MATH 2130 Test 2 June 3, 2025 60 minutes

12 **1.** Find all directions at the point $(1,1)$ in which the rate of change of the function $f(x,y) = xy - x^2y^2$ is equal to $\sqrt{2}$.

The gradient of $f(x,y)$ at $(1,1)$ is

$$\nabla f_{|(1,1)} = (y - 2xy^2, x - 2x^2y)_{|(1,1)} = (-1,-1).$$

If we let $\hat{\mathbf{v}} = (a,b)$ be a unit vector in the required direction, then

$$\sqrt{2} = \nabla f_{|(1,1)} \cdot (a,b) = (-1,-1) \cdot (a,b) = -a - b.$$

Since $\hat{\mathbf{v}}$ is a unit vector, $a^2 + b^2 = 1$. When we solve these equations, we get $a = b = -1/\sqrt{2}$. Thus, $\hat{\mathbf{v}} = (-1,-1,)/\sqrt{2}$. This is the gradient direction, and this should be expected since $\sqrt{2}$ is the length of the gradient.

10 **2.** Find all unit tangent vectors to the curve

$$x^2y^2 + xy^3z = 4, \qquad x\sin(xz) + xy = 2$$

at the point $(1,2,0)$.

If we set $F(x,y,z) = x^2y^2 + xy^3z - 4$ and $G(x,y,z) = x\sin(xz) + xy - 2$, then normals to these surfaces at the point $(1,2,0)$ are

$$\begin{aligned}\nabla F_{|(1,2,0)} &= (2xy^2 + y^3z, 2x^2y + 3xy^2z, xy^3)_{|(1,2,0)} = (8,4,8),\\ \nabla G_{|(1,2,0)} &= (\sin(xz) + xz\cos(xz), x, x^2\cos(xz))_{|(1,2,0)} = (0,1,1).\end{aligned}$$

A tangent vector to the curve of intersection of the surfaces at $(1,2,0)$ is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = (-1,-2,2).$$

All unit tangent vectors are therefore $\pm(1,2,-2)/3$.

14 **3.** The equations

$$x^2 = u^2 - v^2, \qquad xy = 2u^2v + v^3$$

define u and v as functions of x and y. Find $\partial u/\partial x$ when $u = 2$, $v = 1$, and $x > 0$.

If we set $F(x, y, u, v) = u^2 - v^2 - x^2$ and $G(x, y, u, v) = 2u^2v + v^3 - xy$, then

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -2x & -2v \\ -y & 2u^2 + 3v^2 \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ 4uv & 2u^2 + 3v^2 \end{vmatrix}}.$$

When $u = 2$ and $v = 1$, we find $x = \sqrt{3}$ and $y = 3\sqrt{3}$, and

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -2\sqrt{3} & -2 \\ -3\sqrt{3} & 11 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 8 & 11 \end{vmatrix}} = -\frac{-22\sqrt{3} - 6\sqrt{3}}{44 + 16} = \frac{7\sqrt{3}}{15}.$$

12 **4.** Find all critical points of the function

$$f(x, y) = x^2 - 4xy + 4y^2$$

and classify them as yielding relative maxima, relative minima, saddle points, or none of these.

For critical points, we solve

$$0 = f_x = 2x - 4y, \qquad 0 = f_y = -4x + 8y.$$

Every point on the line $x = 2y$ is critical. When we calculate

$$f_{xx} = 2, \qquad f_{xy} = -4, \qquad f_{yy} = 8,$$

we find that $B^2 - AC = 0$ for all critical points, so that the second derivative test fails. However, if write the function in the form $f(x, y) = (x - 2y)^2$, we can conclude that every critical point yields a relative minimum of 0.