Modeling Of Transmission Lines

4.1 Transmission Line Equations in Frequency Domain

In the frequency domain, the equations for an n conductor transmission line are:

$$-\frac{\partial \underline{V}}{\partial x} = Z(\omega)\ \underline{I}$$

$$-\frac{\partial \underline{I}}{\partial x} = Y(\omega)\ \underline{V} \qquad (4.1)$$

where $Z(\omega) = R(\omega) + j\,X(\omega)$

$Y(\omega) = G(\omega) + j\,B(\omega)$

where $\underline{I}(x)$ and $\underline{V}(x)$ are the voltage phasors at a distance x along a

For most frequencies of interest we may approximate the complicated series expansions for Y and Z due to Carson [1] by the formulae due to Wedepohl et al [2] and Deri et al [3]:

$$Z_{ii} = Z_{ii}(\text{int}) + Z_{ii}(\text{ext})$$

where

$$Z_{ii}(\text{int}) = \left[\frac{\rho_i m}{2\pi r_i} \coth(0.7777\, m r_i) + \frac{0.3565\,\rho_i}{\pi r_i^2}\right] \Omega/\text{m} \qquad (4.2a)$$

with $m(\omega) = \sqrt{\dfrac{j\omega\mu_o}{\rho_i}}$

and $$Z_{ii}(\text{ext}) = \frac{j\omega\mu_o}{2\pi} \ln\left[\frac{2h_i + d_e}{r_i}\right] \qquad (4.2b)$$

where $d_e = \sqrt{\dfrac{\rho}{j\omega\mu_o}}$

and $$Z_{ij} = \frac{j\omega\mu_o}{2\pi} \ln\left(\frac{\sqrt{(y_i-y_j)^2 + (h_i+h_j+2d_e)^2}}{\sqrt{(y_i-y_j)^2 + (h_i-h_j)^2}}\right) \Omega/\text{m}$$

with d_e as before. (4.2c)

Here ρ_i is the conductor resistivity (Ω-m), ρ is the earth resistivity, r_i is the conductor radius (or GMR for bundled conductors) for conductor i.

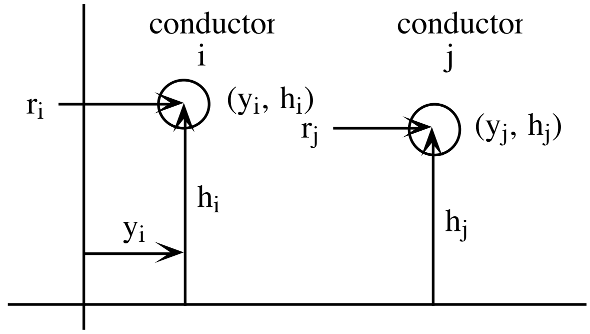


Fig. 4.1: Variables describing the geometry

Similarly $[Y] = (Y_{ij})$

where $[Y] = P^{-1}$

$$P_{ij} = (j2\pi\omega\varepsilon_o)^{-1} \ln\left[\frac{\sqrt{(y_i-y_j)^2 + (h_i+h_j)^2}}{d_{ij}}\right] \tag{4.3}$$

where $d_{ij} = \sqrt{(y_i-y_j)^2 + (h_i-h_j)^2}$ for $i \neq j$

$= r_i$ for $i=j$

4.2 Admittance Matrix Model for One Conductor Lossless Line

Here $\rho_i=0$, $\rho=0$. Assume a single conductor line [4]
Eqn. 4.2 and 4.3

$$Z_{ii}(\text{int}) = 0, \quad Z_{ii}(\text{ext}) = j\omega\left[\frac{\mu_o}{2\pi} \ln\left(\frac{2h_i}{r_i}\right)\right] = j\omega L$$

and $$P_{ii} = (j2\pi\varepsilon_o\omega)^{-1} \ln\left[\frac{2h}{r_i}\right]$$

or $$Y_{ii} = \frac{2j\omega\pi\varepsilon_o}{\ln\left(\frac{2h}{r_i}\right)} = j\omega C$$

where L and C are inductance/capacitance per unit length.
Thus:

$$\left.\begin{aligned} -\frac{\partial V}{\partial x} &= j\omega L\, I \\ -\frac{\partial I}{\partial x} &= j\omega C\, V \end{aligned}\right\} \quad \text{frequency domain form}$$

or:

$$\left.\begin{aligned} -\frac{\partial V}{\partial x} &= L\,\frac{\partial i}{\partial t} \\ -\frac{\partial I}{\partial x} &= C\,\frac{\partial v}{dt} \end{aligned}\right\} \quad \text{time domain form} \tag{4.4}$$

∂ Alembert's solution to (4.4) is:

$$\left\{\begin{array}{l} i(x, t) = f_1(x-ct) + f_2(x+ct) \\ v(x, t) = Z\big(f_1(x-ct) - f_2(x+ct)\big) \\ \text{where } c = 1/\sqrt{LC} \text{ (phase velocity)}, \quad Z = \sqrt{L/C} \text{ (characteristic impedances)} \end{array}\right\} \tag{4.5}$$

Note on rearranging,

$$\left.\begin{aligned} v(x, t) + Z\, i(x, t) &= 2Zf_1(x-ct) \\ v(x, t) - Z\, i(x, t) &= -2Zf_2(x+ct) \end{aligned}\right\} \tag{4.6}$$

From (4.6) it is clear that an observer moving at velocity c along the line will see the quantity v+Zi= constant, because for him, x–ct=constant. Let the observer leave m at time t–τ and

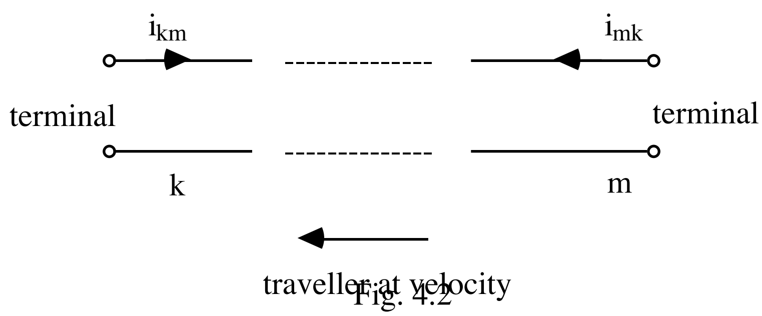


Fig. 4.2

arrive at k at time t, where τ=d/c, d being the line length.
Thus,

$$v_m(t-\tau) + Zi_{m,k}(t-\tau) = v_k(t) + Z(-i_{km}(t))$$

or rearranging

$$i_{km}(t) = \frac{1}{Z}\, v_k(t) + I_k(t-\tau)$$

where $\quad I_k(t-\tau) = -\dfrac{1}{Z}\, v_m(t-\tau) - i_{mk}(t-\tau)$

Similarly, $\hfill (4.7)$

$$i_{mk}(t) = \frac{1}{Z}\, v_m(t) + I_m(t-\tau)$$

where $\quad I_m(t-\tau) = -\dfrac{1}{Z}\, v_k(t-\tau) - i_{km}(t-\tau)$

Thus we have the following Norton representation of the T-line:

