**3132 Solutions to Midterm 1 Summer 2025**

**1.** Set up, but do **NOT** evaluate, a definite integral for the value of the line integral

$$\oint_C (xy + z)ds$$

where $C$ is the curve

$$x^2 + y^2 = 4, \qquad y + z = 2$$

directed counterclockwise as viewed from the origin. You need not simplify the integrand.

With parametric equations

$$x = 2\cos t, \quad y = -2\sin t, \quad z = 2 + 2\sin t, \quad 0 \le t \le 2\pi,$$

$$\oint_C (xy + z)ds = \int_0^{2\pi} [2\cos t(-2\sin t) + (2 + 2\sin t)]\sqrt{(-2\sin t)^2 + (-2\cos t)^2 + (2\cos t)^2}\, dt.$$

**2.** Evaluate the line integral

$$\int_C \left(2xy + \frac{z}{y}\right) dx + \left(x^2 - \frac{xz}{y^2}\right) dy + \left(\frac{x}{y} + 1\right) dz$$

where $C$ is the shorter part of the curve

$$x^2 + y^2 + z^2 = 4, \qquad x^2 + (y - 2)^2 = 1$$

from the point $(\sqrt{15}/4, 7/4, 0)$ to $(0, 1, \sqrt{3})$.

Since

$$\nabla \left(x^2 y + \frac{xz}{y} + z\right) = \left(2xy + \frac{z}{y}\right)\hat{\mathbf{i}} + \left(x^2 - \frac{xz}{y^2}\right)\hat{\mathbf{j}} + \left(\frac{x}{y} + 1\right)\hat{\mathbf{k}},$$

the line integral is independent of path in any domain not containing points on the $xz$-plane where $y = 0$. If $y$ were equal to zero, the equations for the curve would give

$$x^2 + z^2 = 4, \qquad x^2 + 4 = 1.$$

The second is an impossibility. The line integral is independent of path in the domain $y > 0$, and its value is

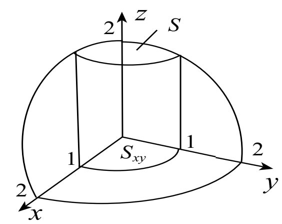
$$\left\{x^2 y + \frac{xz}{y} + z\right\}_{(\sqrt{15}/4, 7/4, 0)}^{(0,1,\sqrt{3})} = \sqrt{3} - \frac{105}{64}.$$

**3.** Set up, but do NOT evaluate, a double iterated integral in polar coordinates for the surface integral

$$\iint_S (x^2 + y^2 + 2z)\, dS$$

where $S$ is that part of $z = 4 - x^2 - y^2$ inside $x^2 + y^2 = 1$.

If we project the surface onto the $xy$-plane,




$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$
$$= \sqrt{1 + 4x^2 + 4y^2}\, dA.$$

Then,

$$\iint_S (x^2 + y^2 + 2z)\, dS = \iint_{S_{xy}} [x^2 + y^2 + 2(4 - x^2 - y^2)]\sqrt{1 + 4x^2 + 4y^2}\, dA$$
$$= 4\int_0^{\pi/2}\int_0^1 (8 - r^2)\sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

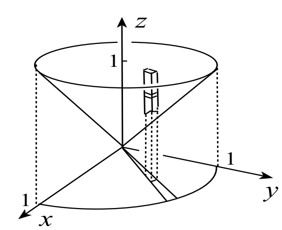
**4.** Evaluate the surface integral

$$\oiint_S (x^3\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

where $S$ is the surface bounding the volume enclosed by the surfaces

$$z = \sqrt{x^2 + y^2}, \qquad z = 1,$$

and $\hat{\mathbf{n}}$ is the inward pointing normal to $S$.




Using the divergence theorem,

$$\oiint_S (x^3\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\iiint_V (3x^2 - 2y)\, dV$$
$$= -3\iiint_V x^2\, dV = -12\int_0^{\pi/2}\int_0^1\int_r^1 r^2\cos^2\theta\, r\, dz\, dr\, d\theta$$
$$= -12\int_0^{\pi/2}\int_0^1 r^3\cos^2\theta\, \{z\}_r^1\, dr\, d\theta$$
$$= -12\int_0^{\pi/2}\int_0^1 (r^3 - r^4)\cos^2\theta\, dr\, d\theta = -12\int_0^{\pi/2}\left\{\frac{r^4}{4} - \frac{r^5}{5}\right\}_0^1 \cos^2\theta\, d\theta$$
$$= -\frac{12}{20}\int_0^{\pi/2}\left(\frac{1 + \cos 2\theta}{2}\right) d\theta = -\frac{3}{10}\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = -\frac{3\pi}{20}.$$