CHAPTER 2 LINEAR TRANSFORMATIONS AND LINEAR OPERATORS

§2.1 Linear Transformations

A **transformation** T from a vector space V to a vector space W is a mapping or function that associates with each vector $\mathbf{v}$ in V a vector $\mathbf{w}$ in W. We can express transformations in various ways. The most compact, but not the most informative, is to use functional notation

$$\mathbf{w} = T(\mathbf{v}). \tag{2.1a}$$

In general, each component of $\mathbf{w}$ (with respect to some basis of W) is a function of all components of $\mathbf{v}$ (with respect to some basis of V). If V is n-dimensional and W is m-dimensional, we could indicate this by writing

$$T: \begin{aligned} w_1 &= f_1(v_1, v_2, \ldots, v_n),\\ w_2 &= f_2(v_1, v_2, \ldots, v_n),\\ &\vdots \quad \vdots \quad \vdots \quad \vdots\\ w_m &= f_m(v_1, v_2, \ldots, v_n). \end{aligned} \tag{2.1b}$$

When $W = V$, we call T an **operator** on V. In this case we often denote the image of $\mathbf{v}$ by $\mathbf{v}'$, and equations 2.1 become

$$\mathbf{v}' = T(\mathbf{v}), \tag{2.2a}$$

and

$$T: \begin{aligned} v_1' &= f_1(v_1, v_2, \ldots, v_n),\\ v_2' &= f_2(v_1, v_2, \ldots, v_n),\\ &\vdots \quad \vdots \quad \vdots \quad \vdots\\ v_n' &= f_n(v_1, v_2, \ldots, v_n). \end{aligned} \tag{2.2b}$$

For *linear* transformations, which are yet to be defined, we find representations 2.1b and 2.2b more informative than 2.1a and 2.2a. Here is an example.

Example 2.1 Vectors $\mathbf{v}$ in $\mathcal{R}^3$ with components (v_1, v_2, v_3) (relative to some basis) are mapped to vectors $\mathbf{w}$ in $\mathcal{R}^2$ with components (w_1, w_2) (relative to some basis) as follows:

$$T: \begin{aligned} w_1 &= v_1^2 + 2v_2 - v_3 - 5,\\ w_2 &= 3v_1 + 4v_1v_2. \end{aligned}$$

Find $T(1, 3, -2)$. In actual fact, we should write this as $T\big((1, 3, -2)\big)$, one set of parentheses for the transformation, and a second set for the vector. No clarity is lost by using only one set, and we will therefore continue this practice.

Solution Since $w_1 = 1^2 + 2(3) + 2 - 5 = 4$ and $w_2 = 3(1) + 4(1)(3) = 15$, we obtain $T(1, 3, -2) = (4, 15)$.●

Here is a physical example.

Example 2.2 When a mass M is moving in space, there is a 3-dimensional vector space V of possible velocities $\mathbf{v}$ of the mass. The momentum $\mathbf{H}$ of m is defined as $\mathbf{H} = m\mathbf{v}$, and there is therefore a 3-dimensional space W of momenta. Set up a transformation from V to W in form 2.1b.

Solution If $\mathbf{H} = \langle H_x, H_y, H_z\rangle$ and $\mathbf{v} = \langle v_x, v_y, v_z\rangle$, then $\langle H_x, H_y, H_z\rangle = m\langle v_x, v_y, v_z\rangle$. If the transformation is denoted by T, then

$$T: \quad \begin{aligned} H_x &= mv_x, \\ H_y &= mv_y, \\ H_z &= mv_z. \bullet \end{aligned}$$

Here is another physical example.

Example 2.3 Suppose forces $\mathbf{F}$ with components $\langle F_x, F_y, F_z \rangle$ act at the point $(1, 2, 3)$ in space. The moments (or torques) $\mathbf{M}$ of these forces about the origin are defined as $\mathbf{M} = \langle 1, 2, 3 \rangle \times \mathbf{F}$. Set this up as a transformation from vectors $\mathbf{F}$ in a vector space of forces to vectors $\mathbf{M}$ in a vector space of moments.

Solution Since

$$\mathbf{M} = \langle 1, 2, 3 \rangle \times \mathbf{F} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ F_x & F_y & F_z \end{pmatrix} = (2F_z - 3F_y)\hat{\mathbf{i}} + (3F_x - F_z)\hat{\mathbf{j}} + (F_y - 2F_x)\hat{\mathbf{k}},$$

we can describe the transformation as follows if we set $\mathbf{M} = \langle M_x, M_y, M_z \rangle$,

$$T: \quad \begin{aligned} M_x &= 2F_z - 3F_y, \\ M_y &= 3F_x - F_z, \\ M_z &= F_y - 2F_x. \bullet \end{aligned}$$

Example 2.4 A transformation T from $M_{2,2}(\mathcal{C})$ to $M_{2,2}(\mathcal{R})$ maps matrices as follows:

$$T\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} |z_1|^2 & z_2 - \overline{z_2} \\ 0 & |z_3| + \operatorname{Im}(z_4) - 5 \end{pmatrix}.$$

Find the transform of the matrix $\begin{pmatrix} 3+4i & 1-6i \\ 7+5i & 2-i \end{pmatrix}$.

Solution The transform is

$$\begin{pmatrix} |3+4i|^2 & (1-6i) - \overline{1-6i} \\ 0 & |7+5i| - 1 - 5 \end{pmatrix} = \begin{pmatrix} 5 & -12i \\ 0 & \sqrt{74} - 6 \end{pmatrix}. \bullet$$

We are only interested in transformations that satisfy the following definition.

Definition 2.1 A transformation L from vector space V to vector space W is **linear** if for every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, and every scalar c, the following two properties are satisfied:

$$L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}), \tag{2.3a}$$

$$L(c\,\mathbf{v}) = c[L(\mathbf{v})]. \tag{2.3b}$$

When $W = V$, L is said to be a **linear operator** on V.

A transformation is linear if it preserves vector addition and scalar multiplication; that is, if two vectors are added before their sum is transformed, then their images are added after they are transformed, and if a vector is multiplied by a scalar before it is transformed, its image is multiplied by the same scalar after it is transformed.

An important point to notice is that on the left side of equation 2.3b, scalar c multiplies a vector in V, whereas on the right side of the equation it multiplies a vector in W. This implies that the set of scalars for V must be a subset of the set of scalars for W. For instance, a linear transformation could map a real vector space to a complex one, but it cannot map a complex vector space to a real one.

We have replaced the letter T for transformations by the letter L for linear transformations to emphasize linearity. The transformation in Example 2.1 is not linear; it violates both conditions. For instance, if $\mathbf{v} = (1, 0, 0)$ and $c = 2$,

$$T(c\,\mathbf{v}) = T(2,0,0) = (0,6) \quad \text{whereas} \quad 2[T(\mathbf{v})] = 2\,T(1,0,0) = 2(-1,3) = (-2,6).$$

The transformations in Examples 2.2 and 2.3 are both linear. We illustrate with Example 2.3. We could use the component representation, but it is easier to use the cross product form. Indeed, for any two force vectors **F** and **G** and any constant c,

$$\begin{aligned} T(\mathbf{F}+\mathbf{G}) &= \mathbf{r}\times(\mathbf{F}+\mathbf{G}) = (\mathbf{r}\times\mathbf{F}) + (\mathbf{r}\times\mathbf{G}) = T(\mathbf{F}) + T(\mathbf{G}),\\ T(c\,\mathbf{F}) &= \mathbf{r}\times(c\mathbf{F}) = c(\mathbf{r}\times\mathbf{F}) = c[T(\mathbf{F})]. \end{aligned}$$

Example 2.5 An operator L on a 3-dimensional space maps vectors with components (v_1, v_2, v_3) (relative to some basis) according to

$$L(\mathbf{v}) = L(v_1, v_2, v_3) = (3v_1 - 2v_2, 4v_1 - 2v_2 + v_3, 3v_3 - 4v_1).$$

Show that L is linear, and find components of $L(1,-3,5)$ relative to the same basis.

Solution If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are any two vectors in the space and c is any scalar, then

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= L(u_1+v_1, u_2+v_2, u_3+v_3)\\ &= \big(3(u_1+v_1) - 2(u_2+v_2), 4(u_1+v_1) - 2(u_2+v_2) + (u_3+v_3), 3(u_3+v_3) - 4(u_1+v_1)\big)\\ &= (3u_1 - 2u_2, 4u_1 - 2u_2 + u_3, 3u_3 - 4u_1) + (3v_1 - 2v_2, 4v_1 - 2v_2 + v_3, 3v_3 - 4v_1)\\ &= L(\mathbf{u}) + L(\mathbf{v}),\\ L(c\mathbf{v}) &= L(cv_1, cv_2, cv_3) = (3cv_1 - 2cv_2, 4cv_1 - 2cv_2 + cv_3, 3cv_3 - 4cv_1)\\ &= c(3v_1 - 2v_2, 4v_1 - 2v_2 + v_3, 3v_3 - 4v_1) = c[L(\mathbf{v})]. \end{aligned}$$

Thus, L is linear.

$$L(1,-3,5) = \big(3(1) - 2(-3), 4(1) - 2(-3) + 5, 3(5) - 4(1)\big) = (9, 15, 11)\bullet$$

Example 2.6 If L is the operator of Example 2.5, find the vector $\mathbf{v}$ that maps to $\mathbf{v}' = (1,-1,2)$.

Solution For $L(\mathbf{v}) = (1,-1,2)$, we must have

$$\begin{aligned} 1 &= 3v_1 - 2v_2,\\ -1 &= 4v_1 - 2v_2 + v_3,\\ 2 &= -4v_1 + 3v_3. \end{aligned}$$

The solution of these equations is $v_1 = -8/7$, $v_2 = -31/4$, and $v_3 - 6/7$, and therefore the required vector is $\mathbf{v} = (-8/7, -31/14, -6/7)$.•

According to the following theorem, a linear transformation always maps the zero vector to the zero vector. It may map other vectors to the zero vector also, but it must map the zero vector to the zero vector.

Theorem 2.1 If L is a linear transformation from vector space V to vector space W, then $L(\mathbf{0}) = \mathbf{0}$.

Proof If we set $c = 0$ in linear property 2.3b, we obtain

$$L(0\mathbf{v}) = 0L(\mathbf{v}) \qquad \Longrightarrow \qquad L(\mathbf{0}) = \mathbf{0}.\blacksquare$$

This can sometimes be used as a quick way to verify that a transformation is not linear.

Example 2.7 Suppose that T is the transformation that adds 2 to one or more of the components of a vector in $\mathcal{R}^n$. Is T linear?

Solution Because the zero vector is not mapped to the zero vector, the transformation is not linear.•

Example 2.8 Is the transformation in Example 2.4 linear?

Solution Since the zero vector (matrix) is mapped to

$$T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix},$$

the transformation is not linear.•

Example 2.9 Is the operator T on the space $\mathcal{R}$ that takes a value of x in $\mathcal{R}$ to $T(x) = ax + b$, where a and b are constants, linear?

Solution A necessary condition for T to be linear is that $T(0) = 0$, but this is only true when $b = 0$. It is straightforward to prove that with $b = 0$, the operator satisfies properties 2.3. It is unfortunate that what we call a linear function $f(x) = ax + b$ in classical algebra is not, in general, a linear operator in linear algebra.•

We now show that when a transformation is linear, then the functions in equations 2.1b must be linear in the components of $\mathbf{v}$.

Theorem 2.2 A transformation L from an n-dimensional vector space V to an m-dimensional vector space W is linear if, and only if, equations 2.1b are linear in the components of $\mathbf{v}$; that is, if, and only if, the equations take the form

$$L: \begin{aligned} w_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n, \\ w_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n, \\ &\vdots \quad\ \vdots \quad\ \vdots \quad\ \vdots \\ w_m &= a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n. \end{aligned} \tag{2.4a}$$

More compactly,

$$w_j = \sum_{k=1}^{n} a_{jk}v_k, \quad j = 1, \ldots, m. \tag{2.4b}$$

Proof: Suppose that L is linear, $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V, and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m\}$ is a basis for W. Since transforms of the basis vectors for V are in W, they must be linear combinations of the basis vectors in W. Denote their components as follows

$$L(\mathbf{b}_k) = \sum_{j=1}^{m} a_{jk}\mathbf{d}_j, \quad k = 1, \ldots, n.$$

Any vector $\mathbf{v}$ with components $(v_1, v_2, \ldots, v_n)$ in V is mapped by L to a vector $\mathbf{w}$ in W. Suppose the components of $\mathbf{w}$ are $(w_1, w_2, \ldots, w_m)$. Since L is a linear transformation, we can write that

$$\begin{aligned} \mathbf{w} &= w_1\mathbf{d}_1 + w_2\mathbf{d}_2 + \cdots + w_m\mathbf{d}_m = \sum_{j=1}^{m} w_j\mathbf{d}_j = L(\mathbf{v}) = L(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) \\ &= v_1\,L(\mathbf{b}_1) + v_2\,L(\mathbf{b}_2) + \cdots + v_n\,L(\mathbf{b}_n) = \sum_{k=1}^{n} v_k\,L(\mathbf{b}_k) = \sum_{k=1}^{n} v_k \sum_{j=1}^{m} a_{jk}\mathbf{d}_j \\ &= \sum_{j=1}^{m}\left(\sum_{j=1}^{n} a_{jk}v_k\right)\mathbf{d}_j. \end{aligned}$$

When we equate $\mathbf{d}_j$-components, we obtain

$$w_j = \sum_{k=1}^{n} a_{jk} v_k, \quad j = 1, \ldots, m.$$

These are equations 2.4b. Conversely, suppose equations 2.4b describe a transformation from V to W. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, be any two vectors in V, and c be any scalar. Then

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= \sum_{j=1}^{m} \left[\sum_{k=1}^{n} a_{jk}(u_k + v_k) \right] \mathbf{d}_j = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} u_k + \sum_{k=1}^{n} a_{jk} v_k \right) \mathbf{d}_j \\ &= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} u_k \right) \mathbf{d}_j + \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} v_k \right) \mathbf{d}_j \\ &= L(\mathbf{u}) + L(\mathbf{v}), \\ L(c\mathbf{v}) &= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} c v_k \right) \mathbf{d}_j = c \sum_{j=1}^{m} \left(\sum_{k=1}^{n} a_{jk} v_k \right) \mathbf{d}_j = c\, L(\mathbf{v}). \end{aligned}$$

This verifies that L is linear.∎

We now have two ways to verify that a transformation is linear. Verify that it satisfies properties 2.3, or show that it can be expressed component-wise in form 2.4.

Example 2.10 Show that the operator L on $\mathcal{G}^3$ that reflects vectors in the yz-plane is linear.