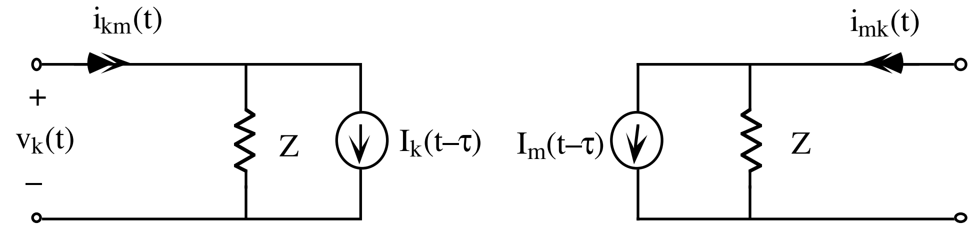
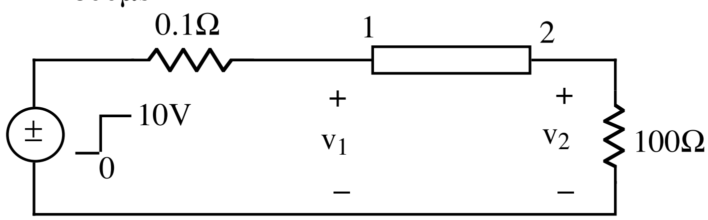


Fig. 4.3: Norton Representation

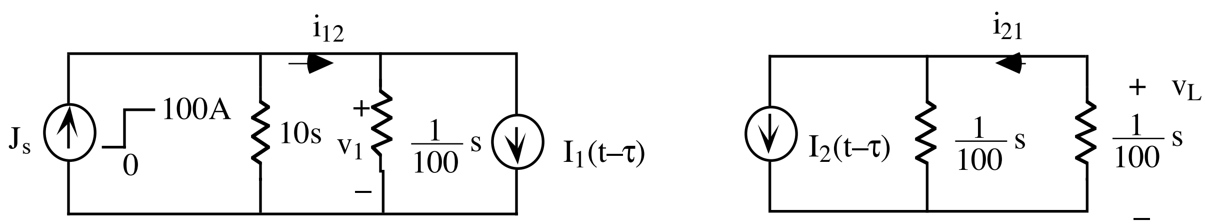
Example: Plot $v_1(t)$, $i_{12}(t)$, $v_2(t)$, $i_{21}(t)$ if

$L = 400$ nH/m, $C = 40$ pf/m, $d = 2\times10^5$m

Thus $\tau = d\sqrt{LC} \quad = 800\mu s$



Thus:



Note: $\quad Z = \sqrt{\frac{L}{C}} = 100\Omega$

where

$$\left.\begin{aligned} I_1(t-\tau) &= -\frac{1}{100}\, v_2(t-\tau) - i_{21}(t-\tau) \\ I_2(t-\tau) &= -\frac{1}{100}\, v_1(t-\tau) - i_{12}(t-\tau) \end{aligned}\right]$$

then

$$\left.\begin{aligned} 10.01\ v_1 &= J_s - I_1(t-\tau) \\ \text{and} \quad 0.02\ v_2 &= -I_2(t-\tau) \end{aligned}\right] \text{ for the two networks}$$

from which $V_1(t)$ and $V_2(t)$ may be evaluated and hence the currents:

$$i_{12} = \frac{v_1}{100} + I_1(t-\tau)$$

$$i_{21} = \frac{v_2}{100} + I_2(t-\tau)$$

Listed below in FORTRAN program to solve for these voltages and currents, and typical results are shown in Figs. 4.4 through 4.6 for various terminations on the line.

```
C****************************************************************************
C Solution of Simple Transmission line problem
C****************************************************************************
       real delt,fintim,prtime,v1,I1(20),I2(20),i12,i21
       real L,C,Js
C____________________________________________________________________
C open datafile for reading and output file for answer:
       open(unit=10,file='inp_file2')
       open(unit=11,file='out_file2')
       write(11,*) 'This is the solution of a simple T-line problem'
C____________________________________________________________________
C Reading the datafile:
C
       read (10,*) delt,fintim,prtime,tup
       read(10,*)L,C,d,Rs,Rl
C Z is the char. imp., Rs is the source res., and Rl the load res.
C____________________________________________________________________
C Initialization
       Z=sqrt(L/C)
       g1=1/Rs+1/Z
       g2=1/Rl+ 1/Z
       z1=1/g1
       z2=1/g2
       v1=0.0
       v2=0.0
      Do i =1,20
       I1(i)=0.0
       I2(i)=0.0
      enddo
       time=0.0
C____________________________________________________________________
C Setting the loop on time:
       nstep=fintim/delt
       nprt = prtime/delt
       ndelay = d*sqrt(L*C)/delt
C____________________________________________________________________
C The numerical solution:
C
       Do 1 J=1,nstep+1
C
C      Setting the source current:
         if(time.LT.tup)then
            Js=0.0
         else
            Js=100.0
         endif
```

```
C (Note: at this time the values of v1  i12,etc,on the right hand side
C are from the previous timestep (t-delt) as the soln for time=t has
C as yet not been found!)
C_____________________________________________________________________
C Now the solution:
         v1 = (Js-I1(ndelay))*z1
         v2 = -I2(ndelay)*z2
C
         i12 = v1/Z+I1(ndelay)
         i21 = v2/Z+I2(ndelay)
C_____________________________________________________________________
C Calculating the history term (for use ndelay timesteps later...).:
      Do i=1,19
         ii=20-i
        I1(ii+1)=I1(ii)
        I2(ii+1)=I2(ii)
      enddo
         I1(1) = -1/Z*v2- i21
         I2(1) = -1/Z*v1- i12
C Writing the soln. to an output file:
C
         if(mod(J-1,nprt).eq.0) write(11,1111) time, v1, i12,v2,i21,
     .                          I1(ndelay),I2(ndelay)
1111     format(f8.5,6G12.5)
C_____________________________________________________________________
C Updating time:
C
         time=time+delt
1      continue
C.....Now loop back for the next timestep.
C_____________________________________________________________________
       write(6,*)'Thats all..folks!'
       stop
       end
```

Results: $R_L = 100\Omega$ (Characteristic impedance)

