Initial Value Problems

1 Euler's Explicit Method (section 10.2.1)

Definition . By a first order initial value problem, we mean a problem such as

$$\frac{dy}{dx} = f(x,y) \qquad\qquad y(a) \text{ is given}$$

in which we are looking a function $y(x)$ that satisfies these condition. Most IVP's cannot be solved analytically, therefore we must come up with numerical solutions for them. As an example consider the following IVP:

$$y' = 2x + y \qquad\qquad y(0) = 1$$

and we may be interested in get an approximate value for $y(1)$. For this, we divide the interval $[0\,,\,1]$ into a number of subintervals of equal length, and get a step size $h = \frac{1-0}{n}$. To be more specific, let us divide the interval into 5 subinterval by taking the following nodes:

$$0\ ,\ 0.2\ ,\ 0.4\ ,\ 0.6\ ,\ 0.8\ ,\ 1$$

Then starting from the value $y(0)$ we construct an approximate value for $y(0.2)$, and then using this value we construct an approximate value to $y(0.4)$, and so on, until we finally construct an approximate value for $y(1)$. In general, take an arbitrary INP

$$\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(a) \ \text{ is given} \end{cases} \quad \Rightarrow \quad \begin{cases} y'(x) = f(x,y(x)) \\ y(a) \ \text{ given} \end{cases}$$

and suppose we want to approximate the value of the function y at some point b bigger than x_1. Then we divide the interval $[a\,,\,b]$ into n subintervals with these nodes:

$$x_1 = a\ ,\ x_2\ ,\ \cdots\ ,\ x_{n-1}\ ,\ x_n = b$$

The tangent line at the point $(x_1, y_1) = (x_1, y(x_1))$ is the line:

$$\begin{aligned} y - y_1 &= y'(x_1)(x - x_1) \\ \hookrightarrow \quad y - y_1 &= f(x_1, y_1)(x - x_1) \\ \hookrightarrow \quad y &= y_1 + f(x_1, y_1)(x - x_1) \end{aligned}$$

By putting x_2 in this equation, we get

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1) \quad \Rightarrow \quad y_2 = y_1 + f(x_1, y_1)h$$

This point (x_2, y_2) is on the tangent line as it satisfies the equation of the line. The value y_2 is taken as an approximation to the true value $y(x_2)$ (Here the details of the figure on page 390 were explained). Now that we have y_2 (an approximate value) we use it to create an approximation y_3 to $y(x_3)$ through :

$$y_3 = y_2 + f(x_2, y_2)h$$

and in general, by having constructed an approximation y_i, the approximate value for $y(x_{n+1})$, which we call y_{i+1} , is found by setting

$$y_{i+1} := y_i + f(x_i, y_i)h$$

This will continue until we find an approximation for $y(x_n)$ which was desired. This method is called the **Euler's Explicit Method**.

Example (from exercise 10.1 of the textbook). Solve the ODE

$$\frac{dy}{dx} = \frac{x^2}{y} \qquad x = 0 \text{ to } x = 2.1 \text{ with } y(0) = 2$$

using the Euler's explicit method with $h = 0.7$

Solution. We have

$$f(x, y) = \frac{x^2}{y}$$

and the nodes are:

$$0 \,,\ 0.7 \,,\ 1.4 \,,\ 2.1$$

Discretization:

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \Rightarrow \quad y_{i+1} = y_i + \frac{x_i^2}{y_i}(0.7)$$

$$\begin{cases} i = 1 \quad \Rightarrow \quad y_2 = y_1 + \frac{x_1^2}{y_1}(0.7) = 2 + \frac{0^2}{2}(0.7) = 2 \\ i = 2 \quad \Rightarrow \quad y_3 = y_2 + \frac{x_2^2}{y_2}(0.7) = 2 + \frac{(0.7)^2}{2}(0.7) = 2.1715 \\ i = 3 \quad \Rightarrow \quad y_4 = y_3 + \frac{x_3^2}{y_3}(0.7) = 2.1715 + \frac{(1.4)^2}{2.1715}(0.7) = \boxed{2.8033} \end{cases}$$

2 Modified Euler's Method (section 10.3)

The two-step algorithm for the **Modified Euler's Method** is as follows:

Step 1. Approximate y_{i+1} using the Explicit Euler' Method on the whatever last approximate value y_i :

$$\hat{y}_{i+1} = y_i + f(x_i, y_i)h$$

Step 2. Take the following y_{i+1} as the Modified Euler's Approximation for the true value of $y(x_{i+1})$

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1})}{2} h$$

Here the figure on page 402 of the textbook was explained.

