Solutions to Midterm #2 MATH 3132 Summer 2025

16 **1.** (a) Show that the Fourier series for the function

$$f(x) = 3x + 2, \quad 0 < x < 4, \quad f(x+4) = f(x),$$

is

$$8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}.$$

Daw a graph of $f(x)$ on the interval $-4 \le x \le 8$, and a separate graph of the function to which the Fourier series converges.

(b) Use the Fourier series in part (a) to find the sum of the series of constants $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$.

(a) Coefficients in the series are

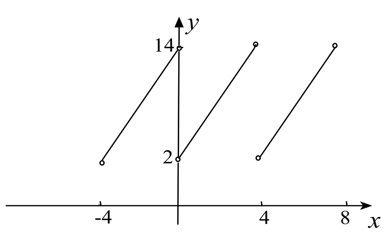
$$a_0 = \frac{1}{2}\int_0^4 (3x+2)\,dx = 16, \quad \text{and for } n \ge 1$$

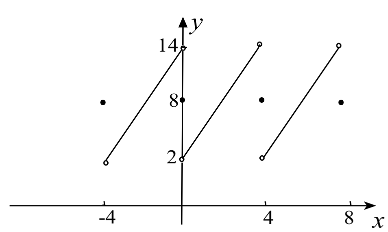
$$a_n = \frac{1}{2}\int_0^4 (3x+2)\cos\frac{n\pi x}{2}\,dx = 0,$$

$$b_n = \frac{1}{2}\int_0^4 (3x+2)\sin\frac{n\pi x}{2}\,dx = \frac{-12}{n\pi}.$$

Hence, the Fourier series is

$$8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}.$$





(b) If we set $x = 1$, then

$$5 = 8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} = 8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi}{2} = 8 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.$$

Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$.

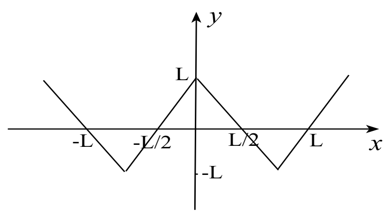
4 **2.** In general, it is not possible to express an arbitrary function $f(x)$ in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}.$$

Give three conditions on $f(x)$ in order that it can be written in this form. Bonus for 4 extra marks: Define a non-trigonometric function that satisfies the three conditions.

Four conditions are:
1. Continuous and piecewise smooth
2. Even
3. $2L$ periodic if defined for all x, but no condition if defined only for $0 \leq x \leq L$.
4. Average value over one full period must be zero.

The function shown below satisfies these conditions.



8 **3.** Find all singular points for the differential equation

$$(x^2 - x)^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x-1)y = 0,$$

and determine whether they are regular or irregular singular points. Justify all statements.

Consider the functions

$$\frac{x}{(x^2-x)^2} = \frac{1}{x(x-1)^2} \quad \text{and} \quad \frac{x-1}{(x^2-x)^2} = \frac{1}{x^2(x-1)}.$$

Since neither has a Maclaurin series or a Taylor series about $x = 1$, $x = 0$ and $x = 1$ are singular points. For $x = 0$, consider the functions

$$\frac{x^2}{(x^2-x)^2} = \frac{1}{(x-1)^2} \quad \text{and} \quad \frac{x^2(x-1)}{(x^2-x)^2} = \frac{1}{(x-1)}.$$

Since both have Maclaurin series, $x = 0$ is regular singular. For $x = 1$, consider

$$\frac{x(x-1)}{(x^2-x)^2} = \frac{1}{x(x-1)} \quad \text{and} \quad \frac{(x-1)^2(x-1)}{(x^2-x)^2} = \frac{x-1}{x^2}.$$

Since the first of these does not have a Taylor series about $x = 1$, $x = 1$ is irregular singular.

12 **4.** If a Frobenius solution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ is substituted into the differential equation

$$(x - x^2)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

the result is

$$0 = x^{r-1}\left\{r(r-4)a_0 + \sum_{n=0}^{\infty}\{(n+r+1)(n+r-3)a_{n+1} + [2-(n+r)(n+r-1)]a_n\}\,x^{n+1}\right\}.$$

Assume this result. Do **NOT** derive it.
(a) Show that the indicial roots differ by an integer.
(b) Find the solution corresponding to the smaller indicial root and express it in sigma notation simplified as much as possible.
(c) Is the solution general? Explain.

(a) The indicial equation is $r(r-4) = 0$ so that indicial roots are $r = 0$ and $r = 4$, differing by an integer.
(b) When $r = 0$, the recursive formula satisfies

$$(n+1)(n-3)a_{n+1} + [2 - n(n-1)]a_n = 0, \quad n \geq 0.$$

Thus,

$$a_{n+1} = \frac{n^2 - n - 2}{(n+1)(n-3)}a_n = \frac{(n+1)(n-2)}{(n+1)(n-3)}a_n = \frac{n-2}{n-3}a_n.$$

For $n = 0$, $a_1 = \frac{2}{3}a_0$.
For $n = 1$, $a_2 = \frac{1}{2}a_1 = \frac{1}{3}a_0$.
For $n = 2$, $a_3 = 0$.
For $n = 3$, we return to $(n-3)a_{n+1} = (n-2)a_n$, which implies that $a_3 = 0$.
For $n = 4$, $a_5 = 2a_4$.
For $n = 5$, $a_6 = \frac{3}{2}a_5 = 3a_4$.
For $n = 6$, $a_7 = \frac{4}{3}a_6 = 4a_4$.
The solution is

$$\begin{aligned} y(x) &= x^0\left[a_0 + \frac{2a_0x}{3} + \frac{a_0x^2}{3} + a_4x^4 + 2a_4x^5 + 3a_4x^6 + \cdots\right] \\ &= a_0\left(1 + \frac{2x}{3} + \frac{x^2}{3}\right) + a_4\sum_{n=4}^{\infty}(n-3)x^n. \end{aligned}$$

10 **5.** If a Maclaurin series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is substituted into the differential equation

$$(x^2+1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0,$$

find the recurrence formula for the a_n simplified as much as possible. Do **NOT** iterate the recurrence formula, just derive it.

When we substitute the Maclaurin series into the differential equation

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 2na_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 2na_n x^n \\ &= \sum_{n=0}^{\infty} [n(n-1)a_n + (n+2)(n+1)a_{n+2} + 2na_n]x^n. \end{aligned}$$

The recurence relation is

$$n(n-1)a_n + (n+2)(n+1)a_{n+2} + 2na_n = 0, \quad n \geq 0,$$

or,

$$a_{n+2} = -\frac{n(n-1)+2n}{(n+2)(n+1)}a_n = -\frac{n(n+1)}{(n+2)(n+1)}a_n = -\frac{n}{n+2}a_n, \quad n \geq 0.$$