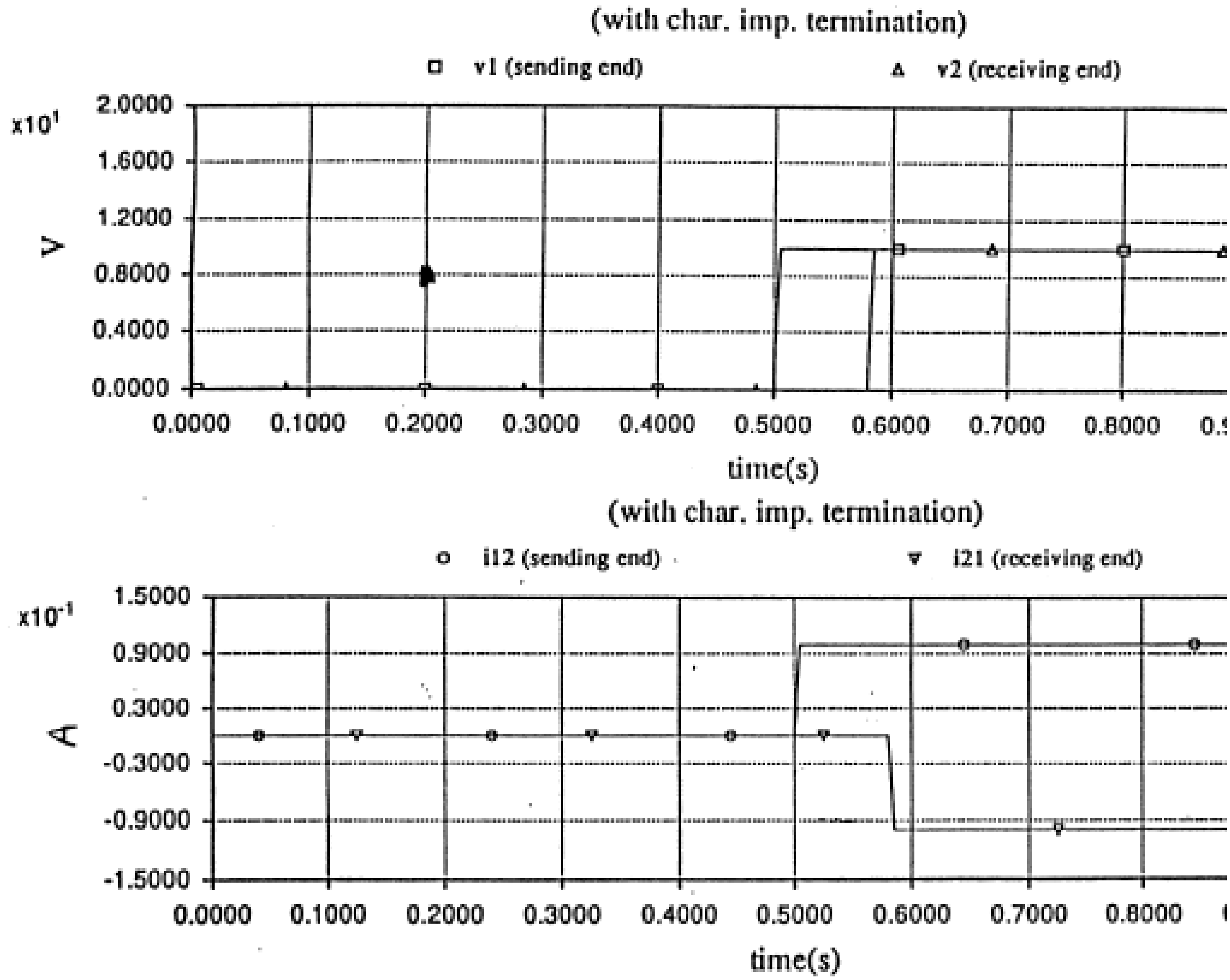


Fig. 4.4. T-line with Characteristic Impedance Termination

Note: a) The 800μs delay due to travel time.
b) The lack of reflections due to characteristic impedance termination.

$R_L = 10^6 \Omega$

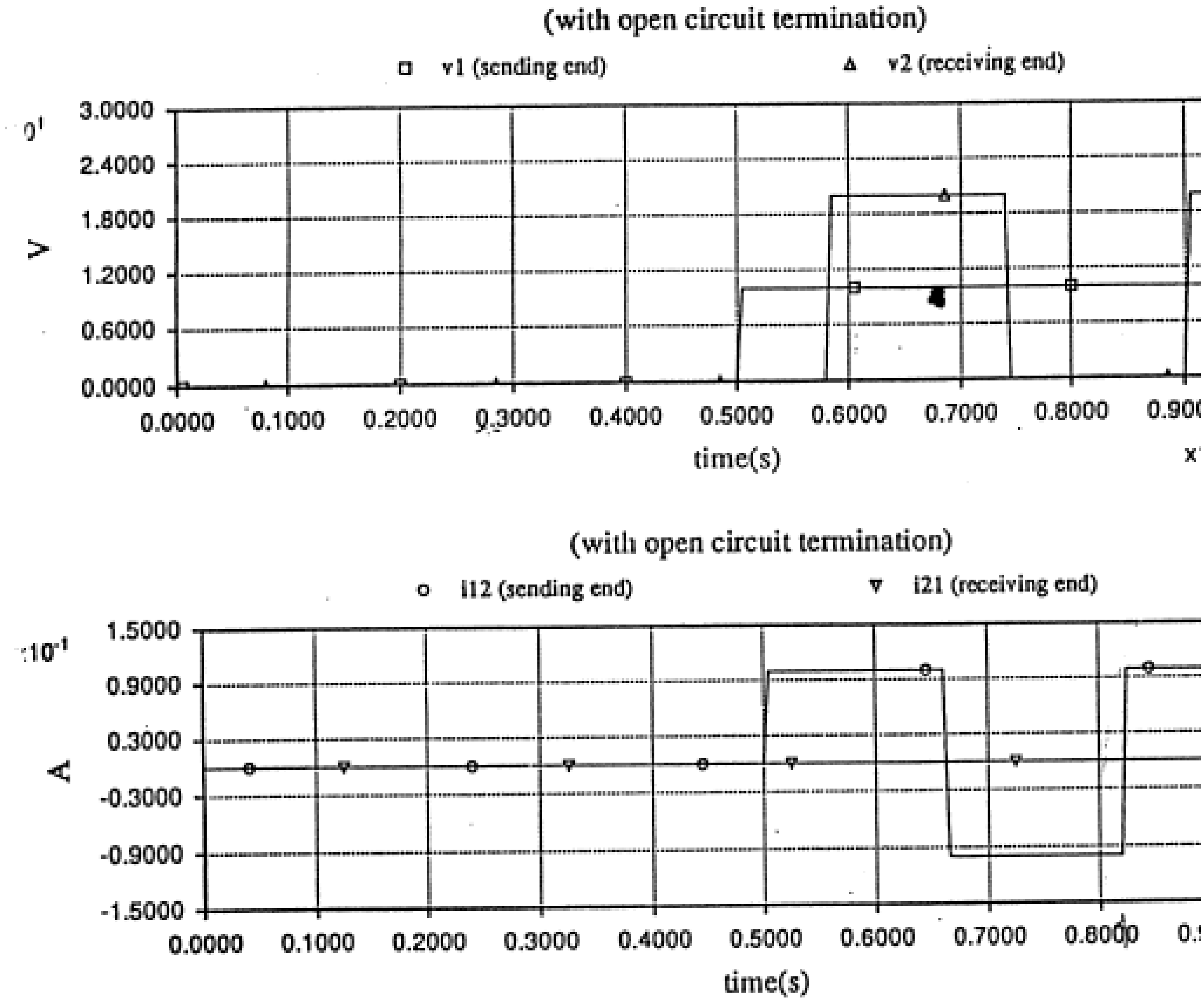


Fig. 4.5. T-line with Open Circuit Termination

Note:
- a) The 800μs delay between the sending and receiving end.
- b) The re-inforcement of the voltage signal due to reflections to twice its original magnitude.
- c) The period of the oscillation = 1.6 ms (2 travel times).