Example (from exercise 10.1 of the textbook). Solve the ODE

$$\frac{dy}{dx} = \frac{x^2}{y} \qquad x = 0 \text{ to } x = 2.1 \text{ with } y(0) = 2$$

using the Modified Euler's Method with $h = 0.7$

Solution. We have

$$f(x, y) = \frac{x^2}{y}$$

and the nodes are:

$$0\ ,\ 0.7\ ,\ 1.4\ ,\ 2.1$$

Interim discretization:

$$\hat{y}_{i+1} = y_i + f(x_i, y_i)h \quad \Rightarrow \quad y_{i+1} = y_i + \frac{x_i^2}{y_i}(0.7)$$

$$
i=1 \quad \Rightarrow \quad \begin{cases}
f(x_1,y_1)=\frac{x_1^2}{y_1}=\frac{0^2}{2}=0 \\[2ex]
\hat{y}_2=y_1+f(x_1\,,\,y_1)\,h=y_1+(0)(0.7)=2+\frac{0^2}{2}(0.7)=2 \\[2ex]
f(x_2,\hat{y}_2)=\frac{x_2^2}{\hat{y}_2}=\frac{0.7^2}{2}=0.2450 \\[2ex]
y_2=y_1+\frac{1}{2}\Big[f(x_1\,,\,y_1)+f(x_2\,,\,\hat{y}_2)\Big]\,h=2+\frac{1}{2}\Big[0+0.2450\Big](0.7)=2.0858
\end{cases}
$$

$$
i=2 \Rightarrow \begin{cases}
f(x_2,y_2)=\frac{x_2^2}{y_2}=\frac{0.7^2}{2.0858}=0.2349 \\[2ex]
\hat{y}_3=y_2+f(x_2\,,\,y_2)\,h=2.0858+(0.2349)(0.7)=2.2502 \\[2ex]
f(x_3,\hat{y}_3)=\frac{x_3^2}{\hat{y}_3}=\frac{1.4^2}{2.2502}=0.8710 \\[2ex]
y_3=y_2+\frac{1}{2}\Big[f(x_2\,,\,y_2)+f(x_3\,,\,\hat{y}_3)\Big]\,h=2.0858+\frac{1}{2}\Big[0.2349+0.8710\Big](0.7)=2.4729
\end{cases}
$$

$$
i=3 \Rightarrow \begin{cases}
f(x_3,y_3)=\frac{x_3^2}{y_3}=\frac{1.4^2}{2.4729}=0.7926 \\[2ex]
\hat{y}_4=y_3+f(x_3\,,\,y_3)\,h=2.4729+(0.7926)(0.7)=3.0277 \\[2ex]
f(x_4,\hat{y}_4)=\frac{x_4^2}{\hat{y}_4}=\frac{2.1^2}{3.0277}=1.4566 \\[2ex]
y_4=y_3+\frac{1}{2}\Big[f(x_3\,,\,y_3)+f(x_4\,,\,\hat{y}_4)\Big]\,h=2.4729+\frac{1}{2}\Big[0.7926+1.4566\Big](0.7)=\boxed{3.2601}
\end{cases}
$$

3 Second Order Runge-Kutta Methods, including the Midpoint Method (sections 10.5.1 and 10.4)

The discretization for the **Runge-Kutta Method of Order 2 (RK2)** is performed in the following two steps:

(i) Walk along the tangent line at (x_i, y_i) as much as α times the vector $\langle h, mh\rangle = h\langle 1, m\rangle$ where $m = f(x_i, y_i)$ is the slope, where $\frac{1}{2} \le \alpha \le 1$. This gets you to the point

$$(x_i, y_i) + \alpha(h, hm) = (x_i + \alpha h, y_i + hm) = (x_i + \alpha h, y_i + \alpha h f(x_i, y_i))$$

If $\alpha = 1$, then we get to the point where the Euler's Explicit Method tells us to go. We call the point (x_i, y_i) the old point, and the point $(u, v) = (x_i + \alpha h, y_i + \alpha h f(x_i, y_i))$ the new point.