Solution The operator changes the sign of the x-component of vectors,

$$L\big(\langle v_x, v_y, v_z \rangle\big) = \langle -v_x, v_y, v_z \rangle.$$

Since the components of $L(\mathbf{v})$ are linear combinations of those of $\mathbf{v}$, the transformation is linear. Alternatively, if $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ and $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ are any two vectors in $\mathcal{G}^3$, and c is a scalar, then

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= L\big(\langle u_x + v_x, u_y + v_y + u_z + v_z \rangle\big) = \langle -(u_x + v_x), u_y + v_y, u_z + v_z \rangle \\ &= \langle -u_x, u_y, u_z \rangle + \langle -v_x, v_y, v_z \rangle = L(\mathbf{u}) + L(\mathbf{v}), \\ L(c\mathbf{v}) &= L\big(\langle cv_x, cv_y, cv_z \rangle\big) = \langle -cv_x, cv_y, cv_z \rangle = c\langle -v_x, v_y, v_z \rangle = c\, L(\mathbf{v}).\bullet \end{aligned}$$

According to the next theorem, the action of taking the vector component of a vector along one subspace as determined by another subspace is a linear transformation.

Theorem 2.3 If vector space V is the direct sum of the subspaces W_1 and W_2, then taking components of vectors in V along W_1 as determined by W_2 is a linear transformation.

Proof Any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V can be expressed in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}_3 + \mathbf{w}_4$ where $\mathbf{w}_1$ and $\mathbf{w}_3$ are in W_1, and $\mathbf{w}_2$ and $\mathbf{w}_4$ are in W_2. If P denotes the action of taking components of vectors in V along W_1 as determined by W_2, then $P(\mathbf{u}) = \mathbf{w}_1$ and $P(\mathbf{v}) = \mathbf{w}_3$. Since $\mathbf{u} + \mathbf{v} = (\mathbf{w}_1 + \mathbf{w}_2) + (\mathbf{w}_3 + \mathbf{w}_4) = (\mathbf{w}_1 + \mathbf{w}_3) + (\mathbf{w}_2 + \mathbf{w}_4)$,

$$P(\mathbf{u}+\mathbf{v}) = \mathbf{w}_1 + \mathbf{w}_3 = P(\mathbf{u}) + P(\mathbf{v}),$$

and if c is a scalar,

$$P(c\mathbf{u}) = P\big(c(\mathbf{w}_1 + \mathbf{w}_2)\big) = P(c\mathbf{w}_1 + c\mathbf{w}_2) = c\mathbf{w}_1 = c\,P(\mathbf{u}).$$

Thus, P is linear.∎

Example 2.11 Let P be the transformation that takes the vector component of vectors $\mathbf{v}$ in $\mathcal{G}^3$ along the plane $y = x$ as determined by the subspace of vectors perpendicular to the plane. We call $P(\mathbf{v})$ the **orthogonal component** of $\mathbf{v}$ along (or in) the plane $y = x$. (Orthogonal is a word that replaces "perpendicular" in spaces that do not have the geometry of $\mathcal{G}^3$. We shall see this in Chapter 5.) According to Theorem 2.4, P is linear. Demonstrate this also by finding the vector component of a vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$.

Solution We can do this geometrically, or algebraically. First the algebraic way. A basis for W_1 is $\{\hat{\mathbf{i}} + \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, and a basis for W_2 is the vector $\hat{\mathbf{i}} - \hat{\mathbf{j}}$. Every vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ in $\mathcal{G}^3$ can be expressed in the form

$$\mathbf{v} = \langle v_x, v_y, v_z \rangle = c_1\mathbf{w}_1 + c_2\mathbf{w}_2,$$

where $\mathbf{w}_1$ is in W_1, and $\mathbf{w}_2$ is in W_2. Vector $\mathbf{w}_1$ can be expressed in terms of the basis vectors for W_1, and $\mathbf{w}_2$ is a multiple of $\hat{\mathbf{i}} - \hat{\mathbf{j}}$,

$$\mathbf{w}_1 = d_1(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + d_2\hat{\mathbf{k}}, \qquad \mathbf{w}_2 = f(\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

When we combine these equations,

$$\langle v_x, v_y, v_z \rangle = c_1(d_1(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + d_2\hat{\mathbf{k}}) + c_2f(\hat{\mathbf{i}} - \hat{\mathbf{j}}).$$

Equating natural components gives

$$v_x = c_1d_1 + c_2f, \quad v_y = c_1d_1 - c_2f, \quad v_z = c_1d_2.$$

Thus, $c_1d_1 = (v_x + v_y)/2$ and $c_1d_2 = v_z$; that is,

$$\mathbf{w}_1 = c_1d_1(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + c_1d_2\hat{\mathbf{k}} = \left(\frac{v_x + v_y}{2}\right)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + v_z\hat{\mathbf{k}}.$$

Since $\mathbf{w}_1$ is the orthogonal component of $\mathbf{v}$ along the plane $y = x$, we can write that

$$P\big(\langle v_x, v_y, v_z \rangle\big) = \left(\frac{v_x + v_y}{2}\right)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + v_z\hat{\mathbf{k}}.$$

Because the components of $P(\mathbf{v})$ are linear combinations of the components of $\mathbf{v}$, P is a linear transformation. Now consider the geometric way. The vector component $P(\mathbf{v})$ is the sum of the vectors $\mathbf{v}$ and $\mathbf{n}$ in Figure 2.1. The direction of $\mathbf{n}$ is perpendicular to the plane and is therefore a multiple of $\hat{\mathbf{i}} - \hat{\mathbf{j}}$. The length of $\mathbf{n}$ is the distance from the point (v_x, v_y, v_z) to the plane,

$$\frac{|v_x - v_y|}{\sqrt{2}}.$$

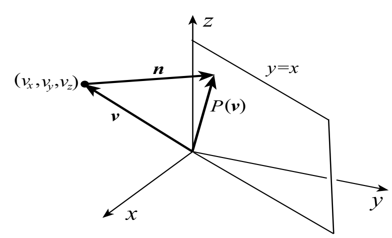

Figure 2.1

Hence,

$$\mathbf{n} = \frac{|v_x - v_y|}{\sqrt{2}}\left(\pm\frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}\right).$$

When $v_x > v_y$, we should use the negative in this equation and when $v_x < v_y$, we should use the positive. In both cases we obtain

$$\mathbf{n} = \frac{v_y - v_x}{\sqrt{2}}\left(\frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}\right).$$

Consequently,

$$P(\mathbf{v}) = \mathbf{v} + \mathbf{n} = (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}) + \frac{v_y - v_x}{\sqrt{2}}\left(\frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}\right) = \left(\frac{v_x + v_y}{2}\right)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + v_z\hat{\mathbf{k}}.\bullet$$

Due to linearity properties 2.3, a linear transformation is completely defined once its action on any set of basis vectors is known. For example, suppose that L is a linear transformation from a 2-dimensional space V with basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a 3-dimensional space W with basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, and the action of L on the basis of V is specified as

$$L(\mathbf{v}_1) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 \qquad \text{and} \qquad L(\mathbf{v}_2) = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3.$$

If $\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2$ is any other vector in V, then

$$\begin{aligned} L(\mathbf{u}) &= L(u_1\mathbf{v}_1 + u_2\mathbf{v}_2) = u_1\,L(\mathbf{v}_1) + u_2\,L(\mathbf{v}_2) \\ &= u_1(c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3) + u_2(d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3) \\ &= (u_1c_1 + u_2d_1)\mathbf{w}_1 + (u_1c_2 + u_2d_2)\mathbf{w}_2 + (u_1c_3 + u_2d_3)\mathbf{w}_3. \end{aligned}$$

In other words, $L(\mathbf{u})$ is known. This is worth stating as a theorem.

Theorem 2.4 A linear transformation on a vector space V is completely determined by its action on any set of basis vectors for V.

Example 2.12 If a linear operator L on $\mathcal{G}^3$ maps $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ as follows

$$L(\hat{\mathbf{i}}) = \langle -2, 1, 4\rangle, \quad L(\hat{\mathbf{j}}) = \langle 0, 5, -1\rangle, \quad L(\hat{\mathbf{k}}) = \langle 1, 1, 8\rangle,$$

find $L(\langle 5, -4, 21\rangle)$.

Solution Using linearity,

$$\begin{aligned} L(\langle 5, -4, 21\rangle) &= 5L(\hat{\mathbf{i}}) - 4L(\hat{\mathbf{j}}) + 21L(\hat{\mathbf{k}}) \\ &= 5\langle -2, 1, 4\rangle - 4\langle 0, 5, -1\rangle + 21\langle 1, 1, 8\rangle \\ &= \langle 11, 6, 192\rangle.\bullet \end{aligned}$$

If V is n-dimensional, then any n linearly independent vectors constitutes a basis for the space. Hence, we can state the following corollary.

Corollary 2.4.1 A linear transformation on an n-dimensional vector space V is completely determined by its action on any n linearly independent vectors in V.

Example 2.13 A linear transformation L from $\mathcal{R}^2$ to $\mathcal{R}^3$ is known to map two vectors $(3, -1)$ and $(2, 5)$ as follows:

$$L(3, -1) = (4, 2, 1), \qquad L(2, 5) = (-2, 3, 1).$$

Find $L(4, 7)$.

Solution We can find the transform of the vector $(4, 7)$ if we first find its components with respect to $(3, -1)$ and $(2, 5)$. If we let (v_1, v_2) be the components, then

$$(4, 7) = v_1(3, -1) + v_2(2, 5).$$

When we equate components, we get

$$4 = 3v_1 + 2v_2, \quad 7 = -v_1 + 5v_2 \qquad \Longrightarrow \qquad v_1 = \frac{6}{17}, \quad v_2 = \frac{25}{17}.$$

Thus, $(4, 7) = \frac{6}{17}(3, -1) + \frac{25}{17}(2, 5)$. Alternatively, the transition matrix from the natural basis to the basis $\{(3, -1), (2, 5)\}$ is

$$T_{bn} = T_{nb}^{-1} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}^{-1} = \frac{1}{17}\begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix},$$

from which the components of $(4, 7)$ with respect to the $\{(3, -1), (2, 5)\}$ basis are

$$\frac{1}{17}\begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 6/17 \\ 25/17 \end{pmatrix}.$$

The transform of the vector is therefore

$$L(4, 7) = \frac{6}{17}L(3, -1) + \frac{25}{17}L(2, 5) = \frac{6}{17}(4, 2, 1) + \frac{25}{17}(-2, 3, 1) = \left(-\frac{26}{17}, \frac{87}{17}, \frac{31}{17}\right).\bullet$$

EXERCISES 2.1

1. A operator T on a 3-dimensional space maps vectors $\mathbf{v} = (v_1, v_2, v_3)$ to $\mathbf{v}' = (v_1', v_2', v_3')$ according to

$$T: \quad \begin{aligned} v_1' &= 3v_1 - 2v_2 + 4v_3, \\ v_2' &= -v_1 + v_2, \\ v_3' &= 2v_1 - 2v_2 + 3v_3. \end{aligned}$$

(a) Find the image of $\mathbf{v} = (1, -3, 4)$.
(b) Is T linear?
(c) Find the vector $\mathbf{v}$, or vectors, that map to $\mathbf{v}' = (-1, 2, 0)$.

2. An operator T on a 3-dimensional space maps vectors $\mathbf{v} = (v_1, v_2, v_3)$ to $\mathbf{v}' = (v_1', v_2', v_3')$ according to

$$T: \quad \begin{aligned} v_1' &= 3v_1 - 2v_2, \\ v_2' &= v_1^2, \\ v_3' &= 2v_1 + v_2 - v_3. \end{aligned}$$

(a) Find the image of $\mathbf{v} = (-1, 1, 2)$.
(b) Is T linear?
(c) Find the vector $\mathbf{v}$, or vectors, that map to $\mathbf{v}' = (1, 1, -2)$.

3. An operator T on a 3-dimensional space maps vectors $\mathbf{v} = (v_1, v_2, v_3)$ to $\mathbf{v}' = (v_1', v_2', v_3')$ according to

$$T: \quad \begin{aligned} v_1' &= v_1 - v_2 + 2v_3, \\ v_2' &= 2v_1 + 3v_3, \\ v_3' &= 3v_1 - v_2 + 5v_3. \end{aligned}$$

(a) Find the image of $\mathbf{v} = (-5, 2, 10)$.
(b) Is T linear?
(c) Find the vector $\mathbf{v}$, or vectors, that map to $\mathbf{v}' = (2, 6, 8)$.

4. An operator T on a 3-dimensional space adds the nonzero vector $\mathbf{u} = (u_1, u_2, u_3)$ with constant components to every vector $\mathbf{v}$. Determine whether T is linear.

5. Determine whether the operator that maps every vector $\mathbf{v}$ in $\mathcal{G}^3$ onto the unit vector in the same direction as $\mathbf{v}$ is linear.

6. A linear operator L on $\mathcal{G}^3$ maps the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ as follows:

$$L(\hat{\mathbf{i}}) = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad L(\hat{\mathbf{j}}) = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad L(\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}.$$

(a) Find $L(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$.
(b) Find the vector $\mathbf{v}$ that has $\mathbf{v}' = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ as its image.

7. An operator T on $\mathcal{G}^3$ maps the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ as follows:

$$T(\hat{\mathbf{i}}) = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad T(\hat{\mathbf{j}}) = \hat{\mathbf{i}} - \hat{\mathbf{k}}, \quad T(\hat{\mathbf{k}}) = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}.$$

Can you find $T(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$?

8. An operator T maps vectors by subtracting 2 from each of its components. Determine whether T is linear.

9. An operator T on $\mathcal{G}^3$ maps a vector $\mathbf{v} = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$ onto

$$T(\mathbf{v}) = \begin{cases} v_1\hat{\mathbf{i}} + 2v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}, & \text{if } v_2 \geq 0 \\ v_1\hat{\mathbf{i}} - v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}, & \text{if } v_2 < 0. \end{cases}$$

Determine whether the operator is linear.

10. A linear operator L on $\mathcal{R}^3$ maps three vectors as follows:

$$L(1,-2,4) = (2,3,4), \quad L(3,1,-5) = (4,1,3), \quad L\big(1,1,1\big) = (0,2,5).$$

Find $L(1,2,-2)$.

11. A transformation T maps vectors in $P_3(x)$ to vectors in $P_2(x)$ according to

$$T(ax^3 + bx^2 + cx + d) = (a+b)x^2 + 2cx + 3d.$$

Is T linear?

12. A transformation T maps vectors in $P_3(x)$ to vectors in $P_2(x)$ according to

$$T(ax^3 + bx^2 + cx + d) = |a|x^2 + 2cx + 3d.$$

Is T linear?

13. Let L be a linear transformation from a vector space V to a vector space W. Suppose that $\mathbf{w}$ is a nonzero vector in W, and let S be the set of vectors in V such $L(\mathbf{v}) = \mathbf{w}$. Is S a subspace of V?

14. Repeat Exercise 13 if $\mathbf{w} = \mathbf{0}$.

15. Let T be the transformation from $M_{m,n}(\mathcal{R})$ to $M_{n,m}(\mathcal{R})$ that takes the transpose of a matrix. Is T linear?

16. Let T be the transformation from $M_{n,n}(\mathcal{R})$ to $\mathcal{R}$ that takes the determinant of a matrix. Is T linear?

17. Let T be the transformation from the space $C^0[a,b]$ of continuous functions on the interval $a \leq x \leq b$ to the space R defined by the definite integral

$$T\big(f(x)\big) = \int_a^b f(x)\,dx.$$

Is the transformation linear?

18. Let T be the transformation from space $P_n(x)$ to space $P_{n+1}(x)$ defined by the integral

$$T\big(f(x)\big) = \int_a^x f(x)\,dx,$$

where a is some fixed constant. Is the transformation linear?

