Appendix B: Numerical Stability of a Discretized System Using Rectangular and Trapezoidal Integration

In this appendix, stability of a discretized system using different integration methods is analyzed. Consider a system described by the following state-space equation, which is presumed to be stable. $\underline{X}, \underline{U}$, and $\underline{Y}$ are vectors of state variables, inputs and outputs.

$$\begin{cases} \dot{\underline{X}} = [A]\underline{X} + [B]\underline{U} \\ \underline{Y} = [C]\underline{X} + [D]\underline{U} \end{cases} \tag{B.1}$$

The discrete form of this set of equations is shown in (B.2). The matrices $[G]$ and $[H]$ can be expressed in terms of $[A]$ and $[B]$ depending on the method of discretization.

$$\begin{cases} \underline{X}(t) = [G]\underline{X}(t-\Delta t) + [H]\underline{U}(t-\Delta t) \\ \underline{Y}(t) = [C]\underline{X}(t) + [D]\underline{U}(t) \end{cases} \tag{B.2}$$

In Appendix B.1, the rectangular rule of integration is used to discretize (B.1) and subsequently the numerical stability of this discretized system is analyzed. Similar analysis is performed for the method of trapezoidal integration in Appendix B.2.

Since the original system is stable, the eigenvalues of matrix $[A]$ are located in the left side of the imaginary axis in the complex plane. It will be shown in this appendix that, after applying the trapezoidal integration, the eigenvalues of matrix $[G]$ will be inside the unity circle regardless of the value of the simulation time-step. This conclusion cannot be made for the eigenvalues of matrix $[G]$ after application of the rectangular integration.

B.1 Stability of the System Using Rectangular Integration

The first equation of (B.1) is integrated for the time interval of $[t-\Delta t \quad t]$:

$$\underline{X}(t) = \int_{\tau=t-\Delta t}^{t} \left([A]\underline{X}(\tau) + [B]\underline{U}(\tau)\right) d\tau + \underline{X}(t-\Delta t) \tag{B.3}$$

Using the rectangular rule of integration:

$$\begin{aligned} \underline{X}(t) &= [A]\Delta t \cdot \underline{X}(t-\Delta t) + [B]\Delta t \cdot \underline{U}(t-\Delta t) + \underline{X}(t-\Delta t) \\ &= [G] \cdot \underline{X}(t-\Delta t) + [H] \cdot \underline{U}(t-\Delta t) \\ &\text{where} \quad [G] = I + [A]\Delta t \quad \text{and} \quad [H] = [B]\Delta t \end{aligned} \tag{B.4}$$

Now, the goal is to find out the condition in which all the eigenvalues of matrix $[G]$ are enclosed in the unity circle. If λ' is one of the eigenvalues of matrix $[G]$, (B.5) applies:

$$\left|\lambda' I - G\right| = 0 \tag{B.5}$$

Equation (B.5) can be expressed in terms of matrix $[A]$:

$$\begin{aligned} &\left|\lambda' I - \left(I + [A]\Delta t\right)\right| = 0 \\ \Rightarrow \quad &\left|\left(\lambda' - 1\right) I - [A]\Delta t\right| = 0 \\ \Rightarrow \quad &\Delta t \left|\frac{\left(\lambda' - 1\right)}{\Delta t} I - [A]\right| = 0 \end{aligned} \tag{B.6}$$

or

$$\begin{aligned} &\left|\lambda I - G\right| = 0 \\ &\text{where} \quad \lambda = \frac{\left(\lambda' - 1\right)}{\Delta t} \quad \text{or} \quad \lambda' = \lambda \cdot \Delta t + 1 \end{aligned} \tag{B.7}$$

According to definition, λ is an eigenvalue of matrix $[A]$ and its relation with λ' is shown in (B.7). The condition for the discretized set of equations (B.4) to be numerically stable is that the eigenvalues of matrix $[G]$ must be in the unity circle:

$$\left|\lambda'\right| \leq 1 \qquad \Rightarrow \qquad \left|\lambda \Delta t + 1\right| \leq 1 \tag{B.8}$$