$R_L = 0.1\Omega$

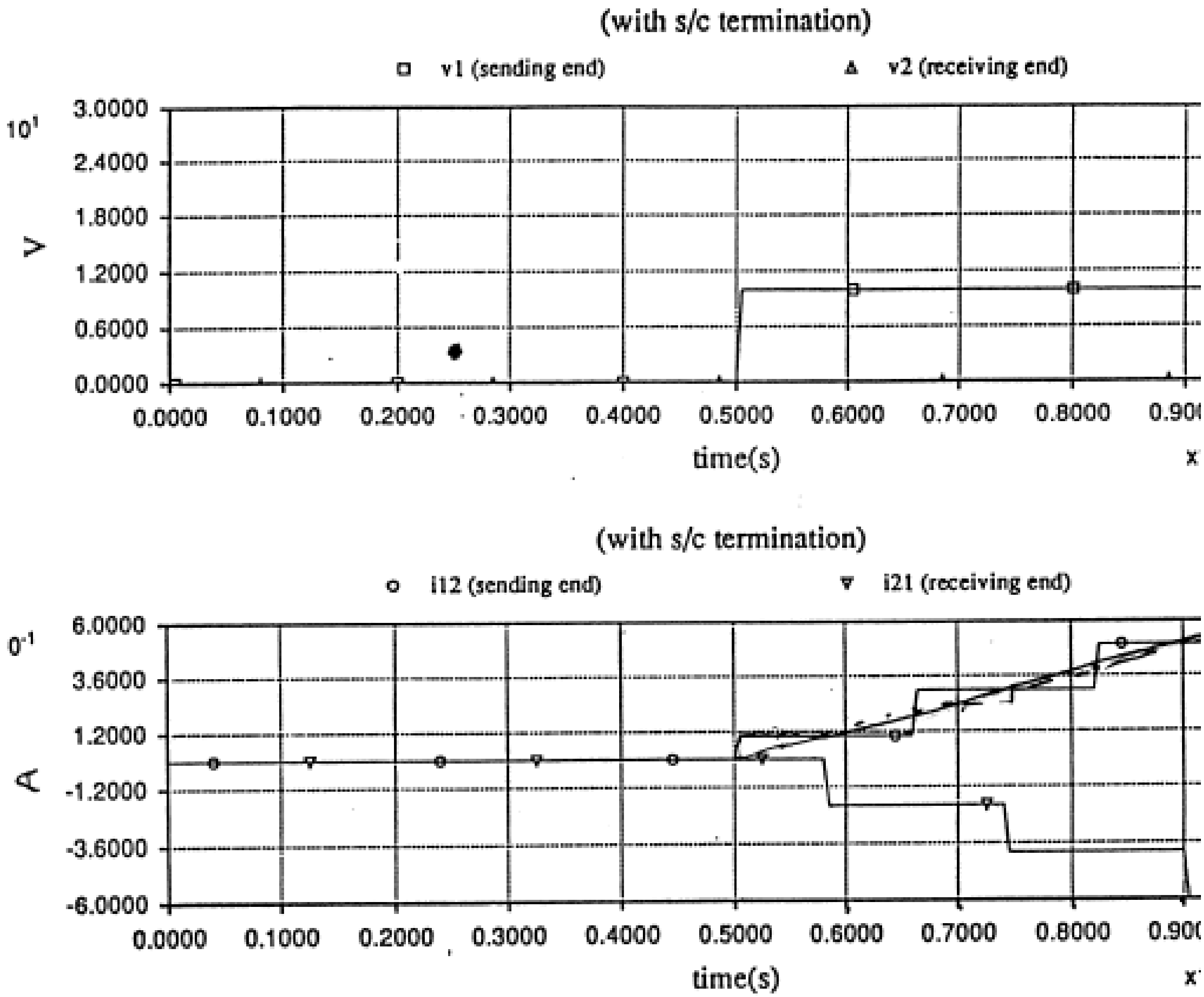


Fig. 4.6. T-line with Short Circuit Termination

Note: a) The zero voltage on the short circuited receiving end.

b) The current shows the same delays and periodic steps seen earlier, but is building up to $\frac{V_1}{R} = \frac{v_1}{0.2\Omega}$ (actually for a true s/c termination it will build to infinity).

4.3 Multiconductor Lossless Line

Some theorems required for this material are now given below:

Theorem 4.1: The eigenvector matrix T for a n×n square matrix A with n distinct eigenvalues $\lambda_1 \ldots \lambda_n$, diagonalizes A, i.e.

$$T^{-1} A\ T = \Lambda \text{ (a diagonal matrix}$$
$$\text{where } \Lambda_{ij} = \delta_{ij}\lambda_j$$

The eigenvector matrix is a matrix whose i^{th} column is an eigenvector corresponding to the eigenvalue λ_i. A proof of this is straightforward and may be looked up in a suitable book on linear algebra.

If A has repeated eigenvalues a strictly diagonal form is sometimes not possible. However in transmission line problems such a situation does not arise.

Example 4.2
Consider the matrix

$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ which has eigenvalues $\lambda_1 = 1, \lambda_2 = 2$ (found by solving $|A - \lambda I| = 0$.

To find the first eigenvector $\underline{p}$ we have $A\underline{p} = \lambda_1 \underline{p}$

or $$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 1 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

i.e. $p_1 = p_1$ and $3p_1 + 2p_2 = p_2$ or $3p_1 = -p_2$. As the two equations are linearly dependent, we may choose any value for p_1 (as long as we don't obtain the singular vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$), and then $p_2 = -3p_1$. Thus $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is a suitable eigenvector. Similarly for the eigenvalue $\lambda_2 = 2$:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 2 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

or $p_1 = 2p_1$ and $3p_1 + 2p_2 = 2p_2$.
Thus $p_1 = 0$, p_2 is arbitrary. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is therefore one suitable eigenvector.

Thus $T = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ is a possible transformation matrix

with $T^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, we have

$$T^{-1} AT = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{, which is diagonal.}$$

Theorem 4.2 If Y and Z correspond to the per unit length admittance and impedance matrices of a transmission line with zero resistivity above a lossless earth, then ZY=YZ is a diagonal matrix.

Proof: From equation 4.2

$Z = [Z_{ij}]$ where (for $i \neq j$)

$$Z_{ij} = \frac{j\omega\mu_o}{2\pi} \ln\left(\frac{D_{ij}}{d_{ij}}\right)$$

where $D_{ij} = \sqrt{(y_i - y_j)^2 + (h_i + h_j)^2}$, $d_{ij} = \sqrt{(y_i - y_j)^2 + (h_i - h_j)^2}$

likewise $Y = P^{-1}$ where $P = [P_{ij}]$

and $P_{ij} = \dfrac{1}{2\pi\epsilon_o j\omega} \ln\left(\dfrac{D_{ij}}{d_{ij}}\right)$

Thus we find

$$Z_{ij} = -\omega^2\mu_o\epsilon_o P_{ij} \qquad i \neq j$$

Similarly $Z_{ii} = -\omega^2\mu_o\epsilon_o P_{ii}$ (4.7)

and thus $Z = -\omega^2\mu_o\epsilon_o P$

or $ZY = -\omega^2\mu_o\epsilon_o \underline{I}$

where I is the identity matrix. Thus ZY is diagonal and similarly so is YZ.

Note: Since $[Z] = j\omega(L)$, $Y = j\omega(C)$ it follows that

$[L][C] = [C][L]$ is diagonal as well.

Theorem 4.3 If $[L][C] = [C][L] = \Lambda$ where Λ is diagonal and [L] and [C] are $n \times n$ nonsingular matrices with distinct eigenvalues, then the matrix T which diagonalizes [L] also diagonalizes [C], i.e. if $T^{-1} [L] T$ is diagonal, then so is $T^{-1} [C] T$.

Proof: Let $T^{-1} [L] T = \Lambda_L$

However $[L] [C] = \Lambda$ so $[L] = \Lambda C^{-1}$

Thus $T^{-1} [\Lambda C^{-1}] T = \Lambda_L$

or $\Lambda T^{-1}[C^{-1}] = \Lambda_L$ as Λ is diagonal.