(ii) Take the number $b = \frac{1}{2\alpha}$. Since $\frac{1}{2}\alpha \le 1$, we have $\frac{1}{2} \le b \le 1$. Then take the combination

$$y_{i+1} = y_i + (1-b)h\,(\text{old point's slope}) + b\,h\,(\text{new point's slope})$$

Since $\frac{1}{2} \le b \le 1$ we have $0 \le 1 - b \le \frac{1}{2}$ therefore are giving more weight to the new slope. The old slope and the new slope are denoted by K_1 and K_2 respectively; so

$$\begin{aligned}
K_1 &= f(x_i, y_i) && \text{old slope} \\
u &= x_i + \alpha h && \text{the first component of the new point} \\
v &= y_i + \alpha h K_1 && \text{the second component of the new point} \\
K_2 &= f(u, v) && \text{new slope} \\
y_{i+1} &= y_i + h\Big\{(1-b)\,K_1 + b\,K_2\Big\}
\end{aligned}$$

The Modified Euler Method is a special case of the RK2 Method by taking the extreme case $\alpha = 1$ (for which we have $b = \frac{1}{2}$) :

$$\begin{aligned}
K_1 &= f(x_i, y_i) && \text{old slope} \\
u &= x_i + h \\
v &= y_i + hK_1 \\
K_2 &= f(u, v) && \text{new slope} \\
y_{i+1} &= y_i + h\Big\{\tfrac{1}{2}K_1 + \tfrac{1}{2}K_2\Big\}
\end{aligned}$$

By taking the extreme case $\alpha = \frac{1}{2}$ we will have $b = 1$ and we get a method called the **Midpoint Method**.

$$\begin{aligned} K_1 &= f(x_i, y_i) \qquad \text{old slope} \\ u &= x_i + \tfrac{1}{2}h \\ v &= y_i + \tfrac{1}{2}hK_1 \\ K_2 &= f(u, v) \qquad \text{new slope} \\ y_{i+1} &= y_i + hK_2 \end{aligned}$$

By taking $\alpha = \frac{2}{3}$ we have $b = \frac{3}{4}$ and the corresponding method is called the **Heun's Method**, whose discretization is as follows:

$$\begin{aligned} K_1 &= f(x_i, y_i) \qquad \text{old slope} \\ u &= x_i + \tfrac{2}{3}h \\ v &= y_i + \tfrac{2}{3}hK_1 \\ K_2 &= f(u, v) \qquad \text{new slope} \\ y_{i+1} &= y_i + h\left\{\tfrac{1}{4}K_1 + \tfrac{3}{4}K_2\right\} \end{aligned}$$

Here the details of the figure on page 405 were explained

Example (from exercise 10.3 of the textbook). Solve the ODE

$$\frac{dy}{dt} = y + t^3 \qquad t = 0 \text{ to } t = 1 \text{ with } y(0) = 1$$

using the Midpoint Method with $h = 0.5$

Solution. We have

$$f(t, y) = t^3 + y$$

and the nodes are:

$$t_1 = 0\ ,\ t_2 = 0.5\ ,\ t_3 = 1$$

$$i=1 \quad \Rightarrow \quad \begin{cases} K_1 = t_1^3 + y_1 = 0+1 = 1 \\ u = t_1 + \frac{1}{2}h = 0 + \frac{1}{2}(0.5) = 0.25 \\ v = y_1 + \frac{1}{2}hK_1 = 1 + \frac{1}{2}(0.5)(1) = 1.25 \\ K_2 = f(u,v) = u^3 + v = (0.25)^3 + (1.25) = 1.2656 \\ y_2 = y_1 + hK_2 = 1 + (0.5)(1.2656) = 1.6328 \end{cases}$$

$$i=2 \quad \Rightarrow \quad \begin{cases} K_2 = t_2^3 + y_2 = (0.5)^3 + 1.6328 = 1.7578 \\ u = t_2 + \frac{1}{2}h = 0.5 + \frac{1}{2}(0.5) = 0.75 \\ v = y_2 + \frac{1}{2}hK_1 = 1.6328 + \frac{1}{2}(0.5)(1.7578) = 2.0723 \\ K_2 = f(u,v) = u^3 + v = (0.75)^3 + 2.0723 = 2.4942 \\ y_3 = y_2 + hK_2 = 1.6328 + (0.5)(2.4942) = \boxed{2.8799} \end{cases}$$