Every complex value like λ can be in the form of (B.9):

$$\lambda = a + jb \tag{B.9}$$

From (B.8) and (B.9):

$$\begin{aligned} &\left|(a + bj)\Delta t + 1\right| \leq 1 \\ \Rightarrow \quad &\sqrt{(a\Delta t + 1)^2 + (b\Delta t)^2} \leq 1 \end{aligned} \tag{B.10}$$

This means:

$$\Delta t \leq \frac{-2a}{a^2 + b^2} \qquad \text{or} \qquad \Delta t \leq \frac{-2\,\mathrm{Re}(\lambda)}{|\lambda|^2} \tag{B.11}$$

Equation (B.11) must be valid for every eigenvalue of matrix $[A]$, therefore the maximum simulation time-step which provides a stable discretized system is:

$$\Delta t_c = \min\left(\frac{-2\,\mathrm{Re}(\lambda)}{|\lambda|^2}\right) \tag{B.12}$$

Based on the complex variable theory, the location of the eigenvalue loci of matrices $[A]$ and $[G]$ are evaluated and plotted in Figure B.1. All the eigenvalues of the matrix $[A]$ are located in a circle on the left hand side of the complex plane. The imaginary axis is tangent to this circle. Multiplying $[A]$ by Δt_c, maps these eigenvalues into another circle on the left hand side half plane with the unity radius. Finally, adding the identity matrix I to $[A]\Delta t_c$, transfers the eigenvalues into the unity circle.

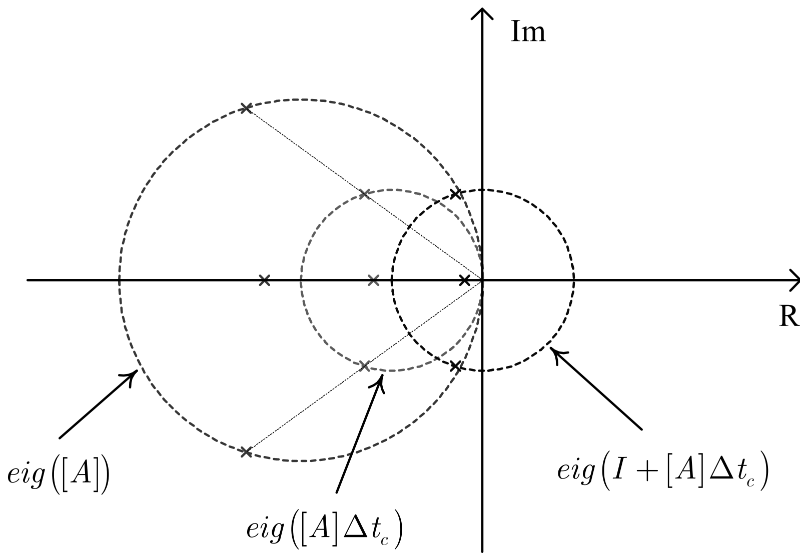

Figure B.1: Eigenvalue loci of matrix [A] and matrix [G] using rectangular integration

B.2 Stability of the System Using Trapezoidal Integration

Application of the trapezoidal integration for discretizing the state-space equations results in the following equation:

$$\underline{X}(t) = [G]\cdot\underline{X}(t-\Delta t)+[H]\cdot\underline{U}^*(t)$$

$$\text{where:}\quad \begin{cases} [G]=\left(I-\dfrac{\Delta t}{2}[A]\right)^{-1}\left(I+\dfrac{\Delta t}{2}[A]\right) \\ [H]=\left(I-\dfrac{\Delta t}{2}[A]\right)^{-1}(\Delta t[B]) \\ \underline{U}^*(t)=\dfrac{\underline{U}(t)+\underline{U}(t-\Delta t)}{2} \end{cases} \tag{B.13}$$