Thus $T^{-1}[C^{-1}] T = \Lambda^{-1}\Lambda_L$ which is still diagonal

Inverting $T^{-1} C T = (\Lambda^{-1}\Lambda_L)^{-1} = \Lambda_C$ which is diagonal.

Consider now a multiphase lossless transmission line, for which we readily derive the time domain equations in a form analogous to the derivation of Eqn. 4.4.

Thus:

$$\left.\begin{aligned} -\frac{\partial \underline{v}}{\partial x} &= [L]\,\frac{d\underline{i}}{dt} \\ -\frac{\partial \underline{i}}{\partial x} &= [C]\,\frac{d\underline{v}}{dt} \end{aligned}\right\} \tag{4.8}$$

The only difference between Eqns. 4.4 and 4.8 being that the latter is a vector-matrix equation. Note that we cannot directly obtain a d'Alembert type solution for Eqn. 4.8 because each component equation is coupled and involves voltages or currents from more than one phase. Our attempt is to transform Eqn. 4.8 into two sets of n decoupled equations which can then be solved by the method of d'Alembert.

Consider the matrix T such that

T^{-1} [L] T = L' where L' is diagonal.

Then by Theorem 4.3, T^{-1} [C] T = C' is also diagonal.

Consider the transformation $\underline{v} = T\,\underline{v}'$. Then equation 4.8 becomes

$$\left.\begin{aligned} -\frac{\partial T\underline{v}}{\partial x} &= [L]\,\frac{dT\underline{i}}{dt} \\ -\frac{\partial T\underline{i}}{\partial x} &= [C]\,\frac{dT\underline{v}}{dt} \end{aligned}\right\} \tag{4.9}$$

or

$$\left.\begin{aligned} -\frac{\partial \underline{v}'}{\partial x} &= \left(T^{-1}\,[L]\,T\right)\frac{d\underline{i}'}{dt} \\ -\frac{\partial \underline{i}'}{\partial x} &= \left(T^{-1}\,[C]\,T\right)\frac{d\underline{v}'}{dt} \end{aligned}\right\}$$

or

$$\left.\begin{aligned} -\frac{\partial \underline{v}'}{\partial x} &= L'\,\frac{d\underline{i}'}{dt} \\ -\frac{\partial \underline{i}'}{\partial x} &= C'\,\frac{d\underline{v}'}{dt} \end{aligned}\right\} \tag{4.9a}$$

Eqn. 4.9a is a collection of pairs of n uncoupled equations because L', C' are diagonal.

$$\left.\begin{aligned} -\frac{\partial \underline{v}_j'}{\partial x} &= L_j'\,\frac{d\underline{i}_i'}{dt} \\ -\frac{\partial \underline{i}_j'}{\partial x} &= C_j'\,\frac{d\underline{v}'}{dt} \end{aligned} \qquad i=1,2,...n \right\} \tag{4.9b}$$

Thus we get the n solution pairs

$$\left.\begin{aligned} i'_j &= f_1(x-c_jt + f_2(x+c_jt) \\ v'_j &= Z_j\left[(f_1(x-c_jt) - f_2(x+c_jt)\right] \end{aligned}\right\} \qquad (4.10)$$

in a manner analogous to equation 4.5.

Here $c_j = 1/\sqrt{L_jC_j}$, $Z_j = \sqrt{L_j/C_j}$

The quantities v and i are called phase quantities and v' and i' are called modal quantities. We thus obtain in the manner analogous to Eqn. 4.7 the equivalent circuit shown in Fig. 4.7.

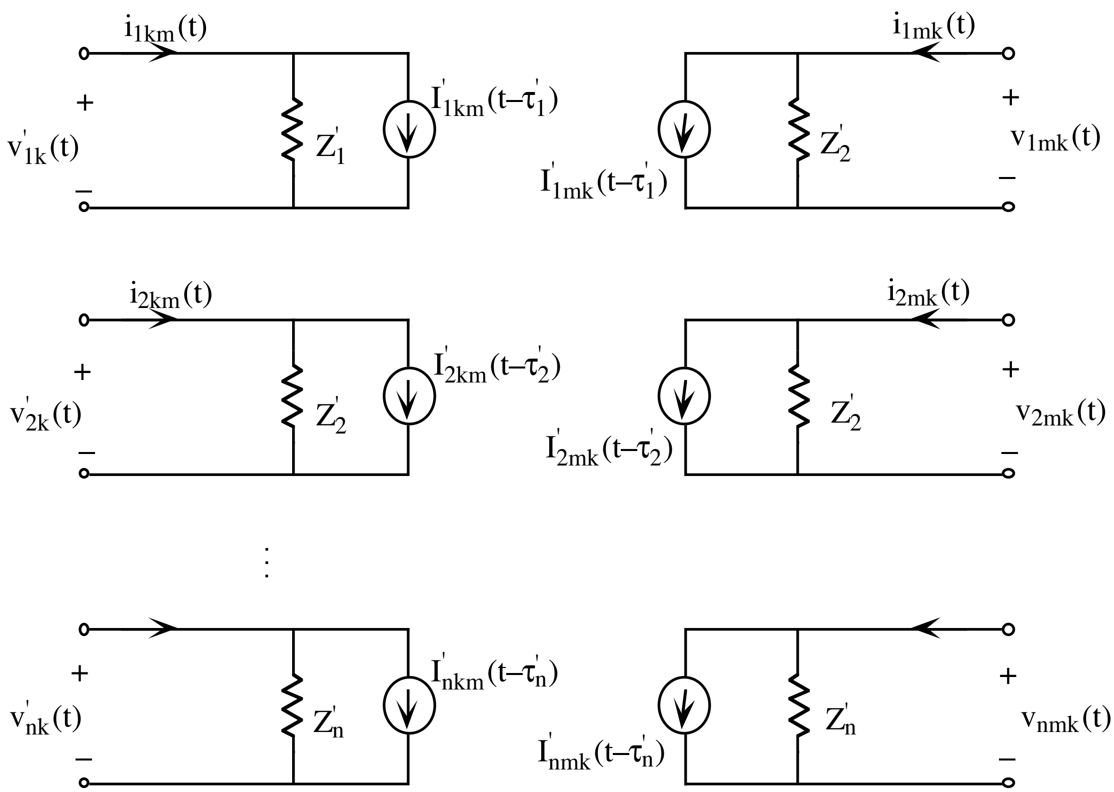


Fig. 4.7: Multiconductor line in Modal Domain

In Fig. 4.7 the history terms are analogous to the ones developed earlier, i.e. for the j[th] mode.

$$\left.\begin{aligned} & I'_{jkm}(t-Z'_j) = -\frac{1}{Z'_j}\, v'_{jm}\,(t-\tau'_j) - i'_{jmk}(t-\tau'_j) \\ & I'_{jmk}(t-Z'_j) = -\frac{1}{Z'_j}\, v'_{jk}\,(t-\tau'_j) - i'_{jkm}(t-\tau'_j) \\ & \text{where } Z'_j = \sqrt{L_j/C_j}\,, \quad \tau'_j = d\,\sqrt{L_jC_j} \;\; \text{d being the length of the line} \end{aligned}\right\} \qquad (4.11)$$

When included in an admittance type formulation the Z' (orY') should be transferred to the phase domain for inclusion in the network's Y matrix, i.e. $Z_o = T^{-1}Z'T$ or $Y_o = T^{-1}\,Y'\,T$

