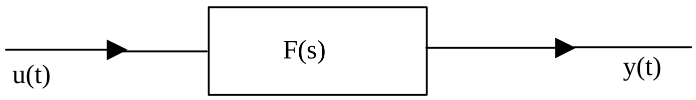
Lecture 6

Modeling Control Systems

Modeling a Transfer Function F(s)



$$Y(s) = F(s)\,U(s) \tag{6.1}$$

F is generally of the form:

$$\frac{b_0 s^n + b_1 s^{n-1} \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} \cdots + a_{n-1} s + a_n} = \frac{N(S)}{D(s)}$$

To convert G(s) into a time domain expression, we re-write F(s)U(s) = Y(s) as:

$$D(s) \bullet Y(s) = N(s) \bullet U(s) \tag{6.2}$$

and then by re-writing $s^k Y(s)$ as its time domain equivalent $\frac{d^k y}{dt^k}$, one obtains,

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^n u}{dt^n} + \cdots + b_n u \tag{6.3}$$

We then select state variables: [1]

$$\begin{aligned}
x_1 &= y - b_0 u \\
x_2 &= \dot{y} - b_0 \dot{u} - b_1 u = \dot{x}_1 - b_1 u \\
x_3 &= \ddot{y} - b_1 \ddot{u} - b_1 \dot{u} - b_2 u = \dot{x}_2 - b_2 u \\
&\vdots \\
x_n &= \overset{(n-1)}{y} - b_0 \overset{(n-1)}{u} - \cdots - b_{n-1} u = \dot{x}_{n-1} - b_{n-1} u
\end{aligned} \tag{6.4a}$$

where

$$\begin{aligned}
b_0 &= b_0 \\
b_1 &= b_1 - a_1 b_0 \\
b_2 &= b_2 - a_1 b_1 - a_2 b_0 \\
&\vdots \\
b_n &= b_n - a_1 b_{n-1} \cdots - a_{n-1} b_1 - a_n b_0
\end{aligned} \tag{6.4b}$$

Note that in Eqn. 6.4a, if we take the derivative of x_n, i.e.

$$\dot{x}_n = \overset{(n)}{y} - b_0 \overset{(n)}{u} - b_1 \overset{(n-1)}{u} - \cdots b_{n-1}\dot{u}$$

$$= b_n - a_n x_1 - a_{n-1} x_2 - a_1 x_n$$

(where we have used Eqns. 6.3 and 6.4b).

Thus we obtain the SV form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\underline{u}$$

and

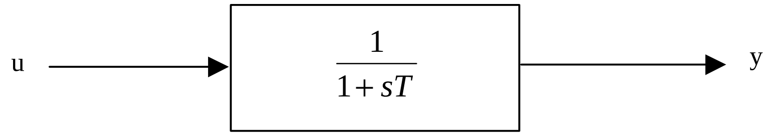
$$y = x_1 + b_0 u = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b_0 \underline{u} \qquad (6.5a)$$

or

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + Bu, \\ y &= C\underline{x} + Du \end{aligned} \qquad (6.5b)$$

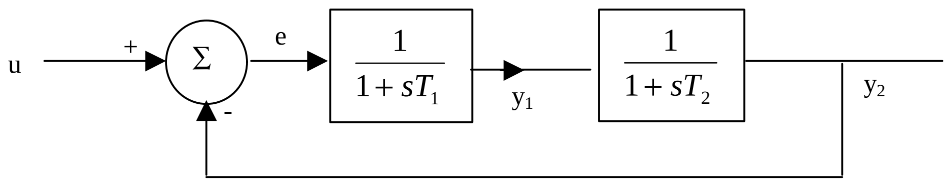
We may integrate Eqn. 6.5a in whichever manner desired.

We can also write functions for standard building blocks, i.e.:



y = REALP2(U, T, U_0)   (where U_0 is an initial condition)

and we may then write cascade blocks like:



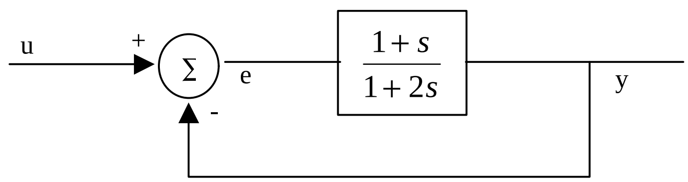
$e = u - y_2$

y_1 = REALP2(e, T_1, e_0)

y_2 = REALP2 (Y, T_1, Y_0)

and many simulation packages such as CSMP and EMTDC have these blocks built in.

