Some examples on domain and range

The sign of a linear form $ax+b$ is determined by the following rule:

		$\frac{-b}{a}$	
$ax+b$	**opposite sign of** a	•	**sign of** a

The sign of a quadratic form ax^2+bx+c is determined by the following rule:

		$\frac{-b-\sqrt{b^2-4ac}}{2a}$		$\frac{-b+\sqrt{b^2-4ac}}{2a}$	
ax^2+bx+c	**sign of** a	•	**opposite sign of** a	•	**sign of** a

To find the range of a quadratic function complete the square

Example. Find the range of the quadratic function $g(x) = -3x^2 + 2x - 1$.

Solution: There is no restriction on x , therefore the domain is $(-\infty, \infty)$. As for the range, we complete the square to get:

$$-3x^2 + 2x - 1 = -3(x^2 - \frac{2}{3}x) - 1 = -3\Big[(x - \frac{1}{3})^2 - \frac{1}{9}\Big] - 1 = -3(x - \frac{1}{3})^2 + \frac{3}{9} - 1$$

$$= -3(x - \frac{1}{3})^2 - \frac{2}{3} \leq -\frac{2}{3} \quad \text{(the maximum value of } -\tfrac{2}{3} \text{ is reached at } x = \tfrac{1}{3}).$$

The maximum for this function is $-\frac{2}{3}$, so the range is $(-\infty\ ,\ -\frac{2}{3}]$. ✓

This inequality shows that the values of the expression $-3x^2 + 2x - 1 = -3(x - \frac{1}{3})^2 - \frac{2}{3}$ is at least $-\frac{2}{3}$. By putting $x = \frac{1}{3}$ it becomes clear the minimum value of the expression is $-\frac{2}{3}$. So the range of this function is $(-\infty\ ,\ -\frac{2}{3}]$.

Example. Find the range of the quadratic function $g(x) = 2x^2 - 3x + 7$.

Solution: There is no restriction on x , therefore the domain is $(-\infty, \infty)$. As for the range, we complete the square to get:

$$2x^2 - 3x + 7 = 2(x^2 - \frac{3}{2}x) + 7 = 2\Big[(x - \frac{3}{4})^2 - \frac{9}{16}\Big] + 7 = 2(x - \frac{3}{4})^2 - \frac{9}{8} + 7$$

$$= 2(x - \frac{3}{4})^2 + \frac{-9 + 56}{8} = 2(x - \frac{3}{4})^2 + \frac{47}{8} \geq \frac{47}{8}$$

This inequality shows that the values of the expression $2x^2 - 3x + 7 = 2(x - \frac{3}{4})^2 + \frac{47}{8}$ is at least $\frac{47}{8}$. By putting $x = \frac{3}{4}$ it becomes clear the minimum value of the expression is $\frac{47}{8}$. So the range of this function is $[\frac{47}{8}\ ,\ \infty)$.

Now , below we find some examples on finding the domain

Example. Find the domain of the function $f(x) = \dfrac{\sqrt{2x^5+64}}{3x-4} + x^3 + 1$

Solution: The restrictions that we have are

$$\begin{cases} 2x^5+64 \geq 0 \\ \text{and} \\ 3x-4 \neq 0 \end{cases} \Leftrightarrow \begin{cases} x^5 \geq -\frac{64}{2} \\ \text{and} \\ x \neq \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} x^5 \geq -32 \\ \text{and} \\ x \neq \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} x \geq \sqrt[5]{-32} \\ \text{and} \\ x \neq \frac{4}{3} \end{cases}$$

$$\Leftrightarrow \begin{cases} x \geq -\sqrt[5]{32} \\ x \neq \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} x \geq -2 \\ x \neq \frac{4}{3} \end{cases}$$

The first restriction gives us the interval $[-2\ ,\ \infty)$ and the second restriction tells us to remove the point $\frac{4}{3}$ from this set. So, the domain is

$$[-2\ ,\ \frac{4}{3}) \cup (\frac{4}{3}\ ,\ \infty)$$

Example (section 1.1 exercise 31). Find the domain of $f(x) = \dfrac{x+4}{x^2-9}$.

Solution: The points $x = \pm 3$ must be excluded as they make the denominator zero.

$$D_f : \quad x \neq \pm 3$$

Example. Find the domain of the function $f(t) = \sqrt{\dfrac{-1+2t}{3t-6}}$

Solution:

We must have $\dfrac{-1+2t}{3t-6} \geq 0$ but under the restriction $t \neq 2$ to make sure that the denominator is non-zero. We must determine the signs of both $-1+2t$ and $3t-6$ to be able to determine where the ration $\dfrac{-1+2t}{3t-6}$ is non-negative.