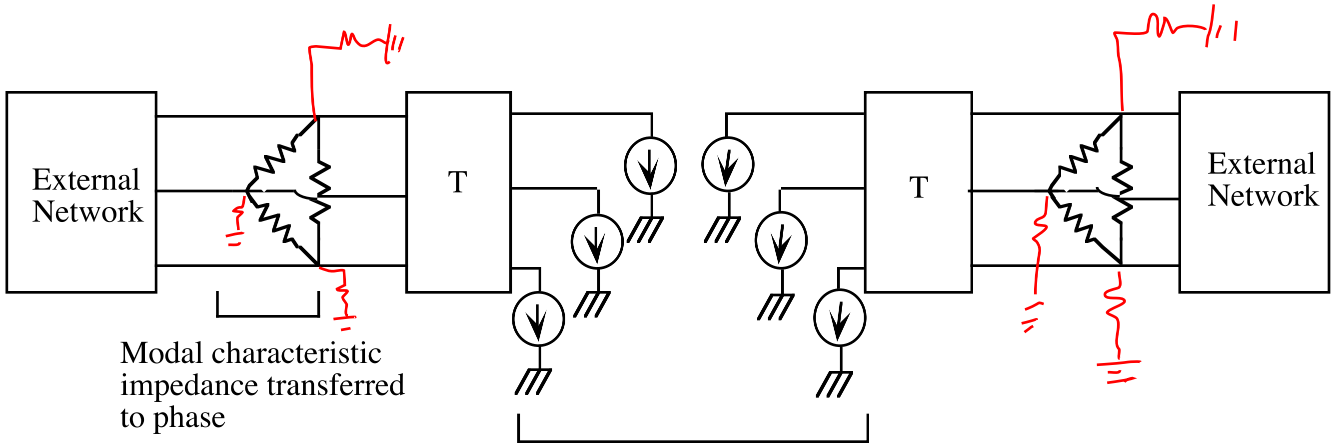


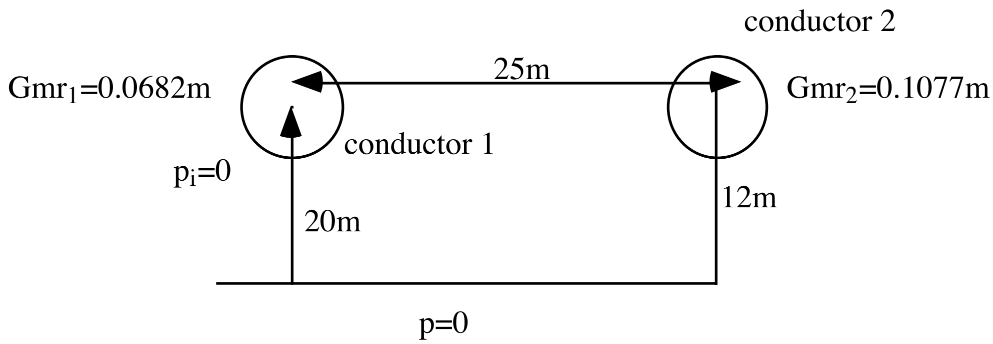
Fig. 4.8: Interfacing to the external network

The method of solving the system with a transmission line now includes the following steps:

1) Calculate the transformation matrix T.
2) Calculate the modal impedance matrix Z' or admittance matrix Y' (Note: $Y_i' = \frac{1}{Z_i'}$) from Eqn. 4.10.
3) Calculate the transferred characteristic impedance matrix Z_o (or Y_o) as seen by the external networks.
4) Form the admittance formulations for these networks using the standard techniques.
5) Calculate the history injections from equations 4.11. Note, you need modal voltages and currents to calculate these, so at every timestep you must evaluate $v' = T^{-1}v$, $i' = T^{-1}i$.
6) The actual injections into the external systems are $[T]\underline{I}_{mk}$ and $[T]\underline{I}_{km}$ respectively.
7) Calculate all voltages and thence currents in the external network and return to Step 5 and then to the next timestep.

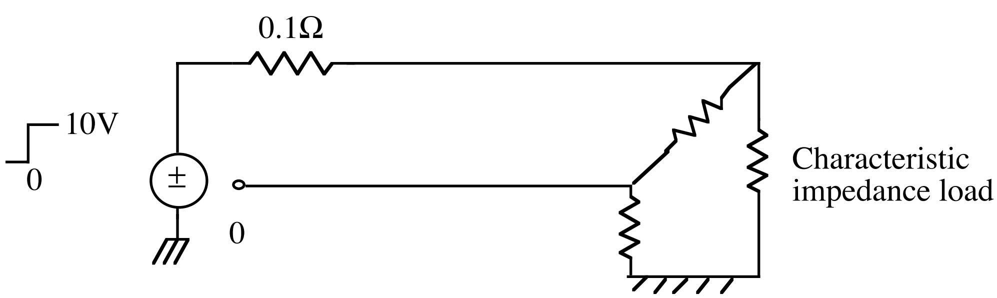
Using transformation matrices Fe and Fi.

Assignment 4:



$\mu_0 = 4\pi \times 10^{-7}$ weber/A-m
$\epsilon_0 = 8.854 \times 10^{-12}$ coul/N-m^2

The 2 conductor transmission line is shown above



Terminate it with a characteristic impedance load and excite it with the 10V step shown. Show voltages and current waveforms on each phase at the sending and receiving ends. Also plot the model quantities.

Repeat with an open circuit termination.

4.4 Inclusion of Line Resistance

If the frequency dependence of line resistance is ignored, the line may be modelled as two lossless sections with lumped resistor values R/4 at each end and a lumped resistor value of R/2 in the centre as shown in Fig. 4.9.

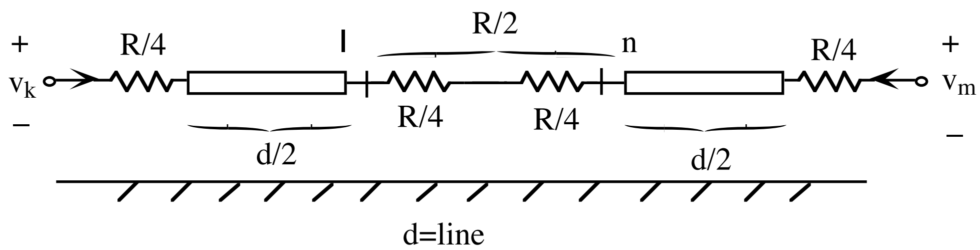


Fig. 4.9: Simple model of lossy line.

Dommel [4] claims that the incremental accuracy obtained by modeling the line with a larger number of sections is only marginal and not worth the effort. Fig. 4.10 shows the resultant current source – admittance formulation for the line in Fig. 4.9.

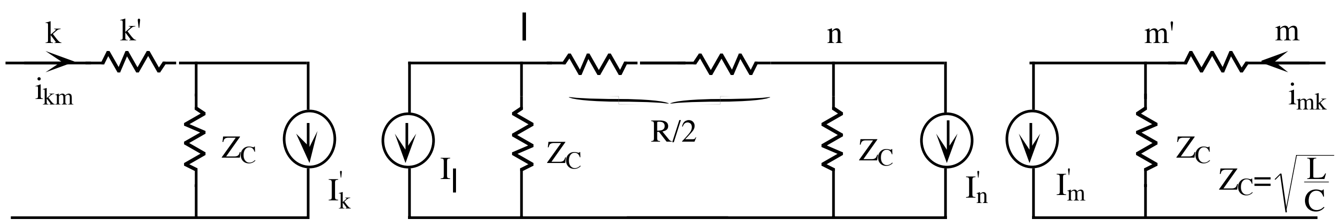


Fig. 4.10: Equivalent circuit of lossy line.