Example . The function $y = e^{-x} + \sin(x) + \cos(x)$ satisfies

$$\begin{cases} y' = -y + 2\cos x \\ y(0) = 2 \end{cases}$$

We want to approximate the value of $y(\pi) = 2$ and compare the result with this true value to demonstrate the superiority of the RK2 method over its rival Euler's Explicit Method. We use the step size $h = \frac{b-a}{n} = \frac{\pi - 0}{10} = 0.314$ corresponding to $n = 10$ subintervals. We apply the two methods "Euler's Explicit" and "Midpoint" to do a comparison. The results are shown below:

Midpoint Method

x_i	approx y_i	exact	error
0.00	2.000000	2.000000	0.000000
0.31	1.992264	1.990476	1.788155e-003
0.63	1.930661	1.930290	3.704116e-004
0.94	1.783842	1.786463	2.621791e-003
1.26	1.538699	1.544683	5.984207e-003
1.57	1.199026	1.207880	8.853566e-003
1.88	0.783220	0.793875	1.065556e-002
2.20	0.321062	0.332133	1.107054e-002
2.51	-0.150235	-0.140229	1.000554e-002
2.83	-0.590440	-0.582875	7.564977e-003
3.14	-0.960802	-0.956786	4.016130e-003

Euler's Explicit Method

x_i	approx y_i	exact	error
0.00	2.000000	2.000000	0.000000
0.31	2.000000	1.990476	9.523798e-003
0.63	1.969248	1.930290	3.895756e-002
0.94	1.858911	1.786463	7.244741e-002
1.26	1.644233	1.544683	9.955006e-002
1.57	1.321843	1.207880	1.139636e-001
1.88	0.906574	0.793875	1.126986e-001
2.20	0.427604	0.332133	9.547117e-002
2.51	-0.076048	-0.140229	6.418116e-002
2.83	-0.560477	-0.582875	2.239783e-002
3.14	-0.981965	-0.956786	2.517843e-002
3.14	-0.960802	-0.956786	2.517843e-002

4 Fourth Order Runge-Kutta Method (section 10.5.3)

The classical **Fourth-Order Runge-Kutta Method (RK4)**

$$
\begin{aligned}
K_1 &= f(x_i, y_i) && \text{old slope} \\
u &= x_i + \tfrac{1}{2}h \\
v &= y_i + \tfrac{1}{2}hK_1 \\
K_2 &= f(u\,,\,v) && \text{first improvement of the slope} \\
v &= y_i + \tfrac{1}{2}hK_2 \\
K_3 &= f(u\,,\,v) && \text{second improvement of the slope} \\
u &= x_i + h \\
v &= y_i + hK_3 \\
K_4 &= f(u\,,\,v) && \text{third improvement of the slope} \\
y_{i+1} &= y_i + h\left(\tfrac{1}{6}K_1 + \tfrac{2}{6}K_2 + \tfrac{2}{6}K_3 + \tfrac{1}{6}K_4\right)
\end{aligned}
$$

Example . For the same function in the previous example, here are the results of applying RK4.

RK4 Method

x node	approx	exact	error
0.00	2.000000	2.000000	0.000000
0.31	1.990464	1.990476	1.219558e-005
0.63	1.930260	1.930290	3.059194e-005
0.94	1.786415	1.786463	4.790234e-005
1.26	1.544624	1.544683	5.936679e-005
1.57	1.207817	1.207880	6.231168e-005
1.88	0.793820	0.793875	5.581790e-005
2.20	0.332093	0.332133	4.042724e-005
2.51	-0.140247	-0.140229	1.784564e-005
2.83	-0.582866	-0.582875	9.375050e-006
3.14	-0.956748	-0.956786	3.817795e-005