		$\frac{1}{2}$		2	
$-1+2t$	$-$	•	$+$	|	$+$
$3t-6$	$-$	|	$-$	•	$+$
qutient	$+$	|	$-$	|	$+$

So the domain on which $\dfrac{-1+2t}{3t-6} \geq 0$ excluding where the denominator is zero is the set

$$D_f = (-\infty\,,\,\frac{1}{2}] \cup (2\,,\,\infty)$$

Example **(section 1.1 exercise 32)**. Find the domain of $f(x) = \dfrac{2x^3-5}{x^2+x-6}$.

Solution: The points satisfying $x^2+x-6=0$ must be excluded as they make the denominator zero.

$$x^2+x-6=0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = \begin{cases} -3 \\ 2 \end{cases}$$

$$D_f : \quad x \neq -3\ ,\ 2$$

Example (section 1.1 exercise 33). Find the domain of $f(t) = \sqrt[3]{2t-1}$.

Solution: As the operator $\sqrt[3]{\cdot}$ can act on all numbers (positive and negative), the value $\sqrt[3]{2t-1}$ can be calculated for all t, so

$$D_f = (-\infty\,,\,\infty)$$

Example (section 1.1 exercise 34). Find the domain of $g(t) = \sqrt{3-t} - \sqrt{2+t}$.

Solution: We must have both $3 - t \geq 0$ and $2 + t \geq 0$. So, we must have both $3 \geq t$ and $t \geq -2$. So the domain is the interval $[-2\;,\;3]$

Example (section 1.1 exercise 35). Find the domain of $h(x) = \dfrac{1}{\sqrt[4]{x^2 - 5x}}$.

Solution: We must have $x^2 - 5x > 0$. But $x^2 - 5x = x(x-5)$ and

		0		5	
x	$-$	•	$+$		$+$
$x-5$	$-$		$-$	•	$+$
product	$+$		$-$		$+$

So, the domain is:

$$D_h = (-\infty\,,\,0) \cup (5\,,\,\infty)$$

Example (section 1.1 exercise 38). Find the domain of the function $h(x) = \sqrt{4 - x^2}$.

Solution:

$$4 - x^2 \geq 0 \quad \Leftrightarrow \quad 4 \geq x^2 \quad \overset{\text{square root}}{\Leftrightarrow} \quad 2 \geq |x| \quad \Leftrightarrow \quad -2 \leq x \leq 2$$

$$D_h = [-2\ ,\ 2]$$

Example Find the domain of the function $\frac{\sqrt{x^2-4}}{e^{5x+7}(x-3)}$

Solution: Since the exponential function return positive values, the quantity e^{5x+7} is positive and therefore it does not cause any problem being in the denominator. So the only restrictions that we have are $x^2 - 4 \geq 0$ and $x - 2 \neq 0$. The restriction $x^2 - 4 \geq 0$ is equivalent to x being in $(-\infty\ ,\ -2] \cup [2\ ,\ \infty)$, and the restriction $x - 3 \neq 0$ is the same as $x \neq 3$. So we should remove the point $x = 3$ from the set $(-\infty\ ,\ -2] \cup [2\ ,\ \infty)$, and the domain is

$$(-\infty\ ,\ -2] \cup [2\ ,\ 3) \cup (3\ ,\ \infty)$$

Example Find the domain of the function $\frac{\sqrt{x^2-3}\cos(2x+1)}{e^{-2x+3}(x-4)}$

Solution: Since the exponential function return positive values, the quantity e^{-2x+3} is positive and therefore it does not cause any problem being in the denominator. Also the function $\cos(2x+1)$ brings along no restriction for x because you can give any value to the function cos as the input. So the only restrictions that we have are $x^2 - 3 \geq 0$ and $x - 4 \neq 0$. The restriction $x^2 - 3 \geq 0$ is equivalent to x being in $(-\infty\ ,\ -\sqrt{3}] \cup [\sqrt{3}\ ,\ \infty)$, and the restriction $x - 4 \neq 0$ is the same as $x \neq 4$. So we should remove the point $x = 4$ from the set $(-\infty\ ,\ -\sqrt{3}] \cup [\sqrt{3}\ ,\ \infty)$, and the domain is

$$(-\infty\ ,\ -\sqrt{3}] \cup [\sqrt{3}\ ,\ 4) \cup (4\ ,\ \infty)$$