1520: Curve Sketching Worked Example

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The following is the completely worked solution for the following problem: Sketch $f(x) = -2x^3 - 9x^2 + 108x - 10$.

This is a polynomial equation, and thus continuous everywhere. It is a cubic, with negative leading coefficient, which means that

$$\lim_{x \to -\infty} f(x) = \infty$$
$$\lim_{x \to \infty} f(x) = -\infty.$$

It also means there will be no asymptotes.

The y-intercept is given by $f(0)$:

$$f(0) = -2(0)^3 - 9(0)^2 + 108(0) - 10 = -10$$

So the y-intercept is the point $(0, -10)$. The x-intercepts are given by solving for x in

$$\underbrace{-2x^3 - 9x^2 + 108x - 10}_{f(x)} = 0,$$

but this looks hard so we'll ignore x-intercepts for now.

Now find f'.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[-2x^3 - 9x^2 + 108x - 10\right] \\ &= -6x^2 - 18x + 108 \end{aligned}$$

Guess that 108 is divisible by 6. Check: Since there are 96 hours in four days, 96 is divisible by 24. Also, divisible by 6, since 24 is divisible by 6. Notice that $108 = 96 + 12$, and 12 is also divisible by 6, thus we have $108 = 96 + 12 = (4 \times 24) + (2 \times 6) = (16 \times 6) + (2 \times 6) = 18 \times 6$.

$$f'(x) = -6\left(x^2 + 3x - 18\right)$$

Does this factor? Sure it does! Since $-3 \times 6 = -18$ and $6 - 3 = 3$

$$f'(x) = -6\,(x-3)\,(x+6)$$

So $f'(x) = 0$ when $x = 3$ or $x = -6$. Since f' is a quadratic with negative leading coefficient, it opens down, which means positive in the middle and negative on the outsides:

$$\begin{array}{c|c|c|c}
f' & - & + & - \\
\hline
f & \searrow & \nearrow & \searrow \\
 & \multicolumn{1}{r}{-6} & \multicolumn{1}{r}{3} &
\end{array}$$

So $x = -6$ corresponds to a minimum and $x = 3$ corresponds to a maximum (by the first derivative test). Now we find the values so we know the critical points:

$$\big(\text{Recall:} \quad f(x) = -2x^3 - 9x^2 + 108x - 10\big)$$

(and also recall that $108 = 6 \times 18 = 3 \times 6^2 = 2^2 \times 3^3$)

$$\begin{aligned}
f(-6) &= -2(-6)^3 - 9(-6)^2 + 108(-6) - 10 \\
&= 2 \times 6^3 - 9 \times 6^2 - 18 \times 6^2 - 10 \\
&= 12 \times 6^2 - 9 \times 6^2 - 18 \times 6^2 - 10 \\
&= (12 - 9 - 18)6^2 - 10 \\
&= (-15)6^2 - 10 \\
&= -2^2 \times 3^3 \times 5 - 10 \\
&= -4 \times 27 \times 5 - 10 \\
&= -4 \times (25 + 2) \times 5 - 10 \\
&= -(100 \times 5 + 4 \times 10) - 10 \\
&= -540 - 10 \\
&= -550
\end{aligned}$$

and similarly,

$$f(3) = 179$$

(Would *I* expect *you* to do this without a calculator? No! But it can be done...) So the local minimum point is $(-6, -550)$ and the local maximum point is $(3, 179)$.

Now find f'':

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[-6x^2 - 18x + 108\right] \\ &= -12x - 18 \\ &= -6(2x+3) \\ &= 0 \qquad \text{when} \quad x = -\frac{3}{2}. \end{aligned}$$

Which is a line of negative slope, and thus:

$$\begin{array}{c|c|c} f'' & + & - \\ \hline f & \cup & \cap \\ & \multicolumn{2}{c}{-\tfrac{3}{2}} \end{array}$$

So the point of inflection has a y-value of

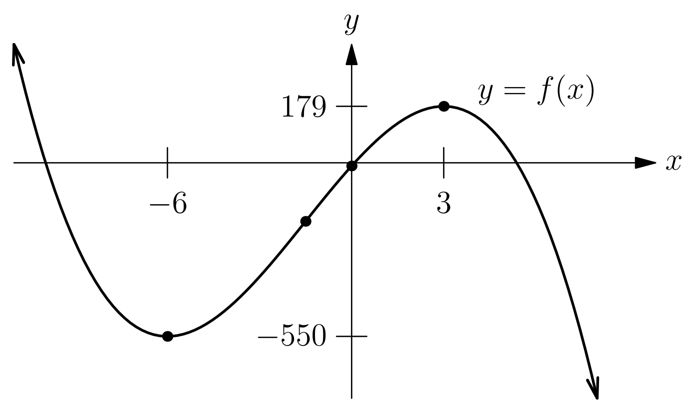
$$f\left(-\frac{3}{2}\right) = -\frac{371}{2}.$$

Does the second derivative test give the same conclusions for the maximum and minimum as the first derivative test did earlier? Yes, it does (if it didn't, you've made a mistake somewhere—go back and try to find it.)

Now draw the axes, plot the four points found in the previous steps

$$(-6, -550), \quad \left(-\frac{3}{2}, -\frac{371}{2}\right), \quad (0, -10), \quad \text{and} \quad (3, 179)$$

and connect the dots.



Remember, despite appearances, this graph *does not* pass through the origin: the y-intercept is $(0, -10)$. However, the scale of the graph makes that difficult to see.