19. Let T be the transformation from space $P_n(x)$ to space $P_{n+1}(x)$ that finds an anti-derivative of a polynomial. Is the transformation linear?

20. Let T be the transformation from space $P_n(x)$ to space $P_{n+1}(x)$ that finds the indefinite integral of a polynomial. Is the transformation linear?

21. If $\mathbf{u}$ is a multiple of vector $\mathbf{v}$ in a vector space V, and L is a linear operator on V, is $L(\mathbf{u})$ the same multiple of $L(\mathbf{v})$?

22. Suppose that three vectors **u**, **v**, and **w** in $\mathcal{G}^3$ are such that their tips are collinear. If L is a linear operator on the space, do the images of the vectors have the same property?

23. Let T be the operator on C^n that takes complex conjugates of components of vectors; that is, if $(c_1, c_2, \ldots, c_n)$ is a vector in C^n, then

$$T(c_1, c_2, \ldots, c_n) = (\overline{c_1}, \overline{c_2}, \ldots, \overline{c_n}).$$

Is T linear?

24. Can a linear transformation map a finite-, nonzero-dimensional, complex vector space to a real vector space?

25. Do linear operators on $\mathcal{G}^3$ preserve parallel lines. Do they preserve perpendicular lines?

26. In Exercise 31 of Section 1.4, we introduced magic squares. The subset of all magic squares in $M_{n,n}(\mathcal{R})$ is a subspace, call it $\text{Magic}_{n,n}(\mathcal{R})$. If w represents the weight of a magic square, show that the transformation that maps a matrix in $\text{Magic}_{n,n}(\mathcal{R})$ to its weight in $\mathcal{R}$ is linear.

27. Can there be a linear transformation from $\mathcal{R}^3$ to $\mathcal{R}^2$ that performs the following mapings?

$$L(1,2,3) = (-4,6), \qquad L(-2,1,4) = (1,-2), \qquad L(-1,3,7) = (-3,5).$$

28. Suppose that L is a linear transformation from space V to space W. If the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent, is the set $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_m)\}$ also linearly independent? If the set $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_m)\}$ is linearly independent, is the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$?

In the remaining exercises, we examine the geometry of some special transformations on $\mathcal{G}^2$ and $\mathcal{G}^3$.

29. An operator L on $\mathcal{G}^2$ or $\mathcal{G}^3$ that transforms vectors according to $L(\mathbf{v}) = a\mathbf{v}$, where a is a positive constant is said to be a **dilation** when $a > 1$ and a **contraction** when $a < 1$. Show that such transformations are linear, and discuss their effect on vectors.

30. An operator L on $\mathcal{G}^2$ or $\mathcal{G}^3$ that transforms vectors according to $L(\mathbf{v}) = \mathbf{v} + \mathbf{b}$, where $\mathbf{b}$ is a constant, nonzero vector is said to be a **translation**. Is L linear?

31. Show that the linear operator L on $\mathcal{G}^2$ that takes orthogonal components of vectors along the line $y = mx$ transforms vectors $\mathbf{v} = \langle v_x, v_y \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{1+m^2}\langle v_x + mv_y, mv_x + m^2 v_y\rangle,$$

or,

$$L(\mathbf{v}) = \frac{1}{2}\langle (1+\cos 2\theta)v_x + (\sin 2\theta)v_y, (\sin 2\theta)v_x + (1-\cos 2\theta)v_y\rangle,$$

where $\tan\theta = m$.

32. Show that the linear operator L on $\mathcal{G}^2$ that reflects vectors in the line $y = mx$ transforms vectors $\mathbf{v} = \langle v_x, v_y \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{1+m^2}\langle (1-m^2)v_x + 2mv_y, 2mv_x + (m^2-1)v_y\rangle,$$

or,

$$L(\mathbf{v}) = \langle v_x \cos 2\theta + v_y \sin 2\theta, v_x \sin 2\theta - v_y \cos 2\theta\rangle,$$

where $\tan\theta = m$.

33. Show that the linear operator L on $\mathcal{G}^3$ that takes orthogonal components of vectors along the plane $Ax + By + Cz = 0$ transforms vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{A^2+B^2+C^2}\langle (B^2+C^2)v_x - ABv_y - ACv_z, -ABv_x + (A^2+C^2)v_y - BCv_z, -ACv_x - BCv_y + (A^2+B^2)v_z\rangle.$$

34. Show that the linear operator L on $\mathcal{G}^3$ that reflects vectors in the plane $Ax + By + Cz = 0$ transforms vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{A^2 + B^2 + C^2} \langle (B^2 + C^2 - A^2)v_x - 2ABv_y - 2ACv_z, -2ABv_x + (A^2 + C^2 - B^2)v_y - 2BCv_z, - 2ACv_x - 2BCv_y + (A^2 + B^2 - C^2)v_z \rangle.$$

35. Show that the linear operator L on $\mathcal{G}^3$ that takes orthogonal components of vectors along the line through the origin in the direction $\langle a, b, c \rangle$ transforms vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{a^2 + b^2 + c^2} \langle a^2 v_x + abv_y + acv_z, abv_x + b^2 v_y + bcv_z, acv_x + bcv_y + c^2 v_z \rangle.$$

36. Show that the linear operator L on $\mathcal{G}^3$ that reflects vectors in the line through the origin in the direction $\langle a, b, c \rangle$ transforms vectors $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ according to

$$L(\mathbf{v}) = \frac{1}{a^2 + b^2 + c^2} \langle (a^2 - b^2 - c^2)v_x + 2abv_y + 2acv_z, 2abv_x + (b^2 - a^2 - c^2)v_y + 2bcv_z, 2acv_x + 2bcv_y + (c^2 - a^2 - b^2)v_z \rangle.$$

37. Show that the linear operator L on $\mathcal{G}^2$ that rotates vectors around the origin through angle θ transforms vectors $\mathbf{v} = \langle v_x, v_y \rangle$ according to

$$L(\mathbf{v}) = \langle v_x \cos\theta - v_y \sin\theta, v_x \sin\theta + v_y \cos\theta \rangle.$$

Answers

1.(a) $(25, -4, 20)$ (b) Yes (c) $(-7/3, -1/3, 4/3)$ **2.**(a) $(-5, 1, -3)$ (b) No (c) $(1, 1, 5)$, $(-1, -2, -2)$
3.(a) $(13, 20, 33)$ (b) Yes (c) $(1, 5/3, 4/3)$ **4.** No **5.** No **6.**(a) $(-18, 21, 11)$ (b) $(10/9, 16/9, -1/3)$
7. No, since it is not stated that the operator is linear. **8.** No **9.** No **10.** $(14/15, 7/3, 26/5)$
11. Yes **12.** No **13.** No **14.** Yes, the kernel **15.** Yes **16.** No **17.** Yes **18.** Yes **19.** No
20. No **21.** Yes **22.** Not necessarily **23.** No **24.** No
25. Sometimes but not always; sometimes but not always **27.** No **28.** Not necessarily, Yes
30. No

§2.2 Matrices Associated With Linear Transformations

We can associate a matrix with linear transformation between finite-dimensional vector spaces, and even some infinite-dimensional vector spaces, so that mapping by the transformation is reduced to matrix multiplication. To see this, recall that a transformation is linear if, and only if, it can be expressed in form 2.4a. For example, the transformation L from vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ in a 4-dimensional space to vectors $\mathbf{w} = (w_1, w_2, w_3)$ in a 3-dimensional space defined by the following equations is linear,

$$\begin{aligned} w_1 &= 2v_1 - 3v_2 + v_3 + v_4, \\ w_2 &= -v_1 + 2v_2 - 3v_3 + 4v_4, \\ w_3 &= 3v_1 + 3v_2 - 5v_3 + 2v_4. \end{aligned} \tag{2.5a}$$

If we define a matrix A containing coefficients of the components of $\mathbf{v}$,

$$A = \begin{pmatrix} 2 & -3 & 1 & 1 \\ -1 & 2 & -3 & 4 \\ 3 & 3 & -5 & 2 \end{pmatrix},$$

then, equations 2.5a can be written in matrix form

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & 1 \\ -1 & 2 & -3 & 4 \\ 3 & 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \tag{2.5b}$$

or, more compactly as

$$\mathbf{w} = A\mathbf{v}. \tag{2.5c}$$

We call A the matrix associated with linear transformation L. To find the image under L of a vector, say $\mathbf{v} = (-1, 2, -3, 6)$, we can use equations 2.5a to find components of $\mathbf{w}$,

$$\begin{aligned} w_1 &= 2(-1) - 3(2) + (-3) + (6) = -5, \\ w_2 &= -(-1) + 2(2) - 3(-3) + 4(6) = 38, \\ w_3 &= 3(-1) + 3(2) - 5(-3) + 2(6) = 30. \end{aligned}$$

Thus, $L(\mathbf{v}) = (-5, 38, 30)$. Alternatively, we can use equation 2.5b,

$$\mathbf{w} = A\mathbf{v} = \begin{pmatrix} 2 & -3 & 1 & 1 \\ -1 & 2 & -3 & 4 \\ 3 & 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ 38 \\ 30 \end{pmatrix}.$$

This example illustrates that both representations of vectors as row vectors (or row matrices) and column vectors (or column matrices) are useful. When working with L itself, it is often convenient to use row vectors; whereas when working with its matrix representation A, the column form of the vector must be used. For instance, $\mathbf{w} = L(\mathbf{v})$ could be written in either of the following forms,

$$\mathbf{w} = (w_1, w_2, w_3) = L(\mathbf{v}) = L(v_1, v_2, v_3, v_4) \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = L(\mathbf{v}) = L\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

On the other hand for $\mathbf{w} = A\mathbf{v}$, it must be

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Context always makes it clear which is to be used; sometimes it makes no difference.

Associating a matrix with a linear transformation is possible because components of $\mathbf{w}$ in equations 2.4a are linear combinations of the components of $\mathbf{v}$. Were these expressions not linear, it would not be possible to associate a matrix with the transformation. But then the transformation would not be linear.

Before formalizing these ideas, we make an important observation. The components (v_1, v_2, v_3, v_4) of vectors in equation 2.5 were with respect to some unmentioned basis. Suppose that basis is $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$. Components of these vectors with respect to them as a basis are $\mathbf{b}_1 = (1,0,0,0)$, $\mathbf{b}_2 = (0,1,0,0)$, $\mathbf{b}_3 = (0,0,1,0)$, and $\mathbf{b}_4 = (0,0,0,1)$. If we calculate images of the basis vectors under L, we obtain

$$L(\mathbf{b}_1) = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad L(\mathbf{b}_2) = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}, \quad L(\mathbf{b}_3) = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}, \quad L(\mathbf{b}_4) = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}.$$

Notice that these are the columns of A. We now formalize these ideas and observation.

According to Theorem 2.4, a transformation $\mathbf{w} = L(\mathbf{v})$ from a vector space V to a vector space W is **linear** if, and only if, components $(w_1, w_2, \ldots, w_m)$ of $\mathbf{w}$ (relative to any basis of W) are linear combinations of the components $(v_1, v_2, \ldots, v_n)$ of $\mathbf{v}$ (relative to any basis of V); that is, if, and only if,

$$\begin{aligned} w_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n, \\ w_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n, \\ &\vdots \\ w_m &= a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n. \end{aligned} \tag{2.6}$$

When we associate a matrix $A = (a_{ij})_{m\times n}$ with the transformation, $L(\mathbf{v})$ is given matrically by

$$\mathbf{w} = L(\mathbf{v}) = A\mathbf{v}. \tag{2.7}$$

The i^{th} column of A is the image of the i^{th} basis vector of V. It is important to realize that the matrix associated with a linear transformation depends on the bases for V and W. Change either basis, or both, and the matrix changes. We will have more to say about this in Section 2.5.

Example 2.14 Find the image of the vector $\mathbf{v} = (-3, 2, 4)$ under the linear operator (on a 3-dimensional space) defined by

$$L: \quad \begin{aligned} v_1' &= 2v_1 - 3v_2 + 4v_3, \\ v_2' &= -3v_1 + 4v_2 + v_3, \\ v_3' &= 2v_1 + 3v_2. \end{aligned}$$

Solution Since $v_1' = 2(-3) - 3(2) + 4(4) = 4$, $v_2' = -3(-3) + 4(2) + 4 = 21$, and $v_3' = 2(-3) + 3(2) = 0$, the image vector is $\mathbf{v}' = L(-3, 2, 4) = (4, 21, 0)$. Alternatively, since the matrix associated with L is

$$A = \begin{pmatrix} 2 & -3 & 4 \\ -3 & 4 & 1 \\ 2 & 3 & 0 \end{pmatrix},$$

we obtain

$$\mathbf{v}' = A\mathbf{v} = \begin{pmatrix} 2 & -3 & 4 \\ -3 & 4 & 1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 21 \\ 0 \end{pmatrix}. \bullet$$

Our observation about the columns of the matrix associated with a linear transformation is very useful when a linear transformation is described by its action on vectors rather than given in component form. Here are three examples.

Example 2.15 Find the matrix (relative to the natural bases) associated with the linear operator L on $\mathcal{G}^3$ that rotates vectors by angle θ around the z-axis where $0 < \theta < 2\pi$, counterclockwise as viewed from a point far up the z-axis. Then find $L\big(\langle 1, 3, -2\rangle\big)$.

Solution Figure 2.2 illustrates that

$$L(\hat{\mathbf{i}}) = \langle \cos\theta, \sin\theta, 0\rangle \qquad \text{and} \qquad L(\hat{\mathbf{j}}) = \langle -\sin\theta, \cos\theta, 0\rangle.$$

Furthermore, $L(\hat{\mathbf{k}}) = \hat{\mathbf{k}}$. Thus, the matrix associated with the transformation is

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is now easy to find $L\big(\langle 1, 3, -2\rangle\big)$,

$$L\big(\langle 1, 3, -2\rangle\big) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos\theta - 3\sin\theta \\ \sin\theta + 3\cos\theta \\ -2 \end{pmatrix}. \bullet$$

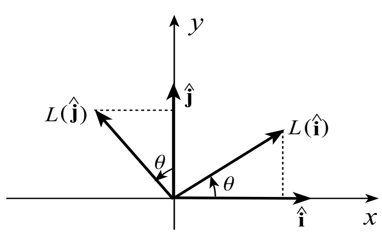

Figure 2.2

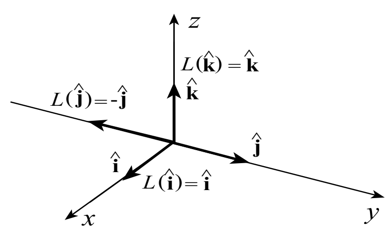

Figure 2.3

Example 2.16 What is the matrix (relative to the natural basis) associated with the linear operator L on $\mathcal{G}^3$ that reflects vectors in the xz-plane? Find $L\big(\langle -1, 2, 5\rangle\big)$.

Solution Since $L(\hat{\mathbf{i}}) = \hat{\mathbf{i}}$, $L(\hat{\mathbf{j}}) = -\hat{\mathbf{j}}$, and $L(\hat{\mathbf{k}}) = \hat{\mathbf{k}}$ (Figure 2.3), the required matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With this matrix,

$$L\big(\langle -1, 2, 5\rangle\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}.$$

We can also use the result of Exercise 37 in Section 2.1 by setting $A = 0$, $B = 1$, and $C = 0$.•

Example 2.17 What is the matrix of the transformation in Example 2.13 relative to the natural basis of $\mathcal{G}^3$.