5 Solving a System of First-Order ODEs Using RK2 Method For Systems (Modeified Euler Version) (section 10.8.2)

Assume a system consisting of two first-order IVPs over an interval $[a, b]$:

$$\begin{cases} \frac{dy}{dx} = f_1(x,y,z) & y(a) \quad \text{is given} \\ \frac{dz}{dx} = f_2(x,y,z) & z(a) \quad \text{is given} \end{cases}$$

and we want to approximate the values $y(b)$ and $z(b)$. As an example:

$$\begin{cases} \frac{dy}{dx} = 3y - 2z - 3x + 1 & y(0) = 0 \\ \frac{dz}{dx} = 4y - 3z - 4x & z(0) = -1 \end{cases}$$

The exact solution to this system is:

$$\begin{cases} y(x) = e^x - e^{-x} + x \\ z(x) = e^x - 2e^{-x} \end{cases}$$

When the exact solution is not known, then an approximate solution can be found using the RK2 method for systems (or other methods):

$$\begin{cases} \begin{cases} K_{y,1} = f_1(x_i, y_i, z_i) \\ K_{z,1} = f_2(x_i, y_i, z_i) \end{cases} \\ \\ u = x_i + \alpha h \\ \\ \begin{cases} v_y = y_i + \alpha h K_{y,1} \\ v_z = z_i + \alpha h K_{z,1} \end{cases} \\ \\ \begin{cases} K_{y,2} = f_1(u, v_y, v_z) \\ K_{z,2} = f_2(u, v_y, v_z) \end{cases} \\ \\ \begin{cases} y_{i+1} = y_i + h\{(1-b)K_{y,1} + bK_{y,2}\} \\ z_{i+1} = z_i + h\{(1-b)K_{z,1} + bK_{z,2}\} \end{cases} \end{cases}$$

Especially, for $\alpha = 1$ we will have $b = \frac{1}{2}$ and then we will have the RK2 method for a system involving two differential equations:

$$\begin{cases} \begin{cases} K_{y,1} = f_1(x_i, y_i, z_i) \\ K_{z,1} = f_2(x_i, y_i, z_i) \end{cases} \\ \\ \quad u = x_i + h \\ \\ \begin{cases} v_y = y_i + hK_{y,1} \\ v_z = z_i + hK_{z,1} \end{cases} \\ \\ \begin{cases} K_{y,2} = f_1(u\,, v_y\,, v_z) \\ K_{z,2} = f_2(u\,, v_y\,, v_z) \end{cases} \\ \\ \begin{cases} y_{i+1} = y_i + h\{\frac{1}{2}K_{y,1} + \frac{1}{2}K_{y,2}\} \\ z_{i+1} = z_i + h\{\frac{1}{2}K_{z,1} + \frac{1}{2}K_{z,2}\} \end{cases} \end{cases}$$

Example . Consider the following system of differential equations:

$$\begin{cases} \frac{dy}{dx} = x + z = f(x, y, z) \qquad y(0) = 2 \\ \frac{dz}{dx} = x - y^2 = g(x, y, z) \qquad z(0) = 1 \end{cases}$$

Use the step size $h = 0.1$ and the RK2 method (for systems) to solve this system.

Solution .

$$\begin{cases} K_{y,1} = f(x_1, y_1, z_1) = x_1 + z_1 = 0 + 1 = 1 \\ K_{z,1} = g(x_1, y_1, z_1) = x_1 - y_1^2 = 0 - 2^2 = -4 \end{cases}$$

$$u = x_1 + h = 0 + 0.1 = 0.1$$

$$\begin{cases} v_y = y_1 + hK_{y,1} = 2 + 0.1(1) = 2.1 \\ v_z = z_1 + hK_{z,1} = 1 + 0.1(-4) = 0.6 \end{cases}$$

$$\begin{cases} K_{y,2} = f(x\,,\, v_y\,,\, v_z) = f(0.1\,,\, 2.1\,,\, 0.6) = 0.1 + 2.1 = 2.2 \\ K_{z,2} = g(x\,,\, v_y\,,\, v_z) = g(0.1\,,\, 2.1\,,\, 0.6) = 0.1 - (2.1)^2 = -4.31 \end{cases}$$

$$\begin{cases} y_2 = y_1 + h\{\tfrac{1}{2} K_{y,1} + \tfrac{1}{2} K_{y,2}\} = 2 + 0.1\{\tfrac{1}{2}(1) + \tfrac{1}{2}(2.2)\} = 2.16 \\ z_2 = z_1 + h\{\tfrac{1}{2} K_{z,1} + \tfrac{1}{2} K_{z,2}\} = 1 + 0.1\{\tfrac{1}{2}(-4) + \tfrac{1}{2}(-4.3)\} = 0.5850 \end{cases}$$