Assuming λ' is one of the eigenvalues of matrix $[G]$, (B.14) applies:

$$\left|\lambda' I - G\right| = 0 \tag{B.14}$$

Equation (B.14) can be expressed in terms of matrix $[A]$:

$$\left|\lambda' I - \left(I - \frac{\Delta t}{2}[A]\right)^{-1}\left(I + \frac{\Delta t}{2}[A]\right)\right| = 0$$

$$\Rightarrow \left|\lambda'\left(I - \frac{\Delta t}{2}[A]\right)^{-1}\left(I - \frac{\Delta t}{2}[A]\right) - \left(I - \frac{\Delta t}{2}[A]\right)^{-1}\left(I + \frac{\Delta t}{2}[A]\right)\right| = 0 \quad \text{(B.15)}$$

$$\Rightarrow \left|\left(\left(I - \frac{\Delta t}{2}[A]\right)^{-1}\right)\cdot\left(\lambda'\left(I - \frac{\Delta t}{2}[A]\right) - \left(I + \frac{\Delta t}{2}[A]\right)\right)\right| = 0$$

The determinant of the product of two matrices is the product of the determinants of the matrices, therefore:

$$\left|\left(I - \frac{\Delta t}{2}[A]\right)^{-1}\right| \cdot \left|\lambda'\left(I - \frac{\Delta t}{2}[A]\right) - \left(I + \frac{\Delta t}{2}[A]\right)\right| = 0 \quad \text{(B.16)}$$

This means:

$$\left|\lambda' I - I - \lambda'\frac{\Delta t}{2}[A] - \frac{\Delta t}{2}[A]\right| = 0$$

$$\Rightarrow \left|(\lambda' - 1)I - (\lambda' + 1)\frac{\Delta t}{2}[A]\right| = 0$$

$$\Rightarrow \left|\frac{(\lambda' - 1)}{(\lambda' + 1)}I - \frac{\Delta t}{2}[A]\right| = 0 \quad \text{(B.17)}$$

$$\Rightarrow \left|\frac{2}{\Delta t}\frac{(\lambda' - 1)}{(\lambda' + 1)}I - [A]\right| = 0$$

Equation (B.17) shows that λ, defined in (B.18), is an eigenvalue of the matrix $[A]$.

$$\lambda = \frac{2}{\Delta t}\frac{(\lambda' - 1)}{(\lambda' + 1)} \quad \text{or} \quad \lambda' = -\frac{\frac{\Delta t}{2}\lambda + 1}{\frac{\Delta t}{2}\lambda - 1} \quad \text{(B.18)}$$

In the following, it is proven that as long as the eigenvalues of $[A]$ are in the left side of the imaginary axis in the complex plane, the eigenvalues of matrix $[G]$ stay in the unity circle. An eigenvalue of matrix $[G]$ is expressed in terms of real and imaginary parts of an eigenvalue of $[A]$ ($\lambda = a + jb$).

$$\lambda' = -\frac{\frac{\Delta t}{2}(a+bj)+1}{\frac{\Delta t}{2}(a+bj)-1} \tag{B.19}$$

And the magnitude of λ' is evaluated in (B.20)

$$\begin{aligned} |\lambda'| &= \left| -\frac{\frac{\Delta t}{2}(a+bj)+1}{\frac{\Delta t}{2}(a+bj)-1} \right| \\ or \qquad |\lambda'| &= \sqrt{\frac{\left(a\frac{\Delta t}{2}+1\right)^2 + \left(b\frac{\Delta t}{2}\right)^2}{\left(a\frac{\Delta t}{2}-1\right)^2 + \left(b\frac{\Delta t}{2}\right)^2}} \end{aligned} \tag{B.20}$$

The condition for λ' to be in the unity circle is:

$$\begin{aligned} |\lambda'| \le 1 \qquad &\Leftrightarrow \qquad \left(a\frac{\Delta t}{2}+1\right)^2 + \left(b\frac{\Delta t}{2}\right)^2 \le \left(a\frac{\Delta t}{2}-1\right)^2 + \left(b\frac{\Delta t}{2}\right) \\ &\Leftrightarrow \qquad \left(a\frac{\Delta t}{2}+1\right)^2 \le \left(a\frac{\Delta t}{2}-1\right)^2 \end{aligned} \tag{B.21}$$

Since the eigenvalues of matrix $[A]$ are on the left hand side of the complex plane (i.e. $a \le 0$), (B.21) is always correct regardless of the value of the time-step Δt . This proves the stability preserving nature of the trapezoidal integration.