Solution In Example 2.13, we showed that

$$P\big(\langle v_x, v_y, v_z\rangle\big) = \left(\frac{v_x + v_y}{2}\right)(\hat{\mathbf{i}} + \hat{\mathbf{j}}) + v_z\hat{\mathbf{k}} = \left(\frac{v_x + v_y}{2}\right)\hat{\mathbf{i}} + \left(\frac{v_x + v_y}{2}\right)\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}.$$

This shows that the matrix is

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Alternatively, the vector components $P(\hat{\mathbf{i}}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/2$, $P(\hat{\mathbf{j}}) = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/2$, and $P(\hat{\mathbf{k}}) = \hat{\mathbf{k}}$ of the basis vectors immediately give the same matrix. We can also use the result of Exercise 36 in Section 2.1 by setting $A = 1$, $B = -1$, and $C = 0$.•

The next few examples are not geometric.

Example 2.18 A basis for the vector space of solutions of the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0$$

is $\{e^{2x}, xe^{2x}, x^2e^{2x}\}$. Find the matrix associated with the linear operator $L = \dfrac{d^2}{dx^2} - 3\dfrac{d}{dx}$ using the same basis for the image space. Use the matrix to find $L(-2xe^{2x} + 10x^2e^{2x})$.

Solution We need images of the basis vectors:

$$\begin{aligned} L(e^{2x}) &= 4e^{2x} - 6e^{2x} = -2(e^{2x}), \\ L(xe^{2x}) &= (4e^{2x} + 4xe^{2x}) - 3(e^{2x} + 2xe^{2x}) = e^{2x} - 2(xe^{2x}), \\ L(x^2e^{2x}) &= (2e^{2x} + 8xe^{2x} + 4x^2e^{2x}) - 3(2x^2e^{2x} + 2xe^{2x}) = 2(e^{2x}) + 2(xe^{2x}) - 2(x^2e^{2x}). \end{aligned}$$

The matrix is therefore

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$

Using the matrix, we find that

$$L(-2xe^{2x} + 10x^2e^{2x}) = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \\ 10 \end{pmatrix} = \begin{pmatrix} 18 \\ 24 \\ -20 \end{pmatrix} = 18e^{2x} + 24xe^{2x} - 20x^2e^{2x}.\bullet$$

In previous examples, linear transformation have been between finite-dimensional vector spaces. The space $P(x)$ is infinite-dimensional, but because it has a basis, we can associate matrices of infinite size when this space is involved. The next example is an illustration.

Example 2.19 Let L be the linear operator that maps polynomials in $P(x)$ to their first derivatives $P'(x)$. Find the matrix of the operator with respect to the natural basis $\{1, x, x^2, \ldots\}$. Use the matrix to find the derivative of $3x - 2x^3 + 4x^4$.

Solution The i^{th} column of the matrix is the transform of the i^{th} basis function. With $d(1)/dx = 0$, and $d(x)/dx = 1$, $d(x^2)/dx = 2x = 2(x)$, $d(x^3)/dx = 3x^2 = 3(x^2)$, and so on, the matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Using the matrix, we find that

$$L(3x - 2x^3 + 4x^4) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ -2 \\ 4 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -6 \\ 16 \\ 0 \\ \vdots \end{pmatrix} = 3(1) - 6(x^2) + 16(x^3).\bullet$$

Example 2.20 Let L be the linear operator on $M_{2,2}(R)$ that adds the transpose of a 2×2 real matrix A to itself; that is, $L(A) = A + A^T$. Find the matrix of the operator with respect to the natural basis of the space. Verify that your matrix is correct by transforming the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Solution The natural basis for $M_{2,2}(R)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Transforms of these matrices are

$$L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad L\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since the columns of the matrix associated with L are the components of the transforms of the basis vectors, we obtain

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The transform of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2a \\ b+c \\ b+c \\ 2d \end{pmatrix};$$

that is,

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix},$$

which is the matrix plus its transpose.●

According to Theorem 2.4, the action of taking the vector component of a vector along a subspace W_1 as determined by a second subspace W_2 is a linear transformation from V to W_1, and as such it has a matrix associated with it. Here is an example.

Example 2.21 Vector space $\mathcal{G}^3 = W_1 \oplus W_2$ where W_1 is the subspace of vectors along the line $y = 2x$, $z = -3x$, and W_2 is the subspace of vectors in the xy-plane. Find the matrix, with respect to the natural basis of $\mathcal{G}^3$, of the linear transformation L that takes components of vectors along W_1 as determined by W_2.

Solution Since $\mathcal{G}^3$ is the direct sum of W_1 and W_2, every vector $\mathbf{v}$ in $\mathcal{G}^3$ can be expressed uniquely in the form

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2. Vector $\mathbf{w}_1$ is the component of $\mathbf{v}$ along W_1 as determined by W_2. Columns of the matrix associated with L with respect to the natural basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ of $\mathcal{G}^3$ are components of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ along W_1 as determined by W_2. Since

$$\overset{\text{Vector in } W_1 \downarrow \quad \downarrow \text{Vector in } W_2}{\hat{\mathbf{i}} = \mathbf{0} + \hat{\mathbf{i}}} \qquad \text{and} \qquad \overset{\text{Vector in } W_1 \downarrow \quad \downarrow \text{Vector in } W_2}{\hat{\mathbf{j}} = \mathbf{0} + \hat{\mathbf{j}}},$$

it follows that $L(\hat{\mathbf{i}}) = \mathbf{0}$ and $L(\hat{\mathbf{j}}) = \mathbf{0}$. To find $L(\hat{\mathbf{k}})$, we use the fact that a basis vector for W_1 is $\langle 1, 2, -3 \rangle$, and write

$$\hat{\mathbf{k}} = c_1 \langle 1, 2, -3 \rangle + c_2 \hat{\mathbf{i}} + c_3 \hat{\mathbf{j}}.$$

When we equate components,

$$0 = c_1 + c_2, \quad 0 = 2c_1 + c_3, \quad 1 = -3c_1.$$

These imply that $c_1 = -1/3$, $c_2 = 1/3$, and $c_3 = 2/3$; that is,

$$\overset{\text{Vector in } W_1 \downarrow \qquad\qquad \downarrow \text{Vector in } W_2}{\hat{\mathbf{k}} = -(1/3)\langle 1, 2, -3 \rangle - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}}),}$$

and therefore $L(\hat{\mathbf{k}}) = -(1/3)\langle 1, 2, -3 \rangle$. The matrix of L with respect to the natural basis of $\mathcal{G}^3$ is

$$\begin{pmatrix} 0 & 0 & -1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ is any vector in $\mathcal{G}^3$, its component along W_1 as determined by W_2 is

$$L(\mathbf{v}) = \begin{pmatrix} 0 & 0 & -1/3 \\ 0 & 0 & -2/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -v_z/3 \\ -2v_z/3 \\ v_z \end{pmatrix};$$

that is, $L(\mathbf{v}) = \langle -v_z/3, -2v_z/3, v_z \rangle = -(v_z/3)\langle 1, 2, -3 \rangle$.●

Example 2.22 A linear operator L on a 3-dimensional space maps vectors with natural components $\mathbf{v} = (v_1, v_2, v_3)$ according to

$$L: \begin{aligned} v_1' &= v_1 + 2v_2 - v_3, \\ v_2' &= 3v_1 + 5v_1 - v_2, \\ v_3' &= -v_1 + 2v_2 - 4v_3. \end{aligned}$$

Find the matrix associated with the operator: (a) with respect to the natural basis, and (b) with respect to the basis consisting of the vectors $\mathbf{b}_1 = (-1, 1, 4)$, $\mathbf{b}_2 = (1, 1, 1)$, and $\mathbf{b}_3 = (0, 3, 5)$. (c) If a vector $\mathbf{v}$ has components $(-2, 4, 1)$ with respect to basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, find its image under L by using both matrices.

Solution (a) The matrix associated with the operator, relative to the natural basis, is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -1 & 2 & -4 \end{pmatrix}.$$

Columns are the images of the basis vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.
(b) Columns of the matrix are images of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$,

$$L(\mathbf{b}_1) = A\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -13 \end{pmatrix}, \quad L(\mathbf{b}_2) = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}, \quad L(\mathbf{b}_3) = A\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ -14 \end{pmatrix}.$$

But these are natural components of the images; we require components with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. The transition matrix from natural components to $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ components is

$$T = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -5 & 3 \\ 7 & -5 & 3 \\ -3 & 5 & -2 \end{pmatrix}.$$

Hence, $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ components of the images of the basis vectors are

$$\mathbf{b}_1 = T\begin{pmatrix} -3 \\ -2 \\ -13 \end{pmatrix} = \begin{pmatrix} -7 \\ -10 \\ 5 \end{pmatrix}, \quad \mathbf{b}_2 = T\begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ -6 \\ 7 \end{pmatrix}, \quad \mathbf{b}_3 = T\begin{pmatrix} 1 \\ 10 \\ -14 \end{pmatrix} = \begin{pmatrix} -18 \\ -17 \\ 15 \end{pmatrix}.$$

The matrix, relative to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, is

$$B = \begin{pmatrix} -7 & -8 & -18 \\ -10 & -6 & -17 \\ 5 & 7 & 15 \end{pmatrix}.$$

(c) Using matrix B, the image of $\mathbf{v}$ is

$$L(\mathbf{v}) = B\mathbf{v} = \begin{pmatrix} -7 & -8 & -18 \\ -10 & -6 & -17 \\ 5 & 7 & 15 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -36 \\ -21 \\ 33 \end{pmatrix}.$$

To use the A matrix, we need the natural components of $\mathbf{v}$,

$$\mathbf{v} = -2\mathbf{b}_1 + 4\mathbf{b}_2 + \mathbf{b}_3 = -2(-1,1,4) + 4(1,1,1) + (0,3,5) = (6,5,1).$$

Hence,

$$L(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -1 & 2 & -4 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 42 \\ 0 \end{pmatrix}.$$

These vectors are the same since the natural components of $(-36,-21,33)$ are

$$-36(-1,1,4) - 21(1,1,1) + 33(0,3,5) = (15,42,0).$$

In Section 2.5, we give a complete discussion on how to change the matrix associated with a linear transformation when a change is made to the basis of the space of vectors to be mapped and/or the basis of the space of mapped vectors .•

EXERCISES 2.2

1. Find the vector $\mathbf{v}$ that has image $\mathbf{v}' = (1,-1,2)$ for the linear operator in Example 2.14.

2. Find the matrix associated with the linear operator in Exercise 6 of Section 2.1 and use it to solve part (a).

3. Find the matrix of the linear operator on $\mathcal{G}^3$ that rotates vectors around the x-axis by angle θ.

4. Find the matrix of the linear operator on $\mathcal{G}^3$ that rotates vectors around the y-axis by angle θ.

5. Find the matrix of of the linear operator L on $\mathcal{G}^3$ that rotates every vector around the y-axis by angle $\pi/3$ radians, clockwise as viewed from the point $(0,2,0)$, and then interchanges its x- and y-components.

6. Use Exercise 34 in Section 2.1 to find the matrix associated with the linear operator that takes orthogonal components of vectors in $\mathcal{G}^2$ along the line $y = mx$.

7. Use Exercise 35 in Section 2.1 to find the matrix associated with the linear operator that reflects vectors in $\mathcal{G}^2$ in the line $y = mx$.

8. Use Exercise 36 in Section 2.1 to find the matrix associated with the linear operator that takes orthogonal components of vectors in $\mathcal{G}^3$ along the plane $Ax + By + Cz = 0$.

9. Use Exercise 37 in Section 2.1 to find the matrix associated with the linear operator that reflects vectors in $\mathcal{G}^3$ in the plane $Ax + By + Cz = 0$.

10. Use Exercise 38 in Section 2.1 to find the matrix associated with the linear operator that takes orthogonal components of vectors in $\mathcal{G}^3$ along the line in the direction $\langle a, b, c\rangle$.

11. Use Exercise 39 in Section 2.1 to find the matrix associated with the linear operator that reflects vectors in $\mathcal{G}^3$ in the line in the direction $\langle a, b, c\rangle$.

12. Show that the matrix of the linear operator on $\mathcal{G}^2$ that reflects vectors in the line making angle θ with the positive x-axis is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

13. Forces $\mathbf{F}$ act at a fixed point (x_0, y_0, z_0) in space. Moments of these forces about the origin are defined as $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r} = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}} + z_0\hat{\mathbf{k}}$. If L is the linear transformation from $\mathbf{F}$ to $\mathbf{M}$, find its matrix.

14. A linear operator L on a 3-dimensional space has matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 3 & 5 \\ 0 & 4 & 6 \end{pmatrix}.$$

It maps vectors $\mathbf{v}$ to $\mathbf{v}' = L(\mathbf{v})$. A second linear operator S has matrix

$$B = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 4 & 2 \\ 5 & -2 & 1 \end{pmatrix}.$$

It maps vectors $\mathbf{v}'$ to $\mathbf{v}'' = L(\mathbf{v}')$. Let SL denote the operator that combines S and L,

$$\mathbf{v}'' = (SL)(\mathbf{v}) = S[L(\mathbf{v})].$$

(a) Show that SL is a linear operator by finding the components of $\mathbf{v}''$ in terms of those of $\mathbf{v}$.
(b) Show that the matrix associated with SL is BA. In other words, composition of linear operators corresponds to matrix multiplication. You might want to try proving this in general.

15. (a) Show that the "shift" operator L on the space $P_3(x)$ of polynomials that maps vectors according to

$$L\big(p(x)\big) = p(x - a),$$

where a is a nonzero constant, is linear.
(b) Find the matrix associated with L relative to the natural basis of the space.

16. (a) Show that the transformation L that maps polynomials $p(x)$ in the space $P_2(x)$ to $\mathcal{R}$ according to

$$L\big(p(x)\big) = \int_0^1 p(x)\,dx$$

is linear.

(b) Find the matrix of L with respect to the natural basis $\{1, x, x^2\}$.

(c) Find the matrix of L with respect to the basis $\{1-x, 3+2x, 4+x^2\}$.

17. (a) Show that the transformation L that maps polynomials $p(x)$ in the space $P_2(x)$ to polynomials in $P_3(x)$ according to

$$L\big(p(x)\big) = \int_a^x p(t)\,dt,$$

where a is a fixed constant, is linear.

(b) Find the matrix of L with respect to the natural basis $\{1, x, x^2\}$.

18. According to Exercise 15 in Section 2.1, taking the transpose of a matrix is a linear operation.

(a) Find the matrix of the linear operator that takes transposes of 2×2 matrices.

(b) Find the matrix of the linear operator that takes transposes of 2×3 matrices.

19. (a) Show that the operator on the space $M_{2,2}(\mathcal{R})$ that adds the transpose of the matrix to itself is linear.

(b) Find the matrix of the operator relative to the natural basis

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Relative to the natural basis 1.7 for $M_{3,3}(\mathcal{R})$, what is the matrix of the linear operator that adds the transpose of a matrix to itself?