Here the code on page 429 of was explained

6 Solving a Higher-Order Initial Value Problem (section 10.9)

The following is an example of a second-order IVP:

$$y'' - y'\sin(x) + xy = 0 \qquad y(0) = -1 \qquad y'(0) = 0.7$$

For a second order IVP we need two initial conditions, for a third-order IVP we need three initial conditions, and so on.

Here is the general idea on how to solve a higher-order IVP: change y' to a new variable z:

$$z = y'$$

Then

$$z' = y''$$

so then:

$$y'' - y'\sin(x) + xy = 0 \quad \Rightarrow \quad z' - z\sin(x) + xy = 0 \quad \Rightarrow \quad z' = z\sin(x) - xy$$

Then the above equation changes to this system:

$$\begin{cases} \frac{dy}{dx} = z = f(x,y,z) & y(0) = -1 \\ \frac{dz}{dx} = z\sin(x) - xy = g(x,y,z) & z(0) = 0.7 \end{cases}$$

Now we may apply the Modified Euler Method for systems to solve this system. Let us approximate $y(0.1)$ by taking $h = 0.1$.

$$\begin{cases} \begin{cases} K_{y,1} = f(x_1, y_1, z_1) = z_1 = 0.7 \\ K_{z,1} = g(x_1, y_1, z_1) = -z_1 \sin(x_1) - x_1 y_1 = 0 \end{cases} \\ \\ \quad u = x_1 + h = 0 + 0.1 = 0.1 \\ \\ \begin{cases} v_y = y_1 + hK_{y,1} = -1 + 0.1(0.7) = -0.93 \\ v_z = z_1 + hK_{z,1} = 0.7 + 0.1(0) = 0.7 \end{cases} \\ \\ \begin{cases} K_{y,2} = f(x\,, v_y\,, v_z) = f(0.1\,, -0.93\,, 0.7) = 0.7 \\ K_{z,2} = g(x\,, v_y\,, v_z) = g(0.1\,, -0.93\,, 0.7) = -0.7\sin(0.1) - (0.1)(-0.93) = 0.1629 \end{cases} \\ \\ \begin{cases} y_2 = y_1 + h\{\frac{1}{2}K_{y,1} + \frac{1}{2}K_{y,2}\} = -1 + 0.1\{\frac{1}{2}(0.7) + \frac{1}{2}(0.7)\} = -0.3 \\ z_2 = z_1 + h\{\frac{1}{2}K_{z,1} + \frac{1}{2}K_{z,2}\} = 0.7 + 0.1\{\frac{1}{2}(0) + \frac{1}{2}(0.1629)\} = 0.7081 \end{cases} \end{cases}$$

Example (from the textbook). Convert the third-order IVP

$$y''' = 2x - 3y + 4y' + xy'' \qquad y(0) = 3 \qquad y'(0) = 2 \qquad y''(0) = 7$$

Solution. Change y' and y'' to some new variables z and w:

$$z = y' \qquad w = z' = y''$$

Then

$$y''' = 2x - 3y + 4y' + xy'' \qquad w' = 2x - 3y + 4z + xw \quad \Rightarrow$$

$$\begin{cases} y' &= z = f(x, y, z, w) & y(0) = 3 \\ z' &= w = g(x, y, z, w) & z(0) = 2 \\ w' &= 2x - 3y + 4z + xw = h(x, y, z, w) & w(0) = 7 \end{cases}$$

Let's go back to the problem

$$y'' - y'\sin(x) + xy = 0 \qquad y(0) = -1 \qquad y'(0) = 0.7$$

and perform one step of the Modified Euler Method for the corresponding system:

So far 194 pages of typed materials, consisting of lecture notes, solutions to the lab questions, and solutions to the homework questions, have been given to the students