There is no loss of accuracy or stability in modeling systems such as the one above either by building blocks or by a single transfer function as in (6.5) unless there are blocks with equal orders of numerator and denominator and feedback is present, i.e.,



This is because there is a direct as well as delayed relationship between y and u. for example a step change in u makes a step change in y, and it is inaccurate to model e = y – y where y is an older value from the previous time-step. In such an instant it is best to model Y(s)/U(s) = G(s)

where G(s) = $\dfrac{\dfrac{1+s}{1+2s}}{1+\dfrac{1+s}{1+2s}} = \dfrac{1+s}{2+3s}$ and using an approach as in Eqn. 6.5 (or using an available block operation on (1+s)/(2+3s)).

Discretization of Eqn. 6.5 for time domain solution [1]

In addition to trapezoidal rule or whatever other methods exist for numerical integration, presented here is a method that relies on the exact solution of Eqn. 6.5.

If $$\dot{x} = Ax + Bu \qquad (6.5a)$$

Then

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-t)}Bu(t)dt \qquad (6.6)$$

is the exact solution of Eqn. (6.5a).

(Here, e^{At} is evaluated as T $e^{\Lambda t}T^{-1}$, where T is the usual diagnonalizing matrix, Λ the diagonal eigenvalue matrix, i.e., $\Lambda = T^{-1}AT$).

If B is constant, and u is assumed constant between steps, then writing u(t) = u(k), x(t - Δt) = x(k-1), we obtain:

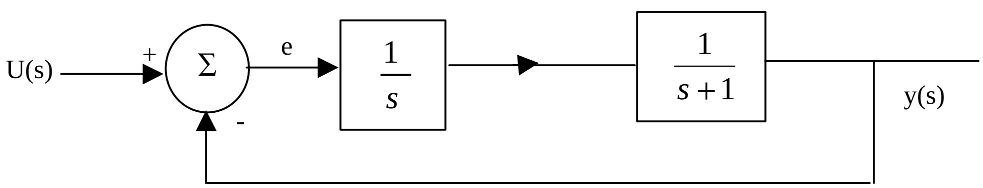
$$\underline{x}(k) = e^{A\Delta t}\,\underline{x}(k-1) + \left[\int_{t-\Delta t}^{t} e^{A(t-t)}dt\right]B\underline{u}(k-1)$$

$$= \left[e^{A\Delta t}\right]x(k-1) + \left[\int_{0}^{\Delta T} e^{At'}dt'\right]B\underline{u}(k-1) \qquad (6.7)$$

$$= Gx(k-1) + Hu(k-1)$$

Thus if the input u is constant over a time-step, (6.7) given an exact answer regardless of the time-step length (a similar approach is possible assuming u varies linearly between t-Δt and t).

Example:



Write G and H using Δt = 0.001 s.

By reduction on the block diagram,

$$y(s) = \frac{1}{s^2 + s + 1} \bullet U(s)$$

Here we notice (comparing with Eqn. 6.1)

$a_1 = 1,\ a_2 = 1,\ b_0 = 0,\ b_1 = 0,\ b_2 = 1$

Thus

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t) \qquad (6.8)$$

Eigenvalues: $\lambda(\lambda+1) + 1 = 0 \rightarrow \lambda = e^{j2\pi/3}, e^{-j2\pi/3}$

Eigenvectors: $a_1 = \begin{bmatrix} 1 \\ e^{j2p/3} \end{bmatrix}\ a_2 \begin{bmatrix} 1 \\ e^{-j2p/3} \end{bmatrix}$

$$\therefore\ \mathrm{T} = \begin{bmatrix} 1 & 1 \\ e^{j2p/3} & e^{-j2p/3} \end{bmatrix}, T^{-1} = \frac{1}{\sqrt{3}} \begin{bmatrix} e^{-j2p/3} & -j \\ e^{j2p/3} & j \end{bmatrix}$$

(check: $T^{-1}AT = \begin{bmatrix} -0.5 + 0.866j & 0 \\ 0 & -0.5 - 0.866j \end{bmatrix} = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} = \Lambda$