20. Let L be the linear transformation from $M_{2,3}(\mathcal{R})$ to $P_2(x)$ defined as follows

$$L\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = (a_{11} + a_{12} + a_{13})x^2 + (a_{21} - a_{22})x - a_{23}.$$

Find its matrix with respect to the natural bases of both spaces.

21. (a) In Example 2.19, we found the matrix for the derivative as a linear operator on $P(x)$. If the space is restricted to $P_3(x)$, what is the matrix, call it A.

(b) What is the matrix associated with the second derivative as a linear operator on $P(x)$? Is it A^2? Would you expect this based on Exercise 14?

(c) Demonstrate that the matrix associated with the third derivative is A^3.

(d) Demonstrate that the matrix associated with the fourth derivative is the zero matrix.

22. The first five Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

They form a basis for the vector space $P_4(x)$, and the first four form a basis for $P_3(x)$. The first derivative $f'(x)$ of polynomials in $P_4(x)$ is a linear transformation to $P_3(x)$. Find the matrix for the transformation using:

(a) Chebyshev polynomials as a basis for $P_4(x)$ and the standard basis for $P_3(x)$;

(b) the standard basis for $P_4(x)$ and Chebyshev polynomials as a basis for $P_3(x)$;

(c) Chebyshev polynomials for both spaces.

23. Repeat Exercise 22, but replace the Chebyshev polynomials with the Legendre polynomials

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3x^2 - 1), \quad p_3(x) = \frac{1}{2}(5x^3 - 3x), \quad p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

24. Repeat Exercise 22 using:

(a) Chebyshev polynomials for $P_4(x)$ and Legendre polynomials for $P_3(x)$;

(b) Legendre polynomials for $P_4(x)$ and Chebyshev polynomials for $P_3(x)$.

25. The differential operator $(1-x^2)\dfrac{d^2}{dx^2} - 2x\dfrac{d}{dx}$ is a linear operator on $P_4(x)$.

(a) Find the matrix of the operator relative to the natural basis for $P_4(x)$.

(b) Find the matrix of the operator using the first five Legendre polynomials.

26. Let L be the following differential operator on $C^\infty(-\infty,\infty)$

$$L = a_n(x)\frac{d^n}{dx^n} + a_n(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x),$$

where $a_n(x) \neq 0$ for any x, and let A be the matrix associated with the first derivative. Based on Exercise 14, what is the matrix associated with L?

27. Repeat part (b) of Exercise 15 when the shift operator acts on (a) $P_4(x)$, and (b) $P_5(x)$. Do you see a pattern emerging?

28. What is the matrix of the linear transformation that takes components of vectors in Example 2.21 along W_2 as determined by W_1?

29. Space $\mathcal{G}^3$ is the direct sum of the subspaces W_1 of vectors in the plane $y = 2x$ and W_2 of vectors along the y-axis. Find the matrix, with respect to the natural basis, of the linear transformation that takes components of vectors along W_1 as determined by W_2.

30. Find the matrix of the linear transformation that takes components of vectors along W_2 as determined by W_1 in Exercise 29.

31. Find the matrix of the linear transformation L that maps polynomials $p(x)$ in $P_2(x)$ to polynomials in $P_4(x)$ according to

$$L\big(p(x)\big) = p(x^2 - x + 2).$$

32. Find the matrix of the linear transformation L from $P_n(x)$ to $\mathcal{R}^n$ that transforms a polynomial $p(x)$ as follows

$$L\big(p(x)\big) = \big(p(0), p(1), p(2), \ldots, p(n-1)\big).$$

33. Find the matrix of the linear transformation L from $P_5(x)$ to $P_5(x)$ that transforms a polynomial $p(x)$ as follows

$$L\big(p(x)\big) = \sum_{n=0}^{5} p^{(n)}(n)x^n.$$

34. Suppose that L is a linear operator on V, and $\mathbf{v}$ is a vector in V such that $L^{m-1}(\mathbf{v}) \neq \mathbf{0}$, but $L^m(\mathbf{v}) = \mathbf{0}$. Prove that the set $\{L(\mathbf{v}), L^2(\mathbf{v}), \ldots, L^{m-1}(\mathbf{v})\}$ is linearly independent.

Answers

1. $(53/80, 9/40, 7/80)$ **2.** $\begin{pmatrix} 2 & 1 & 3 \\ -3 & 0 & -4 \\ 1 & -1 & -2 \end{pmatrix}$ **3.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$ **4.** $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

5. $\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$ **6.** $\dfrac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$ **7.** $\dfrac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

8. $\dfrac{1}{A^2+B^2+C^2}\begin{pmatrix} B^2+C^2 & -AB & -AC \\ -AB & A^2+C^2 & -BC \\ -AC & -BC & A^2+B^2 \end{pmatrix}$

9. $\frac{1}{A^2+B^2+C^2}\begin{pmatrix} B^2+C^2-A^2 & -2AB & -2AC \\ -2AB & A^2+C^2-B^2 & -2BC \\ -2AC & -2BC & A^2+B^2-C^2 \end{pmatrix}$

10. $\frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$ **11.** $\frac{1}{a^2+b^2+c^2}\begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{pmatrix}$

13. $r\begin{pmatrix} 0 & -z_0 & y_0 \\ z_0 & 0 & -x_0 \\ -y_0 & x_0 & 0 \end{pmatrix}$ **15.**(b) $\begin{pmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -2a & 3a^2 \\ 0 & 0 & 1 & -3a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ **16.**(b) $(1 \quad 1/2 \quad 1/3)$ (c) $(1/2 \quad 4 \quad 13/3)$

17.(b) $\begin{pmatrix} -a & -a^2/2 & -a^3/3 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$ **18.**(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

19.(b) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$ **20.** $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

21.(a) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, Yes, Yes (c) $\begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

22.(a) $\begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 4 & 0 & -16 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 32 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 3/2 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$

23.(a) $\begin{pmatrix} 0 & 1 & 0-3/2 & 0 \\ 0 & 0 & 3 & 0 & -15/2 \\ 0 & 0 & 0 & 15/2 & 0 \\ 0 & 0 & 0 & 0 & 35/2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 12/5 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 8/5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 & 5/2 & 0 \\ 0 & 0 & 3 & 0 & 27/2 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$

24.(a) $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 16/5 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 64/5 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 & 9/4 & 0 \\ 0 & 0 & 3 & 0 & 45/8 \\ 0 & 0 & 0 & 15/4 & 0 \\ 0 & 0 & 0 & 0 & 35/8 \end{pmatrix}$

25.(a) $\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & -6 & 0 & 12 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix}$

26. $a_n(x)A^n + a_n(x)A^{n-1} + \cdots + a_1(x)A + a_0(x)I$, where I is the $n \times n$ identity matrix

27.(a) $\begin{pmatrix} 1 & -a & a^2 & -a^3 & a^4 \\ 0 & 1 & -2a & 3a^2 & -4a^3 \\ 0 & 0 & 1 & -3a & 6a^2 \\ 0 & 0 & 0 & 1 & -4a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -a & a^2 & -a^3 & a^4 & a^5 \\ 0 & 1 & -2a & 3a^2 & -4a^3 & -5a^4 \\ 0 & 0 & 1 & -3a & 6a^2 & 10a^3 \\ 0 & 0 & 0 & 1 & -4a & -10a^2 \\ 0 & 0 & 0 & 0 & 1 & 5a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Pascal's triangle is hidden in them.

28. $\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$ **29.** $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **30.** $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **31.** $\begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

32. $\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 4 & 8 & \cdots & 2^n \\ 1 & 3 & 9 & 27 & \cdots & 3^n \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & n-1 & (n-1)^2 & (n-2)^3 & \cdots & (n-1)^n \end{pmatrix}$ **33.** $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 12 & 48 & 160 \\ 0 & 0 & 0 & 6 & 72 & 540 \\ 0 & 0 & 0 & 0 & 24 & 600 \\ 0 & 0 & 0 & 0 & 0 & 120 \end{pmatrix}$

§2.3 Kernel and Range of a Linear Transformation

When L is a linear transformation from a vector space V to a vector space W, the set of vectors that are mapped by L to the zero vector in W, and the set of vectors in W that are images of vectors in V are identified in the following definition.

Definition 2.2 When L is a linear transformation from a vector space V to a vector space W, the set of vectors in V that are mapped to the zero vector in W is called the **kernel** of L, denoted by $\mathrm{Ker}(L)$. The set of vectors in W that are images of vectors in V is called the **range** of L, denoted by $\mathrm{Range}(L)$.

Sometimes the kernel and range of a linear transformation are obvious. Such is the case in the two examples below.

Example 2.23 Determine the kernel and range of the linear transformation in Example 2.3. Find a basis for the range.

Solution Since the cross product of two parallel vectors is the zero vector, it follows that any force vector parallel to $\langle 1,2,3\rangle$ has zero moment; that is, the kernel of the transformation consists of all vectors parallel to $\langle 1,2,3\rangle$. Since the cross product of two vectors yields a vector perpendicular to both of them, and the vector $\langle 1,2,3\rangle$ is fixed, the range of the transformation is the plane of vectors perpendicular to $\langle 1,2,3\rangle$. Any two linearly independent vectors perpendicular to $\langle 1,2,3\rangle$ will form a basis. Two such vectors are $\langle 2,-1,0\rangle$ and $\langle 0,3,-2\rangle$.•

Example 2.24 Let L be the linear transformation that takes orthogonal components of vectors in $\mathcal{G}^3$ along the xy-plane; that is, $L\big(\langle v_x,v_y,v_z\rangle\big)=\langle v_x,v_y,0\rangle$. Find its kernel and range.

Solution Since the component of any vector perpendicular to the xy-plane is the zero vector, the kernel of L is all vectors of the form $\langle 0,0,v_z\rangle$. The range of the transformation is the xy-plane. We can think of this mapping as a linear transformation from $\mathcal{G}^3$ to $\mathcal{G}^2$, or we can think of it as a linear operator on $\mathcal{G}^3$.•

When the kernel and range of a linear transformation are not obvious, they can be obtained from the matrix associated with the transformation, provided that there is such a matrix, and we can find it. To see how, suppose that L is a linear transformation from an n-dimensional space V to an m-dimensional space W, and $A=(a_{ij})_{m\times n}$ is the matrix associated with the linear transformation. Equation 2.7

$$\begin{pmatrix} w_1\\ w_2\\ \vdots\\ w_m\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ \vdots\\ v_n\end{pmatrix}, \tag{2.8a}$$

can also be written in the form

$$\begin{pmatrix} w_1\\ w_2\\ \vdots\\ w_m\end{pmatrix}=v_1\begin{pmatrix} a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{pmatrix}+v_2\begin{pmatrix} a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{pmatrix}+\cdots+v_n\begin{pmatrix} a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}. \tag{2.8b}$$

What this shows is that vector $\mathbf{w}$ is a linear combination of the columns of A, images of the basis vectors in V under L. These vectors span the range of L. In other words, the range of a linear transformation is the column space of its matrix. The kernel of L is the set of vectors in V for which $\mathbf{w}=\mathbf{0}$; that is, vectors for which

$$v_1\begin{pmatrix} a_{11}\\ a_{21}\\ \vdots\\ a_{m1}\end{pmatrix}+v_2\begin{pmatrix} a_{12}\\ a_{22}\\ \vdots\\ a_{m2}\end{pmatrix}+\cdots+v_n\begin{pmatrix} a_{1n}\\ a_{2n}\\ \vdots\\ a_{mn}\end{pmatrix}=\mathbf{0} \qquad\Longrightarrow\qquad A\mathbf{v}=\mathbf{0}.$$

This is the null space of A. In other words, the kernel of a linear transformation is the null space of its matrix. As suggested, then, the matrix associated with a linear transformation yields its range and its kernel. Since the number of columns in the matrix represents the dimension of the space V to be mapped, the Rank-Nullity Theorem contained in equation 1.10 gives the following result.

Theorem 2.5 If L is a linear transformation on an finite-dimensional vector space, then

$$\text{dimension(kernel of } L) + \text{dimension(range of } L) = \text{dimension}(V). \tag{2.9}$$

When a linear transformation is between finite-dimensional vector spaces, we define the **rank** of the transformation to be the dimension of its range. It is therefore the rank of the matrix associated with the transformation.

Example 2.25 Find bases for the kernel and the range of the linear transformation from a 4-dimensional space to a 3-dimensional space defined by

$$L: \begin{aligned} w_1 &= v_1 + 2v_2 + 3v_3 + 4v_4, \\ w_2 &= -2v_1 + 3v_2 + v_3, \\ w_3 &= v_1 + 9v_2 + 10v_3 + 12v_4. \end{aligned}$$

Solution The kernel consists of all vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ that map to the zero vector; it is the null space of its matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 1 & 0 \\ 1 & 9 & 10 & 12 \end{pmatrix}.$$

The augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -2 & 3 & 1 & 0 & 0 \\ 1 & 9 & 10 & 12 & 0 \end{array}\right)$$

has reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 12/7 & 0 \\ 0 & 1 & 1 & 8/7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Solutions are therefore

$$v_1 = -v_3 - \frac{12v_4}{7}, \quad v_2 = -v_3 - \frac{8v_4}{7}, \qquad \text{where } v_3 \text{ and } v_4 \text{ are arbitrary.}$$

Thus, the kernel consists of vectors of the form

$$\begin{pmatrix} -v_3 - 12v_4/7 \\ -v_3 - 8v_4/7 \\ v_3 \\ v_4 \end{pmatrix}.$$

We can find a basis for the kernel by writing these vectors in the form

$$\begin{pmatrix} -v_3 \\ -v_3 \\ v_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -12v_4/7 \\ -8v_4/7 \\ 0 \\ v_4 \end{pmatrix} = -v_3 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \frac{v_4}{7} \begin{pmatrix} 12 \\ 8 \\ 0 \\ -7 \end{pmatrix}.$$

This shows that the vectors $(1, 1, -1, 0)$ and $(12, 8, 0, -7)$ form a basis for the kernel. The range of the transformation is the column space of its matrix. We find the reduced row echelon form for

$$A^T = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 3 & 9 \\ 3 & 1 & 10 \\ 4 & 0 & 12 \end{pmatrix} \qquad \text{which is} \qquad A^T_{\text{rref}} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the columns space is therefore $\{(1,0,3),(0,1,1)\}$.•

Example 2.26 Find bases for the kernel and the range of the linear operator on $\mathcal{C}^3$ defined by

$$L: \begin{aligned} v_1' &= 2v_1 + iv_2 + (1+i)v_3, \\ v_2' &= (1-i)v_1 + v_2 + 2v_3, \\ v_3' &= (1-i)v_2 + 2v_3. \end{aligned}$$

Solution The kernel consists of all vectors $\mathbf{v} = (v_1, v_2, v_3)$ that map to the zero vector; it is the null space of its matrix

$$\begin{pmatrix} 2 & i & 1+i \\ 1-i & 1 & 2 \\ 0 & 1-i & 2 \end{pmatrix}.$$

The augmented matrix

$$\left(\begin{array}{ccc|c} 2 & i & 1+i & 0 \\ 1-i & 1 & 2 & 0 \\ 0 & 1-i & 2 & 0 \end{array}\right) \qquad \text{has reduced row echelon form} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1-i & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Components of vectors in the null space therefore satisfy $v_1 = -v_3$ and $v_2 = -2v_3/(1-i) = -(1+i)v_3$. Hence, the null space consists of vectors of the form

$$\begin{pmatrix} -v_3 \\ -(1+i)v_3 \\ v_3 \end{pmatrix} = -v_3 \begin{pmatrix} 1 \\ 1+i \\ -1 \end{pmatrix},$$

and a basis for the null space is $(1, 1+i, -1)$. Since the reduced row echelon form for the matrix

$$\begin{pmatrix} 2 & 1-i & 0 \\ i & 1 & 1-i \\ 1+i & 2 & 2 \end{pmatrix} \qquad \text{is} \qquad \begin{pmatrix} 1 & 0 & -1+i \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

a basis for the range of the operator is $\{(1,0,-1+i),(0,1,2)\}$.•

Here is a more abstract example wherein there is no matrix associated with the transformation.