If the midpoint information is not required then the circuit in Fig. 4.10 can be collapsed to that shown in Fig. 4.11 which is similar to the one in Fig. 4.3.

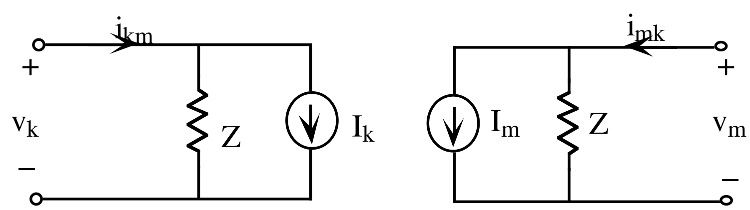


Fig. 4.11: Collapsing Fig. 4.10 to eliminate central node.

The impedances and history currents in Fig. 4.11 are given by the following equations as the reader may show:

$$Z = \sqrt{\frac{L}{C}} + \frac{R}{4} \quad , \quad \tau = d\sqrt{LC}$$ or $Z = Z_C + \frac{R}{4}$

$$I_k(t-\tau) = \left[\frac{1+h}{2}\right] \bullet \left\{-\frac{1}{Z}\, e_m(t-\tau) - h\, i_{mk}(t-\tau)\right\} \tag{4.12}$$

$$+ \left[\frac{1-h}{2}\right] \bullet \left\{-\frac{1}{Z}\, e_k(t-\tau) - h\, i_{km}(t-\tau)\right\}$$

and similarly for I_m with k, m interchanged.

Here $\quad h = \dfrac{Z_c - \frac{1}{4} R}{Z_c + \frac{1}{4} R}$

The question arises as to what value of R should be used. If earth resistivity is ingored only the term Z_{ii} (int) is a function of frequency (see eqn. 4.2), and this model is most valid as the rest of the mutual impedance terms remain frequency independent (thereby keeping the conditions similar to the lossless core).

At dc, eqn. 4.2a for the internal resistance yields

$$Z_{ii}(\text{int}) = \lim_{\omega \to 0} \left[\frac{\rho_i \sqrt{\dfrac{j\omega\mu_o}{\rho_i}}}{2\pi r_i} + \coth\left(0.7777 \sqrt{\frac{j\omega\mu_o}{\rho_i}} \bullet r_i\right) + \frac{0.3565\ \rho_i}{\pi r_i^2} \right]$$

$$= \frac{\rho_i}{\pi r_i^2} \tag{4.13}$$

which is the dc resistance of the conductor (per unit length).

However, at any other frequency Z_{ii}(int) is complex for $\rho_i \neq 0$ and thus yields different resistance values for different frequencies (the so called 'skin-effect'). We may thus choose a value for R that matches the line resistance at the frequency of most interest to our particular study.

For a multiconductor line we use a model such as in Figures 4.9 through 4.11 for each mode of the line. The modes are decomposed and recombined from phase quantities in the manner of section 4.3. This results in a different resistance for each mode. The transformation matrices T and T^{-1} are evaluated as in section 4.3 (without considering the resistive terms, otherwise they would be complex and frequency dependant!) The modal resistance matrix is then the real (and diagonal!) part of T^{-1} [Z]T.

4.4 Transmission Line Parameter Calculations for a Lossy Line

With the inclusion of earth resistance the line parameters do not appear in the convenient form [Z]=jω[L] and Y=jω[C] as discussed in section 4.2. Also it is no longer true that the matrix that diagonizes Z will also diagonalize Y. We still wish to use the treatment discussed in section 4.3 and 4.4, i.e., the assumption of a lossless line, but we should select a set of parameters that gives us most accuracy where desired.

The two parts of eqn. 4.1 can be substituted into each other to form the equations:

$$\left.\begin{aligned}\frac{\partial^2 \underline{V}}{\partial x^2} &= (ZY)\,\underline{V}\\ \frac{\partial^2 \underline{I}}{\partial x^2} &= (YZ)\,\underline{I}\end{aligned}\right\} \tag{4.14}$$

Now consider the Transformation $T_V\underline{V}' = \underline{V}$,m $T_I\underline{I}' = \underline{I}$
where T_V is such that

$$\left.\begin{aligned}T_V^{-1}\,(ZY)\,T_V &= \Lambda_{ZY}\\ T_I^{-1}\,(YZ)\,T_I &= \Lambda_{YZ}\end{aligned}\right\} \tag{4.15}$$

i.e., T_V diagonilizes the first equation of the pair 4.14 and T_I diagonlizes the second.
Note that $(ZY)^T = Y^TZ^T = (YZ)$ because Y and Z are symmetric.
Thus $\Lambda_{ZY} = \Lambda_{YZ} = \Lambda$ because symmetric matrices have the same eigenvalues.
Thus let us call

$$\left.\begin{aligned}A&=ZY\\ A^T&=YZ\end{aligned}\right\} \tag{4.16}$$

Now $T_V^{-1}\ A\ T_V = \Lambda$
on transposing:

$$T_V^T A^T (T_V^{-1})^T = \Lambda^T = \Lambda \quad \text{(from eqn. 4.15a)}$$

But this is the same as the second equation in 4.15 if we consider $T_V^T = T_I^{-1}$.

Thus we may choose $T_I = (T_V^{-1})^T$ to satisfy equation 4.15b.

Now consider

$$T_V^{-1}\ ZY\ T_V = \Lambda$$

Hence $T_V^{-1}\ Z\,(T_I T_I^{-1})\ Y\ T_V = \Lambda$

or $$\left(T_V^{-1}\ Z\ T_I\right)\left(T_I^{-1}\ Y\ T_V\right) = \Lambda \tag{4.17a}$$

Similarly by considering the second equation in 4.15,

$$\left(T_V^{-1}\,Y\,T_V\right)\left(T_I^{-1}\ Z\ T_I\right) = \Lambda \tag{4.17b}$$

It can be shown that if $PQ=QP=\Lambda$ where the eigenvalues are distinct the P and Q themselves must be diagonal. Thus,

$$T_I^{-1}\ Y\ T_V \quad \text{and } T_V^{-1}\ Z\ T_i \quad \text{are diagonal}$$

or $$\left.\begin{aligned}T_I^{-1}\ Y\ T_V &= Y_m\\ T_V^{-1}\ Z\ T_i &= Z_m\end{aligned}\right\} \tag{4.18}$$

Now Y_m and Z_m are not of the form jωC', jωL', but for small losses are approximately so. We thus select a frequency of interest ω_o and evaluate

$$\left.\begin{aligned} C' &= \frac{[Y_m]}{j\omega_o} \\ L' &= \frac{[Z_m]}{j\omega_o} \\ T_{VO} &= T_V(\omega_o) \\ T_{IO} &= T_I(\omega_o) \end{aligned}\right\} \quad (4.19)$$

We now (follow an identical procedure to that explained in section 4.3) but, use matrices T_{VO} and T_{IO} instead of the matrix T and T^{-1} in the calculations between phase and mode quantities.

