(b)

$$V = \int_{-2}^{0}\int_{x^2}^{x^3+12} 2\pi \frac{|x-y-10|}{\sqrt{2}}\,dy\,dx$$

$$= \sqrt{2}\pi \int_{-2}^{0}\int_{x^2}^{x^3+12} (10+y-x)\,dy\,dx$$



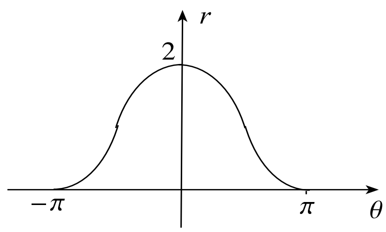
9. Do part (a) or part (b), but not both.

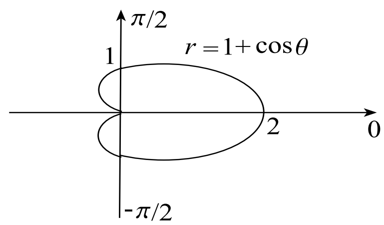
(a) Set up. **but do NOT evaluate**, a double iterated integral in polar coordinates to find the area bounded by the curve

$$x^2 + y^2 = x + \sqrt{x^2 + y^2}.$$

(b) Set up. **but do NOT evaluate**, a double iterated integral in polar coordinates to find the area of that part of the surface $x^2 + y^2 + z^2 = 4$ inside $z = x^2 + y^2$.

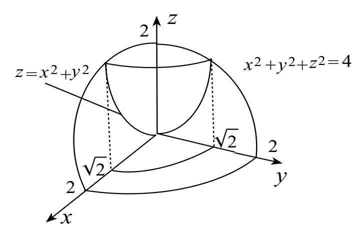
(a)





$$\text{Area} = 2\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta.$$

(b) Area $= 4\displaystyle\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$



Differentiating $x^2 + y^2 + z^2 = 4$ implicitly with respect to x gives $2x + 2z\dfrac{\partial z}{\partial x} = 0 \quad\Longrightarrow\quad \dfrac{\partial z}{\partial x} = -\dfrac{x}{z}$.

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$. Hence,

$$A = 4\iint_{S_{xy}} \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\,dA = 4\iint_{S_{xy}} \sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dA$$

$$= 8\iint_{S_{xy}} \frac{1}{\sqrt{4 - x^2 - y^2}}\,dA = 8\int_0^{\pi/2}\int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - r^2}}\,r\,dr\,d\theta.$$

10. Set up. **but do NOT evaluate**, triple iterated integrals in (a) Cartesian, (b) cylindrical, and (c) spherical coordinates for the value of the triple integral

$$\iiint_V x^2 z \, dV$$

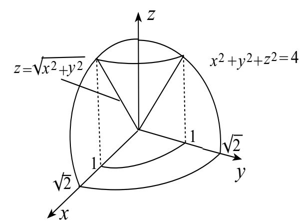
where V is the volume in the first octant bounded by the surfaces

$$x^2 + y^2 + z^2 = 2, \quad z = \sqrt{x^2 + y^2}, \quad x = 0, \quad y = 0.$$

If we denote the integral by I, then

(a) $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} x^2 z \, dz \, dy \, dx$

(b) $I = \int_0^{\pi/2} \int_0^1 \int_r^{\sqrt{2-r^2}} r^2 \cos^2\theta \, z \, r \, dz \, dr \, d\theta$



(c) $I = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} (\mathcal{R} \sin\phi \cos\theta)^2 (\mathcal{R} \cos\phi)(\mathcal{R}^2 \sin\phi \, d\mathcal{R} \, d\phi \, d\theta$