Example 2.27 Let L be the linear transformation that maps functions $y(x)$ in the space $C^2(a,b)$ of twice continuously differentiable functions on the interval $a < x < b$ to $L(y) = y'' + 3y' + 2y$. Determine the kernel of the transformation.

Solution The kernel of the transformation consists of all functions satisfying the homogeneous differential equation $y'' + 3y' + 2y = 0$. Since the auxiliary equation associated with the differential equation is $0 = m^2 + 3m + 2 = (m+2)(m+1)$, all solutions of the differential equation are of the form $y(x) = c_1e^{-x} + c_2e^{-2x}$, where c_1 and c_2 are constants. In other words, functions of this form constitute the kernel of the transformation. •

According to the following theorem, the kernel and range of a linear transformation are subspaces of the spaces in which they reside.

Theorem 2.6 The kernel of a linear transformation from V to W is a subspace of V, and the range is a subspace of W.

Proof: Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors in the kernel of a linear transformation L from V to W, and a is a scalar. Because $L(\mathbf{v}_1) = \mathbf{0}$, $L(\mathbf{v}_2) = \mathbf{0}$, and L is linear,

$$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}, \qquad L(a\mathbf{v}_1) = a\,L(\mathbf{v}_1) = a\mathbf{0} = \mathbf{0}.$$

Thus, the kernel is closed under vector addition and scalar multiplication. According to Theorem 1.1, it is a subspace of V.

Now let $\mathbf{w}_1$ and $\mathbf{w}_2$ be any two vectors in the range of W, and a be a scalar. There exists vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$. Because L is linear, it follows that the vectors

$$L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2 \quad \text{and} \quad L(a\mathbf{v}_1) = a\,L(\mathbf{v}_1) = a\mathbf{w}_1$$

are in the range of W. Thus, the range of W is closed under vector addition and scalar multiplication, and is therefore a subspace of W.∎

Onto and 1-1 Transformations

Linear transformations can sometimes be categorized as *onto* and *1-1* (pronounced one-to-one). Before introducing these adjectives in the context of linear transformations, we review them as applied to functions. An integral part of a real-valued function $y = f(x)$ is its domain D. The set of y-values obtained by substituting all values of x (in the domain) into the function is the range of the function. The function can be considered as a mapping from D to its range, or it can be considered as a mapping from D to the reals $\mathcal{R}$. Whether a function is onto or 1-1 depends both on its domain and the prescribed set in which its images are to be found.

As an example to illustrate these ideas, consider the function $y = f(x) = x^2$ with domain $\mathcal{R}$ and range $y \geq 0$. Because every y in the range is the image of at least one x in the domain, we say that the function maps the domain **onto** the range. If we were to consider the function from domain $\mathcal{R}$ to $\mathcal{R}$, then negative real numbers are not images of any values of x in the domain. In this case, we say that the function maps the domain **into** $\mathcal{R}$, but not onto. Suppose now that the domain of the function is all nonnegative reals, $D : x \geq 0$. As a mapping to $\mathcal{R}$, it is still into, but it has an an added attribute. Every real number that is the image of an x in D is the image of exactly one value of x in D. Such a function is said to be **1-1**. The same function is both onto and 1-1 from D to the range $y \geq 0$. Thus, functions that are into can be 1-1 or not, functions that are onto can be 1-1 or not. We have illustrated this for the function $f(x) = x^2$ in Table 2.1.

Domain	Image Set	Into/Onto	1-1/Not 1-1
$-\infty < x < \infty$	$-\infty < y < \infty$	Into	Not 1-1
$x \geq 0$	$-\infty < y < \infty$	Into	1-1
$-\infty < x < \infty$	$y \geq 0$	Onto	Not 1-1
$x \geq 0$	$y \geq 0$	Onto	1-1

Table 2.1

The situation is similar for linear transformations, but there are also some differences. Formal requirements for a linear transformation to be into, onto, and 1-1 are contained in the following definition.

Definition 2.3 A linear transformation L from vector space V to vector space W is said to be **into** if there exists at least one vector $\mathbf{w}$ in W that is not the image of some vector $\mathbf{v}$ in V; it is said to be **onto** if every vector $\mathbf{w}$ in W is the image of at least one vector $\mathbf{v}$ in V; and it is said to be **1-1** if every vector $\mathbf{w}$ in W, that is the image of a vector $\mathbf{v}$ in V, is the image of exactly one such vector[†].

Another way to describe 1-1 is to say that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in V, with $\mathbf{v}_1 \neq \mathbf{v}_2$, then $L(\mathbf{v}_1) \neq L(\mathbf{v}_2)$.

Into and onto are mutually exclusive; a linear transformation cannot be both into and onto. Every linear transformation is one or the other, not both. We shall see that just like functions, linear transformations that are into can be 1-1 or not, transformations that are onto can be 1-1 or not. Which situation prevails is intimately connected to the dimension of the space V being mapped and the dimension of its range. We first consider the case that the dimension of V is less than that of W. Here are two examples to illustrate.

Example 2.28 Is the linear transformation that maps every vector in the xy-plane to the zero vector in xyz-space into, onto, and/or 1-1?

Solution Since there are vectors in xyz-space that are not images of vectors in the xy-plane, the mapping is into. Because every vector in the xyz-plane maps to the same vector in xyz-space, the transformation is not 1-1.●

Example 2.29 Suppose that L is the linear transformation that takes a vector in the xy-plane, maps it to the same vector in xyz-space, and then doubles its length. Is the transformation into, onto, and/or 1-1?

Solution Since there are vectors in xyz-space that are not images of vectors in the xy-plane, the mapping is into. Because every vector in xyz-space that is the image of a vector in the xy-plane is the image of only one such vector, the mapping is 1-1. Alternatively, the mapping is 1-1 because different vectors in the xy-plane map to different vectors in space.●

These examples are special cases of the following theorem.

Theorem 2.7 If L is a linear transformation from a vector space V with dimension n to a vector space W with dimension m, where $n < m$, then L is into (but not therefore onto). It could be 1-1, but it might not be.

Proof All that we need prove is that the transformation is into. If $A = (a_{ij})_{m\times n}$ is the matrix of the linear transformation, then equation 2.8b shows that the range of L is the column space of A; that is, the range is the space spanned by the n columns of A. Since this is at most an n-dimensional subspace of W, there are vectors in W that are not images of vectors in V. Hence L must be into.∎

We now consider the case when the dimension of V is greater than that of W. Two examples illustrate the situation.

Example 2.30 If the transformation in Example 2.24 is regarded as a mapping from $\mathcal{G}^3$ to $\mathcal{G}^2$, is it into, onto, and/or 1-1?

Solution As a transformation from $\mathcal{G}^3$ to $\mathcal{G}^2$, it is onto; every vector in the xy-plane is the orthogonal component of some vector in xyz-space. It is not 1-1 because many vectors in space have the same vector component in the xy-plane.●

Example 2.31 A linear transformation L from $\mathcal{R}^3$ to $\mathcal{R}^2$ has matrix $\begin{pmatrix} 1 & 5 & -3 \\ -2 & -10 & 6 \end{pmatrix}$. Is the transformation into, onto, and/or 1-1?

Solution If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector in $\mathcal{R}^3$ and $\mathbf{w} = (w_1, w_2) = L(\mathbf{v})$ is its image, we can write that

† A mapping is also called **injective** if it is 1-1, **surjective** if it is onto, and **bijective** if it is both 1-1 and onto.

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -3 \\ -2 & -10 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

This shows that $w_2 = -2w_1$; images are multiples of $(1, -2)$. In other words, the transformation is into. Since many vectors in $\mathcal{R}^3$ will get mapped to the same vector in $\mathcal{R}^2$, the transformation is not 1-1. If this is not clear, consider the system of equations

$$w_1 = v_1 + 5v_2 - 3v_3,$$
$$w_2 = -2v_1 - 10v_2 + 6v_3,$$

for v_1, v_2, and v_3 for given values of w_1 and w_2. The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & w_1 \\ -2 & -10 & 6 & w_2 \end{array}\right)$$

has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & w_1 \\ 0 & 0 & 0 & w_2 + 2w_1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 5 & -3 & w_1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This demonstrates that there is an infinity of solutions to the system.•

The last two examples are special cases of the next theorem.

Theorem 2.8 If L is a linear transformation from a vector space V with dimension n to a vector space W of dimension m, where $n > m$, then L is never 1-1. It is onto if the range of L has dimension m, and it is into if the range has dimension less than m.

Proof Once again we consider equation 2.8b, where A is the matrix of the transformation. At most m of the columns of A can be linearly independent. When m columns are independent, they will span W, and the transformation is onto. If less than m columns are linearly dependent, then they will not span W and the transformation is into (and not 1-1). Because the dimension of V is larger than that of W, the transformation cannot be 1-1. Equations 2.8a show this algebraically. When $\mathbf{w}$ is a vector in the range of L, this is a nonhomogeneous system of m equations in the n unknown components $\mathbf{v}_j$, which has a solution. Since there are fewer unknowns than equations, there must be an infinity of solutions.∎

The only case left to discuss is when V and W have the same dimension. This time, we state the result, and then illustrate with examples.

Theorem 2.9 Suppose that L is a linear transformation from a vector space V to a vector space W, both with dimension n.

(1) If the range of L has dimension less than n, then L is into (and not 1-1).

(2) If the range of L has dimension n, then L is 1-1 and onto.

Proof (1) If the range of L has dimension less than n, then there are vectors in W that are not images of vectors in V, and L is therefore into.

(2) If the range of L is n-dimensional, then the matrix A of L must have n linearly independent columns. Hence, its determinant is nonzero. Given any vector $\mathbf{w}$ in W, there is a vector $\mathbf{v}$ in V that maps to $\mathbf{w}$, if and only if, its components satisfy equation 2.8a with $m = n$. Because the determinant of the coefficient matrix is nonzero, there is a unique solution. In other words, the transformation is 1-1 and onto.∎

Example 2.32 If the transformation in Example 2.24 is regarded as an operator on $\mathcal{G}^3$, is it into, onto, and/or 1-1?

Solution Since the range of the operator is the xy-plane with dimension 2, Theorem 2.9 indicates that the operator must be into, and not onto or 1-1. This is equally obvious without the theorem. As an operator on $\mathcal{G}^3$, the transformation is into because any vector not in the xy-plane is not the image of any vector in $\mathcal{G}^3$.●

Example 2.33 The matrix of a linear transformation L from a vector space with dimension 3 to a vector space W with dimension 3 is

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix}.$$

Is the transformation into, onto, and/or 1-1?

Solution Since the determinant

$$\det \begin{pmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix} \neq 0,$$

the column vectors are linearly independent, and the dimension of the range is 3. Theorem 2.9 then indicates that the transformation is 1-1 and onto.●

There are other ways to determine whether a transformation is 1-1, one of which is contained in the next theorem.

Theorem 2.10 A linear transformation is 1-1 if, and only if, the only vector in the kernel is the zero vector.

Proof If L maps V to W, then the zero vector $\mathbf{v} = \mathbf{0}$ always maps to the zero vector $\mathbf{w} = \mathbf{0}$. If the transformation is 1-1, then only $\mathbf{v} = \mathbf{0}$ can map to $\mathbf{w} = \mathbf{0}$. In other words, the kernel consists only of $\mathbf{v} = \mathbf{0}$. Conversely, suppose that $\mathbf{v} = \mathbf{0}$ is the only vector in $\mathrm{Ker}(L)$. Suppose that there exist two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $L(\mathbf{v}_1) = L(\mathbf{v}_2)$. Because L is linear,

$$L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{0}.$$

In other words, $\mathbf{v}_1 - \mathbf{v}_2$ must be in the kernel of L. Since the kernel consists only of the zero vector, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, from which $\mathbf{v}_1 = \mathbf{v}_2$. Hence, L is 1-1.∎

Example 2.34 Is the linear operator on $\mathcal{G}^3$ that reflects vectors in the plane $y = x$ 1-1?

Solution Since the only vector that reflects to the zero vector is the zero vector, the transformation is 1-1.●

Example 2.35 Let L be the operator that multiplies matrices in $M_{2,2}(\mathcal{C})$ by the matrix $A = \begin{pmatrix} i & 2+i \\ 3-i & 4 \end{pmatrix}$. Show that L is linear and then determine whether it is 1-1.

Solution If B and C are two matrices in the space, and c is any complex scalar, then

$$L(B + C) = A(B + C) = AB + AC = L(B) + L(C), \quad L(cB) = A(cB) = c(AB) = c[L(B)].$$

In other words, the transformation is linear. To use Theorem 2.10 to decide whether L is 1-1, we determine the kernel of the transformation. It consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} i & 2+i \\ 3-i & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If multiply the matrices on the left and equate entries, we obtain

$$ai + (2+i)c = 0, \quad bi + (2+i)d = 0, \quad (3-i)a + 4c = 0, \quad (3-i)b + 4d = 0.$$

The determinant of the coefficient matrix of this system of four homogeneous equations in a, b, c, and d has nonzero value $40-42i$ implying that only solution is $a=b=c=d=0$. In other words, the only vector that maps to the zero vector is the zero vector. The operator is therefore 1-1.•

EXERCISES 2.3

In Exercises 1–9 the matrix is associated with a linear transformation from an n-dimensional real vector space to an m-dimensional real vector space for appropriate values of m and n. Determine:
(a) a basis for the kernel of the transformation,
(b) a basis for the range of the transformation,
(c) whether the transformation is into, onto, and/or 1-1.

1. $\begin{pmatrix} 3 & 3 & 2 & -1 \\ 3 & 0 & 2 & 1 \\ 1 & 3 & 5 & 6 \\ 7 & 6 & 9 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 3 & 2 & 1 \\ -2 & 4 & 1 \\ 5 & -2 & 0 \\ 9 & 6 & 3 \end{pmatrix}$

4. $\begin{pmatrix} 2 & 1 & -2 & 3 \\ 5 & 2 & 3 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 3 & 3 & 2 \\ 3 & 0 & 2 \\ 1 & 3 & 5 \\ 7 & 6 & 9 \end{pmatrix}$

6. $\begin{pmatrix} 3 & 3 & 1 & 7 \\ 3 & 0 & 3 & 6 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{pmatrix}$

9. $\begin{pmatrix} -1 & 2 & 0 & -1 & 3 \\ 3 & 8 & -4 & -5 & 13 \\ 3 & 1 & -2 & -1 & 2 \\ -9 & 4 & 4 & -1 & 5 \end{pmatrix}$

In Exercises 10–11 the matrix is associated with a linear transformation from an n-dimensional complex vector space to an m-dimensional complex vector space for appropriate values of m and n. Determine:
(a) a basis for the kernel of the transformation,
(b) a basis for the range of the transformation,
(c) whether the transformation is into, onto, and/or 1-1.