Then from 6.7,

$$e^{A\Delta t} = Te^{\Lambda\Delta t}T^{-1} = T \begin{bmatrix} e^{\left(e^{j2p/3}\right)\Delta t} & 0 \\ 0 & e^{\left(e^{-j2p/3}\right)\Delta t} \end{bmatrix} T^{-1} \qquad (6.9a)$$

and

$$\left[\int_0^{\Delta t} e^{At}\,dt\right]B = \underbrace{T \begin{bmatrix} \frac{1}{l_1}\left(e^{l_1\Delta t} - 1\right) & 0 \\ 0 & \frac{1}{l_2}\left(e^{l_2\Delta t} - 1\right) \end{bmatrix}}T^{-1} \quad B$$

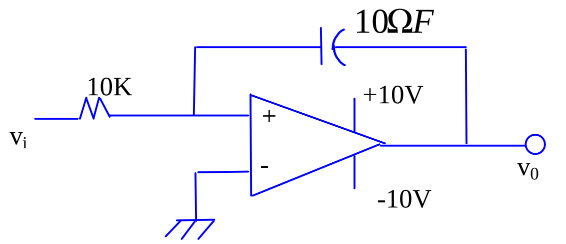
For ΔT = 1ms, we have

$$x(t) = \underbrace{\begin{bmatrix} 1 & 9.995\times10^{-4} \\ -9.995\times10^{-4} & 0.999 \end{bmatrix}}_{G} \underline{x}(t - \Delta t) + \underbrace{\begin{bmatrix} 5\times10^{-7} \\ 9.995\times10^{-4} \end{bmatrix}}_{H} u(t - \Delta t)$$

and $y(t) = x_1(t)$.

Controls with Non-linearities

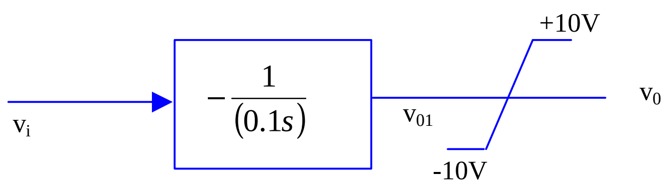
When non-linearities exist it is no longer possible to compress the block diagrams into one block as before. Also, care must be taken as to how the non-linearities affect the internal dynamics of the control system, i.e.,



$$\frac{v_0}{v_i} = -\frac{1}{0.1s}$$

Note that V_0 can only reach a maximum voltage of –10 V or a minimum voltage of + 10V.

Our first attempt may be to model the circuit as follows:



But note, as the –1/s block is ideal, if v_i = 1V say, v_{01} will keep dropping even below –10V, although v_0 will be clamped at –10. Now if v_i is reversed in polarity to –1V, nothing will change at the output until v_{01} rises above –10V as shown below (Fig. 1(a)).

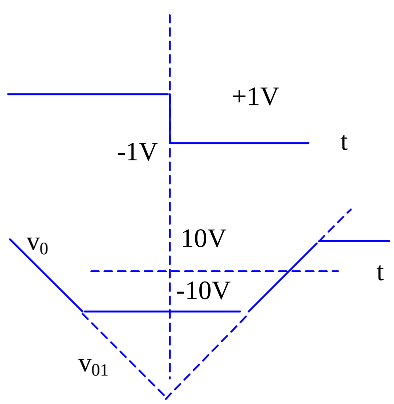


Fig. 1a

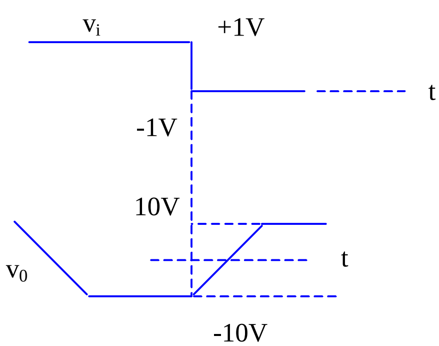


Fig. 1b

As a second attempt, we might construct a "nonstick" integrator as in Fig. 1b, in which we hold the value of the state variable clamped at the limit, i.e., if we are modeling

$$v_0 = \frac{-1}{0.1}\int v_i dt$$

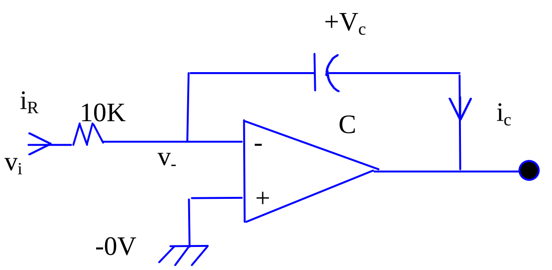
by say, rectangular rule, then in pseudocode:

(rectangular integration)

$$v_0 = v_0 - 10.\Delta t.v_i$$
$$\text{if } (v_0 > 10V) v_0 = 10V$$
$$\text{if } (v_0 < -10V) v_0 = -10V$$

This approximation is usually ok, because there are diodes that breakdown at the op-amp input thereby not allowing the inverting terminal from making large excursions from 0V. (Limiting it to ± 0.7V say.) However, if such diodes were not present (or if the 0.7v drop was to be accounted for), we must use a state transition diagram, realizing that the op-amp could be in 3 possible states: active, saturated on 10V rail, or saturated on the –10V rail. Also to be noted is that V_c, the capacitor voltage must be continuous when we make state transitions.

Thus: State 1: Active state



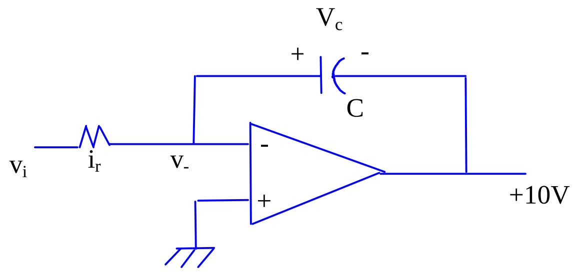
$$i_c = i_R = \frac{v_i}{R} \qquad v_c = \frac{1}{c}\int i_c \, dt$$

$$v_0 = -v_c$$

Transition: if $v_0 > 10V \rightarrow$ to saturation state 2

$v_0 < -10V \rightarrow$ to saturation state 3

State 2: Saturation state (10V)



Now $\dot{v}_c = \dfrac{v_i - 10V - v_c}{RC}$

$v_- = 10V + v_c$.

Note: transition to normal state occurs with a positive going zero crossing of v_-.

State 3: Similar to state 2, except +10V is replaced by –10v, and the negative transition of v_- causes change of state. Thus:

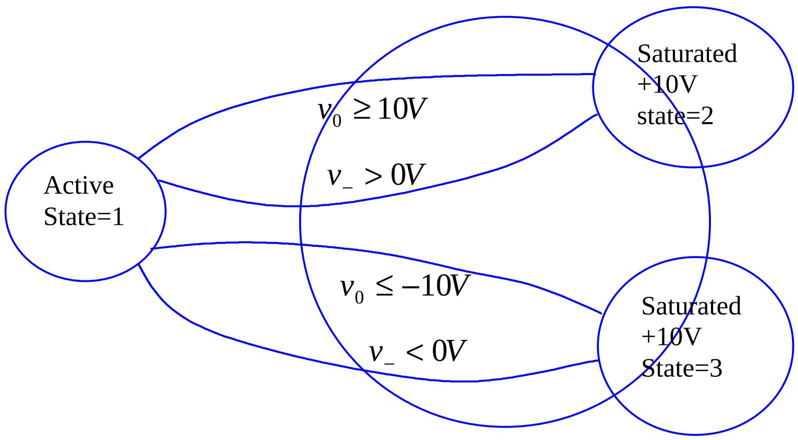


Fig. 2 State Transition Diagram

Note that now the integrator is sticky but not in the manner of our first attempt. Often such detail of modeling is necessary in the presence of non-linearities.

Sometimes it is necessary to introduce some hysterisis is necessary in the transition conditions. Also, interpolation of the capacitor voltage to the exact instant of switching (as was the case for switched networks earlier) can be applied. Often we do not go to that extent because unlike thyristor circuits, control circuits do not constantly change state during a typical simulation (although this is not necessarily true).

Another situation that often arises is with a P-I controller with the feedback from a remote point as in Fig. 3.