4.6 Treatment of Frequency Dependence

As can be seen from eqns. 4.2 and 4.3, Z(ω) and Y(ω) are in general functions of frequency. Carson [1] has solved such equations for linear terminations using frequency domain solution methods, in an exact manner. Unfortunately modeling in the time domain has proved to be quite difficult. The treatment below is due to Marti [5]. For a single conductor line with equations (4.1) (repeated here), we have:

$$\left.\begin{aligned} -\frac{\partial \underline{V}}{\partial x} &= Z(\omega)\ \underline{I} \\ -\frac{\partial \underline{I}}{\partial x} &= Y(\omega)\ \underline{V} \end{aligned}\right\} \text{Z, Y = impedance/admittance per unit length (4.1)}$$

with $\underline{I}$ and $\underline{V}$ as phasors, ([5], [6]).

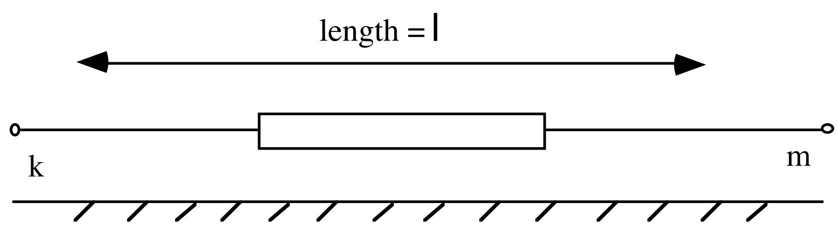


Figure 4.11. Transmission Line

$$\begin{aligned} V_k(\omega) &= \cosh[\gamma(\omega)l]V_m(\omega) - Z_C(\omega)\sinh[\gamma(\omega)l]I_m(\omega) \\ I_k(\omega) &= \frac{1}{Z_C(\omega)}\sinh[\gamma(\omega)l]V_m(\omega) - \cosh[\gamma(\omega)l]I_m(\omega) \end{aligned} \quad (4.20)$$

and conversely for V_m, I_m

Here $$Z_C(\omega) = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$$

$$\gamma(\omega) = \sqrt{ZY} = \sqrt{(R+j\omega L)\,(G+j\omega C)}$$

As in eqn. (4.6) we form the functions

$$\left.\begin{aligned} F_k(\omega) &= V_k(\omega) + Z_C(\omega)\, I_k(\omega) \\ B_k(\omega) &= V_k(\omega) - Z_C(\omega)\, I_k(\omega) \end{aligned}\right\} \qquad (4.21)$$

and likewise for m, we have

$$\left.\begin{aligned} F_m(\omega) &= V_m(\omega) + Z_C\, I_m(\omega) \\ B_m(\omega) &= V_m(\omega) - Z_C\, I_m(\omega) \end{aligned}\right\} \qquad (4.22)$$

We can eliminate V_m, I_m, V_k, I_k to get:

$$\left.\begin{aligned} B_k(\omega) &= A_1(\omega) F_m(\omega) \\ B_m(\omega) &= A_1(\omega) F_k(\omega) \\ \text{where } A_1(\omega) &= e^{-\gamma(\omega)\ell} = \frac{1}{\cosh[\gamma(\omega)\ell] + \sinh[\gamma(\omega)\ell]} \end{aligned}\right\} \qquad (4.23)$$

$A_1(\omega)$ is called the weighting function and a_1 (Fourier inverse of $A_1(\omega)$ is called the impulse response function.
Note that eqns. 4.21 and 4.22 can be used to solve for the Transmission line. The schematic representation of eqn. 4.21 (for B_k B_m only) is as in Fig. 4.12.

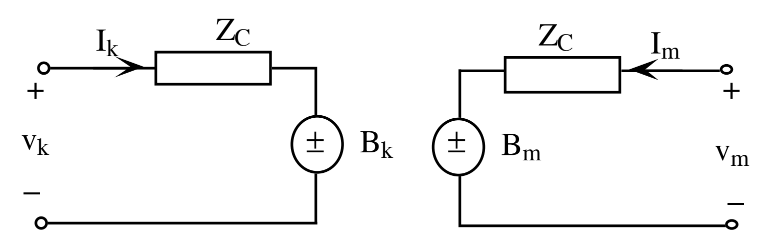


Fig. 4.12: Schematic representation of Eqn. 4.23 for B_k, B_m.

When the circuit of Fig. 4.12 is drawn in the time domain, it is as in Fig. 4.13.

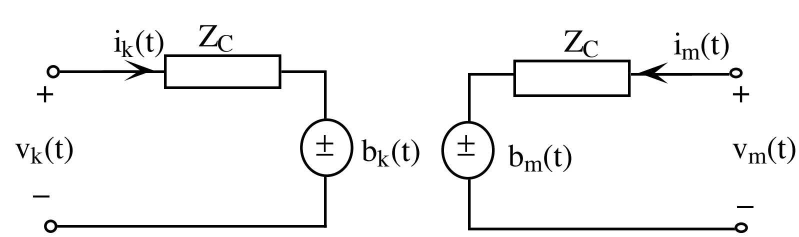


Fig. 4.13: Time domain Form.

Now

$$\left.\begin{aligned} b_k(t) &= F^{-1}(A_1(\omega) \bullet F_m(\omega)) \\ &= \int_{\tau}^{\infty} f_m(t-x)a_1(x)\,dx \\ b_m(t) &= \int_{\tau}^{\infty} f_k(t-x)a_1(x)\,dx \end{aligned}\right\} \qquad (4.24)$$

Note that the old values of $f_m(t)$ and f_k in eqn. 4.24 are old (history) values and are known from previous timesteps.

Z_C is approximated by a Foster I type realization (approximation) [5] of the true Z_C; and is of the form

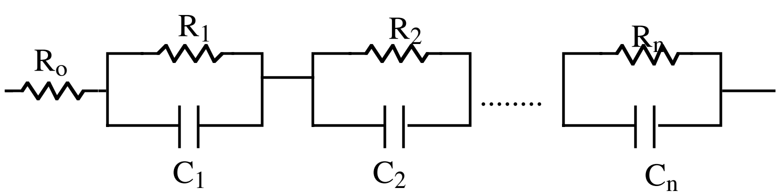


Fig. 4.14: Foster I Realization of Z_C

It is not required to do an infinite integral as suggested by eqn. 4.24. This is so because $a_1(t)$ has the form shown in Fig. 4.15 and only has any significant value between $t=\tau$ and $t=\tau+p$ (say). Thus only values of $f_m(t-x)$ between $x=\tau$ and $x=\tau+p$ need be considered in the integration to obtain $b_k(t)$.

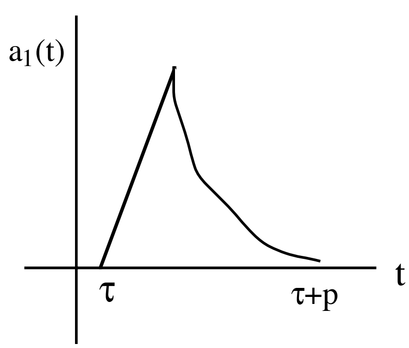


Fig. 4.15: Graph of $a_1(t)$.

We have only shown the single conductor case. For the multiconductor case we also obtain frequency dependent transformation matrices. For transmission lines we can often ignore the frequency dependence and equate T_V and T_I to those calculated for the most important frequency of interest to us. The treatment for cables however requires the use of frequency dependent matrices and is a topic of continuing research.

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