10. $\begin{pmatrix} 1-i & i \\ 2 & -1+i \end{pmatrix}$

11. $\begin{pmatrix} i & 0 & 1 & 0 \\ 0 & -i & 0 & -2 \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 2 \end{pmatrix}$

12. Let L be the linear transformation that maps functions $y(x)$ in $C^3(-\infty,\infty)$ to $L(y)=y'''-y''+4y'-4y$. Find a basis for the kernel of L.

13. Let L be the linear transformation that maps functions $y(x)$ in $C^4(-\infty,\infty)$ to $L(y)=y''''+16y$. Find a basis for the kernel of L.

14. Let L be the linear operator on $M_{n,n}(\mathcal{R})$ that subtracts from a matrix its transpose. What is the kernel of L?

15. Let L be the linear transformation that maps a polynomial in space $P_2(x)$ to the value of its first derivative at $x = 2$. What is the kernel of L?

16. Let L be the linear transformation from $P_3(x)$ to $P_2(x)$ that maps vectors according to $L(ax^3 + bx^2 + cx + d) = (a + b)x^2 + (c + 2d)x$. Find bases for (a) the kernel of L, and (b) the range of L?

17. What happens in Exercise 16 if $a + b$ is replaced by $a^2 + b^2$?

18. Let $A_{m\times n}$ be the matrix of a linear transformation from $\mathcal{R}^n$ to $\mathcal{R}^m$. Show that the dimension of the range of the transformation is equal to the dimension of the range of the transformation from $\mathcal{R}^m$ to $\mathcal{R}^n$ defined by A^T.

19. (a) Show that the polynomials $p_1(x) = x(1 - x)/2$, $p_2(x) = 1 - x^2$, and $p_3(x) = x(1 + x)/2$ constitute a basis for the space $P_2(x)$.
(b) Let L be the linear operator on the space defined by its action on the basis in part (a) by

$$L\big(p_1(x)\big) = x + x^2, \quad L\big(p_2(x)\big) = x^2 - 1, \quad L\big(p_3(x)\big) = 1 + x.$$

Find Ker(L).

20. (a) Let L be the transformation from $P_2(x)$ to $M_{2,2}(\mathcal{R})$ defined by

$$L(a + bx + cx^2) = \begin{pmatrix} ra & c \\ c & sb \end{pmatrix},$$

where r and s are nonzero real numbers. Show that L is linear.
(b) Find the matrix for L using the natural bases for both vector spaces.
(c) Find the kernel of L.

21. Repeat Exercise 20 if $s = 0$.

22. Find a basis for the kernel of the linear transformation in Exercise 20 of Section 2.2.

23. (a) Find a basis for the kernel of the linear transformation L that maps polynomials $p(x)$ in $P_2(x)$ to $\mathcal{R}$ according to

$$L\big(p(x)\big) = \int_0^1 p(x)\,dx.$$

(b) What is the range of the transformation?

24. Repeat Exercise 23 if the integral is from a to b.

25. (a) Does a linear transformation preserve linear independence; that is, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of linear independent vectors in a vector space V, and L is a linear transformation from V to W, is the image set $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ in W linearly independent?
(b) Does it do so if the transformation is 1-1?

26. Suppose that L is a linear transformation from an n-dimensional space V to a space W. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V such that $\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_n\}$ is a basis for the kernel of L. Show that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_r)\}$ is a basis for the range of L.

27. Let L be the linear transformation from $P_n(x)$ to $P_{n-1}(x)$ that differentiates polynomials in $P_n(x)$. Prove that the transformation is onto. Is it 1-1?

28. Let L be the linear transformation from $C^n(a, b)$ to $C^0(a, b)$ defined by

$$L\big(f(x)\big) = \sum_{i=0}^{n} a_j f^{(j)}(x),$$

where the a_j are constants with $a_n \neq 0$. Determine the dimension of the kernel.

29. In Exercise 29 of Section 2.1, we defined the weight map for magic squares (see Exercise 31 in Section 1.4 for the definition of a magic square). Find a basis for the kernel of the weight map of the subspace $\text{Magic}_{3,3}(\mathcal{R})$ of 3×3 magic squares.

30. Consider the set S of matrices in $M_{3,3}(\mathcal{R})$ that commute with the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

under matrix multiplication; that is, matrices A in $M_{3,3}(\mathcal{R})$ such that $AB = BA$. Show that S is a subspace and find a basis for it.

Answers

1.(a) $(33, 26, -69, 39)$ (b) $(1,0,0,1)$, $(0,1,0,1)$, $(0,0,1,1)$ (c) Into and not 1-1
2.(a) Kernel $= \mathbf{0}$ (b) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (c) Onto and 1-1
3.(a) $(2,5,-16)$ (b) $(1,0,1,3)$, $(0,1,-1,0)$ (c) Into and not 1-1
4.(a) $(11,-23,7,5)$ (b) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (c) Onto and not 1-1
5.(a) Kernel $= \mathbf{0}$ (b) $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (c) Into and 1-1
6.(a) $(-3,2,3,0)$, $(6,1,0,-3)$ (b) $(1,0)$, $(0,1)$ (c) Onto and not 1-1
7.(a) $-5,1,1)$ (b) $(1,0,1)$, $(0,1,1)$ (c) Into and not 1-1
8.(a) $(-2,1,0)$, $(-3,0,1)$ (b) $(1,2,-3)$ (c) Into and not 1-1
9.(a) $(4,2,1,0,0)$, $(1,4,0,7,0)$, $(-1,-11,0,0,7)$ (b) $(2,0,-3,12)$, $(0,2,1,-2)$ (c) Into and not 1-1
10.(a) $(1-i,2)$ (b) $(1,1+i)$ (c) Into and not 1-1
11.(a) $(0,2i,0,1)$ (b) $(1,0,1,0)$, $0,1,0,0)$, $(0,0,0,1)$ (c) Into and not 1-1
12. $\{e^x, \sin 2x, \cos 2x\}$ **13.** $\{e^{\sqrt{2}x}\cos\sqrt{2}x, e^{\sqrt{2}x}\sin\sqrt{2}x, e^{-\sqrt{2}x}\cos\sqrt{2}x, e^{-\sqrt{2}x}\sin\sqrt{2}x\}$
14. Symmetric $n \times n$ matrices. **15.** Polynomials of the form $ax^2 - 4ax + c$
16.(a) $\{x^3 - x^2, 1 - 2x\}$ (b) $\{x^2, x\}$ **15.** Transformation is not linear.
19. Constant polynomials **20.**(b) $\begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & s & 0 \end{pmatrix}$ (c) $p(x) = 0$ **21.**(b) $\begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $p(x) = bx$
22. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
23.(a) $3x^2 - 1$, $2x - 1$ (b) $\mathcal{R}$ **24.**(a) $x^2 - (a^2 + ab + b^2)/3$, $x - (a+b)/2$ (b) $\mathcal{R}$
25.(a) Not necessarily (b) Yes **27.** No **28.** n **29.** $\begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$
30. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

§2.4 Inverse Linear Transformations

The reader should be familiar with the fact that a function has an inverse if, and only if, it is 1-1. For instance, the function $f(x) = x^2$ with domain R does not have an inverse because it is not 1-1 on this domain. It does have an inverse on the domain $x \geq 0$, and the inverse is $f^{-1}(x) = \sqrt{x}$. On the domain $x \leq 0$, it has inverse $f^{-1}(x) = -\sqrt{x}$. The sine function $g(x) = \sin x$ has an inverse on the domain $-\pi/2 \leq x \leq \pi/2$, namely $g^{-1}(x) = \text{Sin}^{-1}x$ with principal values $-\pi/2 \leq \text{Sin}^{-1}x \leq \pi/2$. An inverse function $f^{-1}(x)$ must undo whatever the function $f(x)$ does, and as such the domain of $f^{-1}(x)$ is taken to be the range of $f(x)$.

The situation is similar for linear transformations.

Definition 2.4 Suppose that L is a linear transformation from a vector space V to a vector space W. The inverse transformation, if it exists, is denoted by L^{-1}. It is the mapping from W to V such that $\mathbf{v} = L^{-1}(\mathbf{w})$ if $\mathbf{w} = L(\mathbf{v})$.

The fact that L^{-1} must map every vector $\mathbf{w}$ in W to a vector $\mathbf{v}$ in V means that L must be onto. Furthermore, L must be 1-1 else it would be impossible to determine $\mathbf{v}$ so that $\mathbf{w} = L(\mathbf{v})$. In Section 2.3, we saw that a linear transformation between spaces can only be 1-1 and onto if V and W have the same dimension, say n, and, in addition, the range of the transformation has dimension n also. For instance, the linear transformation L in Example 2.33 with matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix}$$

is 1-1 and onto. Given any vector $\mathbf{w}$ in W, there is one, and only one, vector $\mathbf{v}$ in V such that

$$\mathbf{w} = L(\mathbf{v}) \quad \Longleftrightarrow \quad \mathbf{w} = A\mathbf{v} \quad \Longleftrightarrow \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 3 & 0 \\ 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

We can find $\mathbf{v}$ by solving this system of equations. Because matrix A has a nonzero determinant, its inverse exists, and we can write that

$$\mathbf{v} = A^{-1}\mathbf{w}.$$

What this says is that the matrix associated with L^{-1} is the inverse of the matrix associated with L. That this is always true is contained in the following theorem.

Theorem 2.11 A linear transformation has an inverse if, and only if, its matrix has an inverse.

Example 2.36 Show that the rotation operator in Example 2.15 has an inverse and find its matrix.

Solution Because the operator is 1-1, it has inverse which clearly must be rotation by angle $-\theta$ around the z-axis. It matrix is therefore

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \bullet$$

Example 2.37 Show that the linear transformation in Example 2.18 has an inverse, and that it maps $18e^{2x} + 24xe^{ex} - 20x^2e^{2x}$ back to $-2xe^{2x} + 10x^2e^{2x}$.

Solution The matrix of the transformation L is

$$A = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{which has inverse} \quad A^{-1} = -\frac{1}{4}\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

This must be the matrix of the inverse transformation L^{-1}. It maps $18e^{2x}+24xe^{ex}-20x^2e^{2x}$ to

$$L^{-1}(18e^{2x}+24xe^{2x}-20x^2e^{2x}) = -\frac{1}{4}\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 18 \\ 24 \\ -20 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 0 \\ 8 \\ -40 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -2 \\ 10 \end{pmatrix} = -2xe^{2x}+10x^2e^{2x}.\bullet$$

EXERCISES 2.4

in Exercises 1–6 the matrix represents a linear transformation from some n-dimensional space to another n-dimensional space for an appropriate value of n. Determine whether the transformation has an inverse.

1. $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 3 & -5 \\ 2 & 4 & 1 \\ 3 & 7 & -4 \end{pmatrix}$

4. $\begin{pmatrix} 3 & 4 & -2 \\ 1 & 4 & 2 \\ -5 & 3 & 7 \end{pmatrix}$

5. $\begin{pmatrix} 0 & 2 & 1 & -2 \\ 3 & 4 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 4 & 9 & 9 & 13 \end{pmatrix}$

7. Show that when a linear operator L has an inverse L^{-1}, then L^{-1} is also linear.

8. (a) Find the matrix A of the linear operator L on $\mathcal{G}^3$ that reflects every vector in the xy-plane and then triples its length.
(b) Find the image of the vector $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ under L.
(c) Find the matrix associated with L^{-1}.

9. (a) Find the matrix A of the linear operator L on $\mathcal{G}^3$ that takes orthogonal components of vectors along the xz-plane.
(b) Find the image of the vector $\mathbf{v} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ under L.
(c) Does L have an inverse?

10. (a) Find the matrix A of the linear operator L on $\mathcal{G}^3$ that reflects every vector in the plane $y = x$.
(b) Find the image of the vector $\mathbf{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ under L.
(c) Find the matrix associated with L^{-1}.

11. Prove that a 1-1 linear transformation preserves linearly independence of vectors.

12. What is the matrix of the inverse of the mapping in Exercise 15 in Section 2.2?

Answers

1. Yes **2.** No **3.** No **4.** Yes **5.** Yes **6.** No

8. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ (b) $\langle 6, -9, -3 \rangle$ (c) $\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

9.(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\langle -3, 0, 1 \rangle$ (c) No

10. (a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\langle -2, 1, 3 \rangle$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **12.** $\begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 0 & 0 & 1 & 3a \\ 0 & 0 & 0 & 1 \end{pmatrix}$

§2.5 Changing Bases

Suppose that

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \end{pmatrix}$$

is the matrix of a linear transformation L from $\mathcal{R}^3$ to $\mathcal{R}^2$ where natural bases have been used in both spaces. We question what the matrix would be if we change either basis, or both bases. We consider some specific examples before stating general results. First, suppose we wish to use the vectors $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (-2,1,4)$, and $\mathbf{v}_3 = (3,1,5)$ as the basis for $\mathcal{R}^3$. We know that the columns of the matrix representing a linear transformation are images of its basis vectors. These are

$$L(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix},$$

$$L(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -19 \end{pmatrix},$$

$$L(\mathbf{v}_3) = A\mathbf{v}_2 = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 16 \\ -13 \end{pmatrix}.$$

Hence, the matrix of the linear transformation using the above basis for $\mathcal{R}^3$ and the natural basis for $\mathcal{R}^2$ is

$$B = \begin{pmatrix} 6 & 8 & 16 \\ -8 & -19 & -13 \end{pmatrix}.$$

Now consider using the natural basis for $\mathcal{R}^3$, but the basis $\{(2,1),(3,2)\}$ for $\mathcal{R}^2$. We need the components of the images of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ under L relative to this basis. In Section 1.6, we learned how to do this. The transition matrix from the natural basis in $\mathcal{R}^2$ to the basis $\{(2,1),(3,2)\}$ is

$$T = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The images of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ with respect to the basis $\{(2,1),(3,2)\}$ are

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -11 \end{pmatrix}.$$

The matrix of the transformation using the natural basis for $\mathcal{R}^3$ and the basis $\{(2,1),(3,2)\}$ for $\mathcal{R}^2$ is

$$C = \begin{pmatrix} -4 & -7 & 18 \\ 3 & 4 & -11 \end{pmatrix}.$$

Finally, suppose we change both bases. We need the components of $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$, and $L(\mathbf{v}_3)$ with respect to the basis $\{(2,1),(3,2)\}$ for $\mathcal{R}^2$. They are

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 36 \\ -22 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -19 \end{pmatrix} = \begin{pmatrix} 73 \\ -46 \end{pmatrix}, \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ -13 \end{pmatrix} = \begin{pmatrix} 71 \\ -42 \end{pmatrix}.$$

The matrix of the linear transformation using the bases $\{(1,2,3),(-2,1,4),(3,1,5)\}$ for $\mathcal{R}^3$ and $\{(2,1),(3,2)\}$ for $\mathcal{R}^2$ is

$$D = \begin{pmatrix} 36 & 73 & 71 \\ -22 & -46 & -42 \end{pmatrix}.$$

To see that these actually work, consider taking the transform of the vector with natural components $\mathbf{v} = (-2, 3, 4)$. Natural components of $L(\mathbf{v})$ are

$$A\mathbf{v} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -17 \end{pmatrix}.$$

The components of vector $\mathbf{v}$ relative to the basis $\{(1,2,3), (-2,1,4), (3,1,5)\}$ for $\mathcal{R}^3$ are $(22/15, 11/15, -2/3)$. Natural components of $L(\mathbf{v})$ are

$$B\mathbf{v} = \begin{pmatrix} 6 & 8 & 16 \\ -8 & -19 & -13 \end{pmatrix} \begin{pmatrix} 22/15 \\ 11/15 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 4 \\ -17 \end{pmatrix}.$$

Components of $L(\mathbf{v})$ relative to the basis $\{(2,1), (3,2)\}$ for $\mathcal{R}^2$ using natural components of $\mathbf{v}$ are

$$C\mathbf{v} = \begin{pmatrix} -4 & -7 & 18 \\ 3 & 4 & -11 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 59 \\ -38 \end{pmatrix}.$$

Natural components of this vector are

$$59 \begin{pmatrix} 2 \\ 2 \end{pmatrix} - 38 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -17 \end{pmatrix}.$$

Finally, components of $L(\mathbf{v})$ relative to the basis $\{(2,1), (3,2)\}$ for $\mathcal{R}^2$ using the components $(22/15, 11/15, -2/3)$ of $\mathbf{v}$ relative to the basis $\{(1,2,3), (-2,1,4), (3,1,5)\}$ for $\mathcal{R}^3$ are

$$D\mathbf{v} = \begin{pmatrix} 36 & 73 & 71 \\ -22 & -46 & -42 \end{pmatrix} \begin{pmatrix} 22/15 \\ 11/15 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 59 \\ -38 \end{pmatrix}.$$

Natural components of this vector are once again $(4, -17)$.