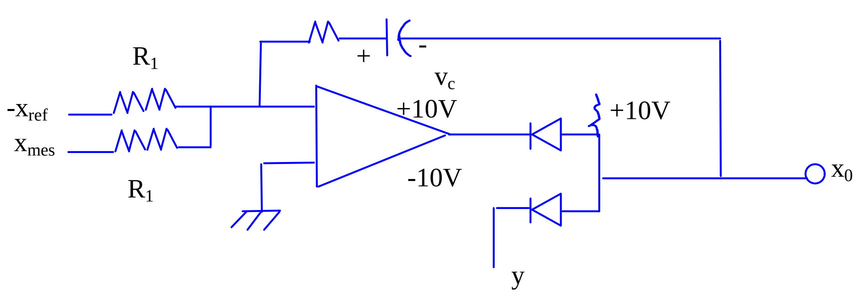


Fig. 3 Feedback to PI circuit after another block

In this circuit, if y is selected to be a very large voltage, then the circuit behaves like a conventional PI controller. If y is a smaller voltage, i.e. $y < x_2$, then the select min block passes y to the output, and the op-amp output x_2 is +10V (power supply rails). Thus we may draw the following state transition diagram. (Note that if either x_{ref}, x_{mes} or y change so as to make v_- change sign, then x_2 drops from its saturated value and assumes control again.)

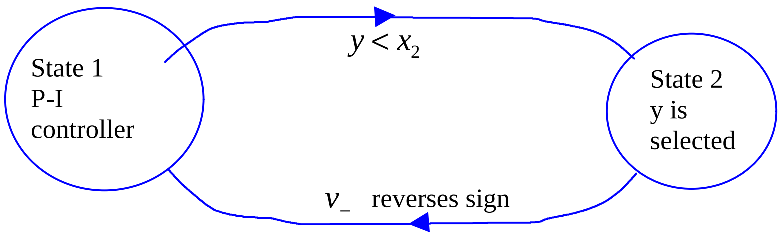
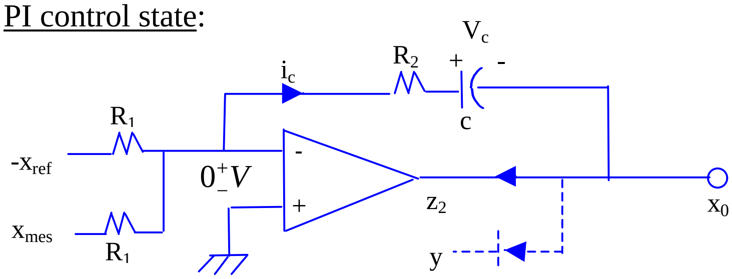


Fig. 4 State transition diagram (partial)

In this diagram we should ideally include more states such as when the op-amp saturates on its rails (as in the earlier example) even when y is not selected.

Thus the relevant equations are:

<u>PI control state</u>:



v_- = 0V because negative feedback exists.

$$i_c = \frac{x_{mes}}{R_1} - \frac{x_{ref}}{R_1} \quad , \qquad \dot{v}_c = \frac{1}{C} i_c (\leftarrow \text{state equation})$$

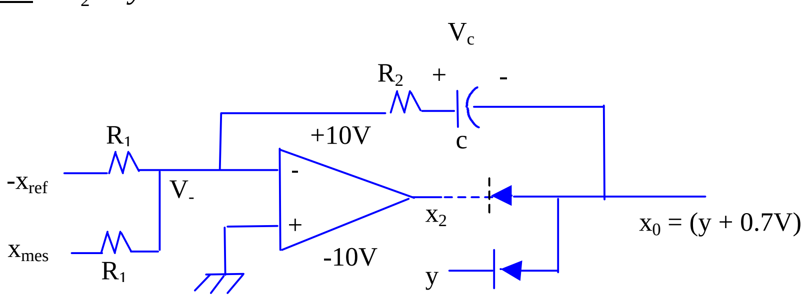
$$\boxed{x_0 = -i_c R_2 - v_c}$$

and

$$x_2 = x_0 - 0.7V$$

<u>Transition condition</u>: $x_2 < y$

<u>Deselected state</u>



$$x_0 = y + 0.7V \qquad , \qquad x_2 = 10V$$

$$v_- = \frac{\dfrac{x_{mes} - x_{res}}{R_1} + \dfrac{y + v_c}{R_2}}{\dfrac{2}{R_1} + \dfrac{1}{R_2}}$$

$$i_c = \frac{\left(x_{mes} - x_{ref}\right) - 2v_-}{R_1}, \qquad \dot{v}_c = \frac{1}{C} i_c$$

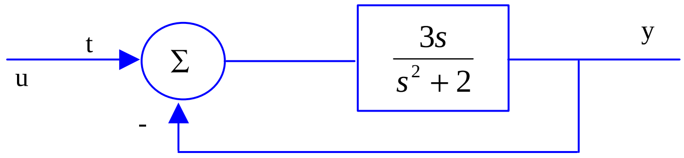
<u>Note</u>: integration for v_c must be continued in this state because we require v_c to calculate v_- and our escape from this state depends on v_- changing sign.