We now develop a method for changing the matrix A of a linear transformation between any two vector spaces. First we deal with an operator on a vector space since this involves only two bases.

Theorem 2.12 Let L be a linear operator on an n-dimensional vector space V. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ be bases for V. If A_b and A_d are matrices associated with L relative to the two bases, then

$$A_d = T_{db} A_b T_{db}^{-1}, \tag{2.10a}$$

or,

$$A_d = T_{bd}^{-1} A_b T_{bd}, \tag{2.10b}$$

where T_{db} is the transition matrix from the **b**-basis to the **d**-basis of Section 1.6.

Proof Let $\mathbf{v}$ be any vector in V. Denote it by $\mathbf{v_b}$ when its components are with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, and by $\mathbf{v_d}$ when its components are with respect to the basis $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$. To verify equation 2.10, we show that both matrices multiply every vector in V to give the same result.

$$\begin{aligned}(T_{db}A_bT_{db}^{-1})\mathbf{v_d} &= T_{db}A_b(T_{db}^{-1}\mathbf{v_d}) && \text{(matrix multiplication is associative)}\\ &= T_{db}A_b\mathbf{v_b} && (T_{db}^{-1} \text{ changes } \mathbf{v} \text{ from } \mathbf{d}\text{- to } \mathbf{b}\text{-components})\\ &= T_{db}(A_b\mathbf{v_b}) && \text{(matrix multiplication is associative)}\\ &= T_{db}L(\mathbf{v_b}) && (\text{where } L(\mathbf{v_b}) \text{ has } \mathbf{b}\text{-components})\\ &= L(\mathbf{v_d}) && (T_{db} \text{ changes from } \mathbf{b}\text{- to } \mathbf{d}\text{-components})\\ &= A_d\mathbf{v_d}. \blacksquare\end{aligned}$$

Example 2.38 The matrix of a linear operator L on a 3-dimensional space relative to the natural basis is

$$A_n = \begin{pmatrix} 2 & 1 & -4 \\ 5 & 1 & 0 \\ 3 & 4 & -5 \end{pmatrix}.$$

Find its matrix relative to the basis $\{(1,2,3),(-3,2,6),(4,-1,2)\}$. Find the transform of the vector $(1,0,0)$ using both matrices and confirm that results are the same.

Solution Suppose we denote the natural basis by $\mathbf{n}$ and the other basis by $\mathbf{b}$. Then the transition matrix from the $\mathbf{b}$ basis to the natural basis is

$$T_{nb} = \begin{pmatrix} 1 & -3 & 4 \\ 2 & 2 & -1 \\ 3 & 6 & 2 \end{pmatrix}.$$

The inverse of this matrix is

$$T_{nb}^{-1} = \frac{1}{55}\begin{pmatrix} 10 & 30 & -5 \\ -7 & -10 & 9 \\ 6 & -15 & 8 \end{pmatrix}.$$

Using equation 2.10b, the matrix of the operator relative to the $\mathbf{b}$ basis is

$$\begin{aligned}A_b = T_{nb}^{-1}A_nT_{nb} &= \frac{1}{55}\begin{pmatrix} 10 & 30 & -5 \\ -7 & -10 & 9 \\ 6 & -15 & 8 \end{pmatrix}\begin{pmatrix} 2 & 1 & -4 \\ 5 & 1 & 0 \\ 3 & 4 & -5 \end{pmatrix}\begin{pmatrix} 1 & -3 & 4 \\ 2 & 2 & -1 \\ 3 & 6 & 2 \end{pmatrix}\\ &= \frac{1}{55}\begin{pmatrix} 150 & -515 & 570 \\ -50 & 47 & -201 \\ -185 & -221 & -307 \end{pmatrix}.\end{aligned}$$

Using matrix A_n, the transform of $(1,0,0)$ is $L\big((1,0,0)\big) = (2,5,3)$. To transform $(1,0,0)$ using matrix A_b, we require its $\mathbf{b}$-components,

$$(1,0,0)_{\mathbf{b}} = \frac{1}{55}\begin{pmatrix} 10 & 30 & -5 \\ -7 & -10 & 9 \\ 6 & -15 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 10 \\ -7 \\ 6 \end{pmatrix}.$$

The transform of this vector is

$$L\big((1,0,0)_{\mathbf{b}}\big) = \frac{1}{55}A_b\begin{pmatrix} 10 \\ -7 \\ 6 \end{pmatrix} = \frac{1}{55^2}\begin{pmatrix} 150 & -515 & 570 \\ -50 & 47 & -201 \\ -185 & -221 & -307 \end{pmatrix}\begin{pmatrix} 10 \\ -7 \\ 6 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 155 \\ -37 \\ -39 \end{pmatrix}.$$

These are $\mathbf{b}$-components of the vector. Its natural components are

$$\frac{1}{55}\begin{pmatrix} 1 & -3 & 4 \\ 2 & 2 & -1 \\ 3 & 6 & 2 \end{pmatrix}\begin{pmatrix} 155 \\ -37 \\ -39 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}. \bullet$$

To handle a situation like that in our introductory example, we need the extension of Theorem 2.12 in the next theorem.

Theorem 2.13 Let L be a linear transformation from an n-dimensional vector space V to an m-dimensional space W. Let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ be bases for V, and $\{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_m\}$ and $\{\mathbf{d}'_1, \mathbf{d}'_2, \ldots, \mathbf{d}'_m\}$ be bases for W. If A_b is the matrix of L relative to the **b**- and **b′** bases, and A_d is the matrix relative to the **d** and **d′** bases, then

$$\begin{aligned} A_d &= T_{d'b'} A_b T_{bd}, && (2.11a)\\ &= T_{d'b'} A_b T_{db}^{-1}, && (2.11b)\\ &= T_{b'd'}^{-1} A_b T_{bd}, && (2.11c)\\ &= T_{b'd'}^{-1} A_b T_{db}^{-1}. && (2.11d) \end{aligned}$$

Proof The proof is similar to that for Theorem 2.12. Let $\mathbf{v}$ be any vector in V. Denote it by $\mathbf{v_b}$ when its components are with respect to the basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, and by $\mathbf{v_d}$ when its components are with respect to the basis $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$. Let $\mathbf{w}$ be the transform of $\mathbf{v}$. Denote it by $\mathbf{w_{b'}}$ when its components are with respect to the basis $\{\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_m\}$, and by $\mathbf{w_{d'}}$ when its components are with respect to the basis $\{\mathbf{d}'_1, \mathbf{d}'_2, \ldots, \mathbf{d}'_m\}$. To verify equation 2.11, we show that both matrices multiply every vector in V to give the same result.

$$\begin{aligned} (T_{d'b'} A_b T_{bd})\mathbf{v_d} &= T_{d'b'} A_b (T_{bd}\mathbf{v_d}) && \text{(matrix multiplication is associative)}\\ &= T_{d'b'} A_b \mathbf{v_b} && (T_{bd} \text{ changes } \mathbf{v} \text{ from } \mathbf{d}\text{- to } \mathbf{b}\text{-components})\\ &= T_{d'b'} (A_b \mathbf{v_b}) && \text{(matrix multiplication is associative)}\\ &= T_{d'b'} L(\mathbf{v_{b'}}) && (\text{where } L(\mathbf{v_{b'}}) \text{ has } \mathbf{b}'\text{-components})\\ &= L(\mathbf{v_{d'}}) && (T_{db} \text{ changes from } \mathbf{b}'\text{- to } \mathbf{d}'\text{-components})\\ &= A_d \mathbf{v_{d'}}. \blacksquare \end{aligned}$$

Example 2.39 A linear transformation L from a 3-dimensional space to a 4-dimensional space maps three linearly independent vectors as follows

$$L(1, 3, -2) = (-1, 2, 1, -1), \quad L(-2, -5, 4) = (5, -3, -1, 4), \quad L(3, 7, -5) = (1, 4, 0, 2),$$

where components of all vectors are natural components. Find the matrix of the transformation relative to natural bases of the spaces.

Solution Let us denote the basis $\{(1, 3, -2), (-2, -5, 4), (3, 7, -5)\}$ of the 3-space by **b**. The matrix of the transformation relative to the natural basis in the 4-space and the **b**-basis is

$$A = \begin{pmatrix} -1 & 5 & 1\\ 2 & -3 & 4\\ 1 & -1 & 0\\ -1 & 4 & 2 \end{pmatrix}.$$

Because components of the 4-space vectors are natural ones, and we want the matrix with respect to natural basis, we do not need the transition matrix $T_{d'b'}$ in equation 2.11. But we do need the transition matrix T_{bn} from the natural basis to the **b**-basis. It is

$$T_{bn} = T_{nb}^{-1} = \begin{pmatrix} 1 & -2 & 3\\ 3 & -5 & 7\\ -2 & 4 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 2 & 1\\ 1 & 1 & 2\\ 2 & 0 & 1 \end{pmatrix}.$$

Hence, the matrix of the transformation relative to natural bases in the spaces is

$$B = AT_{bn} = \begin{pmatrix} -1 & 5 & 1\\ 2 & -3 & 4\\ 1 & -1 & 0\\ -1 & 4 & 2 \end{pmatrix} \begin{pmatrix} -3 & 2 & 1\\ 1 & 1 & 2\\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 3 & 10\\ -1 & 1 & 0\\ -4 & 1 & -1\\ 11 & 2 & 9 \end{pmatrix}. \bullet$$

EXERCISES 2.5

In Exercises 1–2 the matrix is associated with a linear operator relative to the natural basis in a 2-dimensional space. Find the matrix for the operator with respect to the given basis.

1. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\{(1,1),(1,-1)\}$

2. $\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$, $\{(3,-3),(1,2)\}$

In Exercises 3–4 the matrix is associated with a linear operator relative to the natural basis in a 3-dimensional space. Find the matrix for the operator with respect to the given basis.

3. $\begin{pmatrix} 2 & 4 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\{(1,1,1),(2,-1,0),(0,1,3)\}$

4. $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$, $\{(2,-2,1),(1,0,1),(0,2,3)\}$

5. The matrix of a linear operator relative to the basis $\{(-3,-2),(-1,2)\}$ of a 2-dimensional space is $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$. Find the matrix of the operator relative to the basis $\{(-1,-6),(2,4)\}$.

In Exercises 6–7 the matrix is associated with a linear transformation relative to the natural bases of the spaces. Find the matrix for the transformation with respect to the given bases.

6. $\begin{pmatrix} 2 & 4 & 1 \\ 1 & -1 & 3 \end{pmatrix}$, $\{(2,3),(4,-1)\}$, $\{(1,1,1),(2,-1,0),(0,2,3)\}$

7. $\begin{pmatrix} -1 & 3 \\ 2 & 5 \\ -1 & 3 \end{pmatrix}$, $\{(3,-3),(1,2)\}$, $\{(-1,1,2),(2,0,3),(1,2,0)\}$

8. What would you have done in Example 2.39 if the three mapped vectors were not linearly independent?

9. A linear transformation L from a 3-dimensional space to a 4-dimensional space maps three vectors as follows:

$$L(-1,2,1) = (-4,-3,1,3), \quad L(-2,0,3) = (1,-8,-5,3), \quad L(4,1,-2) = (0,8,-1,-1),$$

where components of all vectors are natural. Find $L(4,1,-3)$.

10. (a) Verify that $\{1+x+x^2, 2+3x+x^2, 1+2x+x^2\}$ is a basis for the space $P_2(x)$.
(b) If L is the linear operator on the space that takes the first derivative of polynomials with respect to x, find the matrix of L with respect to the basis in part (a).

11. Repeat Exercise 10 with the shift operator of $L\big(p(x)\big) = p(x-1)$ of Exercise 15 in Section 2.2.

12. (a) Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a basis for $M_{2,2}(\mathcal{R})$.
(b) Let L be the linear operator on $M_{2,2}(\mathcal{R})$ that maps a matrix A according to

$$L(A) = 2A - 3A^T.$$

Find the matrix of L with respect to the basis in part (a) in two ways: (i) by finding images of the basis vectors in part (a); (b) by transforming the matrix of the operator with respect to the natural basis of $M_{2,2}(\mathcal{R})$ according to Theorem 2.12.

13. (a) Show that the matrices

$$\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix}, \quad \begin{pmatrix} -3 & 2 \\ 4 & -3 \end{pmatrix}, \quad \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$$

form a basis for $M_{2,2}(\mathcal{R})$.

(b) Let L be the linear operator on $M_{2,2}(\mathcal{R})$ that maps a matrix A according to

$$L(A) = 4A + 2A^T - 3I.$$

Find the matrix of L with respect to the basis in part (a).

Answers

1. $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **2.** ( ) **3.** ( ) **4.** ( ) **5.** $\begin{pmatrix} -2 & 7 \\ -3 & 10 \end{pmatrix}$ **6.** ( ) **7.** ( )

8. Nothing. The transformation would not be defined. **9.** $(-1, 10, 2, -2)$ **10.** $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & -4 & -2 \end{pmatrix}$

11. $\begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$ **12.**(b) $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ **13.**(b) $\begin{pmatrix} -11 & -9 & -4 & -2 \\ 17 & 15 & 4 & 2 \\ 0 & 0 & 6 & 0 \\ 19 & 15 & 2 & 4 \end{pmatrix}$