<u>Transition Condition</u>: v_- changes sign

<u>Note</u>: When the mode changes, x_2 makes a jump from its previous value to 10V. But v_c remains a continuous function of time. The equation to evaluate $\dot{v}_c$ changes, but that only means that we use a new equation to <u>update</u> v_c. Thus using the variables of state in our equations we can introduce non-linearities and mode switchings quite easily.

Assignment 6: (Very last Assignment of the course!!!)

a)



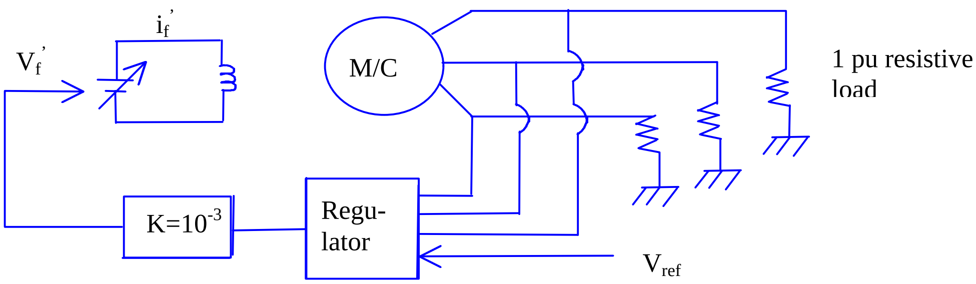
Write the discretized form of state equations of the above control block that makes use of the exact solution with a constant u.

Simulate the block with various time-steps given that

i) u is a step input.

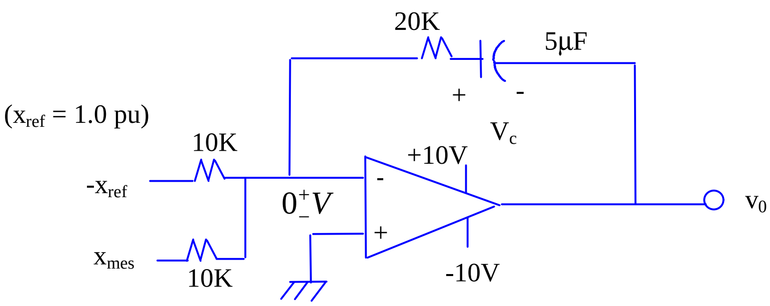
ii) u is a sinusoidal input of frequency 10 Hz.

b) i) Connect the machine model of Assignment 5 to a 1 pu resistive load as in the figure below. (You will have to modify the problem of Assignment 5 to include an admittance matrix type solution at its terminals.)



Make sure this model is working before proceeding further.

ii) Use the model shown below for the regulator. (All inputs to the regulator are in pu. So make sure that voltage is measured in pu.)

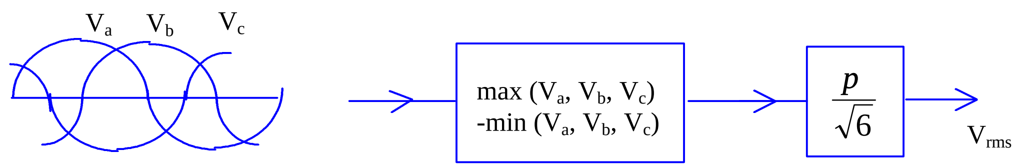


Test the working of this regulator by applying say a 1 Hz square-wave on V_{mes} of magnitude ± 1V with V_{ref} held at 0V.

Connect the regulator (after a multiplication by 10^{-3} to obtain a suitable voltage level on the machines field) because field voltage in the machine model is in pu).

Observe the action of the regulator in bringing the terminal voltage to 1 pu. Plot v_0, v_f, v_{mes} and v_a, v_b, v_c.

Hint: for ac voltage measurement



A further multiplication by $1/(V_{rms\ rated})$ (=1/13.8 kV) is necessary per unitization.

References

[1] Ogata, K., Modern Control Engineering, Prentice Hall, 1970.